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# Systems of Linear Equations in Generalized $b-$Metric Spaces

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**Abstract.** This paper introduces a pioneering concept in the realm of metric spaces, specifically focusing on a novel category termed controlled generalized b-metric spaces (CGbMS). The study delves into the investigation of fixed points within CGbMS for self-mappings that exhibit both linear and non-linear contraction characteristics. The analysis establishes the existence and uniqueness of such fixed points, contributing valuable insights into the properties of these spaces. Moreover, the paper extends its impact by exploring diverse applications and implementations derived from the established results. One notable application is the application of these findings in solving systems of linear equations. The comprehensive examination of these applications not only underscores the practical significance of the proposed concept but also offers a broader understanding of its potential utility in various mathematical contexts. In summary, this research not only introduces and rigorously defines the concept of controlled generalized b-metric spaces but also provides a robust theoretical foundation by establishing the existence and uniqueness of fixed points. The exploration of applications, with a focus on solving linear equations, further highlights the practical implications and versatility of the proposed framework within the broader mathematical landscape.

## 1. Introduction

Over the past two decades, the scholarly community has dedicated a substantial volume of research to the exploration of fixed-point theories within quasi-symmetric spaces, commonly denoted as b-metric spaces [13–16]. Recent strides in fixed-point theory, particularly in its application to solving fractional differential and integral equations, have marked a significant leap forward. The inherent challenges posed by the nonlocal nature of these equations have necessitated the development of innovative techniques and methodologies, aimed at establishing both the existence and uniqueness of solutions [21–23]. This has, in turn, propelled fixed-point theory into a

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position of trust and prominence within the academic realm, attracting the attention and scrutiny of numerous researchers [2, 17, 21].

Beyond its theoretical foundations, the utility of fixed-point theory extends into a broad array of applications spanning diverse fields. Researchers have diligently explored the underlying principles of the theory, seeking to leverage its potential for effective problem-solving within their respective domains. As a result, fixed-point theory has emerged not merely as an abstract mathematical concept but as a valuable and versatile tool, increasingly recognized for its contribution to the advancement of research and practical applications.

The historical roots of fixed-point theory trace back to 1922 when Banach pioneered the initial proof of its existence and uniqueness [1, 2, 5]. Banach's seminal contribution, encapsulated in the Banach Contraction Principle (BCP), harnessed the power of contraction mapping over entire metric spaces, leading to an array of developments in non-linear analysis [6, 17, 20]. Despite the impressive strides made in this direction, two critical concerns have come to the forefront of scientific inquiry: the precise definition of proper contraction states and the exploration of plausible abstract metric spaces [8, 9].

In the ongoing quest for generalizations of metric spaces, several noteworthy concepts have emerged, each with a distinct focus on enriching fixed-point theory. Notable among these are the concepts of b-metric space and partial metric space. Czerwic, in collaboration with Bakhtine, introduced the concept of b-metric space in 1993 [3, 17, 18]. This innovative concept, characterized by preserving symmetry while modifying the triangular inequality of metric spaces, has become a cornerstone for proving the existence of fixed points in a myriad of studies across various disciplines [1, 4–7, 10, 13, 14, 16–20].

Matthews, in 1994, contributed significantly to this field by introducing partial metric spaces to study the semantic implications of data flow networks [13]. Acknowledged for their pivotal role in computational theory model-building, partial metric spaces have played a crucial part in extending Banach's contraction theorem [13]. Researchers have since embarked on an exploration of the topological properties of partial metric spaces, leading to remarkable results in fixed-point theories for both single and multivalued mappings [9, 10, 12, 15].

This article marks a new frontier in the extension of generalized metric spaces, a concept introduced by Jleli and Samet in 2015 [1]. Building upon the foundational principles of fixed-point theory, our contribution involves the incorporation of new and robust assumptions, culminating in the creation of a novel metric space known as Controlled Generalized b-Metric Space (CGbMS). While upholding the self-distance and symmetry conditions, we introduce modifications to the inequality condition, thereby establishing a framework that not only enhances the theory's theoretical foundation but also broadens its applicability in diverse mathematical contexts.

The organizational structure of this paper is outlined as follows. In the first section, we present a comprehensive introduction encompassing the definition, background, assumptions, and topology of CGbMS. The second section is dedicated to the rigorous proof of fixed-point theorems for self-mappings on CGbMS, emphasizing both linear and nonlinear contractions. Following the theoretical exploration, the third section delves into practical examples and applications of our results, particularly within the realm of solving systems of linear equations. Additionally, we conclude our discourse by leaving the door open for future avenues of research and exploration, proposing thought-provoking questions to stimulate further inquiry within the scientific community.

## 2. Preliminaries

**Definition 2.1.** *Let $\Omega$ be a nonempty set and, $\Theta : \Omega^2 \to [0,\infty)$, $\theta(\delta,\nu) \to [0,\infty)$. We say that $(\Omega,\Theta)$ Controlled Generalized b-Metric Spaces (CGbMS) if the following conditions are satisfies:*

(A1) $\Theta(\delta,\nu) = 0 \Rightarrow \delta = \nu$ *for any* $\delta,\nu \in \Omega$,

(A2) $\Theta(\delta,\nu) = \Theta(\nu,\delta)$ *for any* $\delta,\nu \in \Omega$,

(A3) *If* $\delta_n \in S(\Theta,\Omega,\delta)$,

$$S(\Theta,\Omega,\delta) = \{\{\delta_n\} \subset \Omega : \lim_{n\to\infty}(\delta_n,\delta) = 0\}$$

*we have*

$$\Theta(\delta,\nu) \leq \theta(\delta,\nu) \limsup_{n\to\infty} \Theta(\delta_n,\nu), \tag{2.1}$$

**Remark 2.1.** *from the definition, we can note that every generalized metric space is a controlled generalized b-metric space we just have to replace*

$$\theta(\delta,\nu) = b$$

*for all $\delta,\nu \in \Omega$ but the converse is not always true*

**Definition 2.2.** *Let $(\Omega,\Theta)$ be a controlled generalized b-metric space and let $\delta_n \subset \Omega$, we proposed that $\delta_n$ converges to $\delta$ if $\{\delta_n\} \in S(\Theta,\Omega,\delta)$*

**Proposition 2.1.** *Let $(\Omega,\Theta)$ be a controlled generalized b metric space if $\delta_n \to \delta$, for any $\nu \in \Omega$, and for $\theta(\delta,\nu) < \infty$ then $\delta = \nu$*

*Proof.* from the definition of CGbMS

$$\Theta(\delta,\nu) < \theta(\delta,\nu) \limsup_{n\to\infty} \Theta(\delta_n,\nu).$$

since $\theta(\delta,\nu)$ is finite and $\theta(\delta,\nu) < \infty$, also,

$$\limsup_{n\to\infty} \Theta(\delta_n,\nu) = 0$$

So, $\Theta(\delta,\nu) = 0 \Longrightarrow \delta = \nu$ □

**Hypothesis 2.1.** *We say that $(\Omega,\Theta)$ satisfies the hypothesis $(H_1)$ if for every sequence say $\{\delta_n\}$ that converges to some $u$ in $\Omega$ we have $\theta(u,\nu) < \infty$ for all $\nu \in \Omega$.*

**Definition 2.3.** *Let $(\Omega,\Theta,\theta)$ be a controlled generalized b-metric space, we say that $\delta_n$ is Cauchy if*

$$\lim_{n,m\to\infty} \Theta(\delta_n,\delta_m) = 0$$

**Definition 2.4.** *Let $(\Omega, \Theta)$ be a controlled generalized b-metric space we say that $(\Omega, \Theta, \theta)$ is complete, if every Cauchy sequences converge to some element in $\Omega$.*

Next, we intreduce an example of controlled generalized b-metric spaces.

**Example 2.1.** *Let $\Omega = [1, \infty)$ and $\Theta(\delta, \nu) = |\delta| + |\nu|$ for all $\delta, \nu \in \Omega$. and let $\theta : \Omega^2 \to [0, \infty)$, where $\theta(\delta, \nu) = max(|\delta|, |\nu|)$ Note that, $\Theta(\delta, \nu) = 0$ implies that $|\delta| + |\nu| = 0$ which is true only if $|\delta| = |\nu| = 0$. Also, note that $\Omega$ is symmetric. Next, we prove the triangle inequality. Let $\delta, \nu \in \Omega$ and $\delta_n$ a convergent sequence to some $\delta \in \Omega$. Hence, we have*

$$\begin{aligned}
\Theta(\delta, \nu) &= |\delta| + |\nu| \\
&\leq |\delta||\nu| + |\nu||\delta_n| \\
&= |\nu|(|\delta| + |\delta_n|) \\
&\leq max(|\delta|, |\nu|) \limsup_{n\to\infty} \Theta(\delta, \delta_n) \\
&= \theta(\delta, \nu) \limsup_{n\to\infty} \Theta(\delta, \delta_n).
\end{aligned}$$

*Therefore, $(\Omega, \Theta)$ be a controlled generalized b-metric space as desired (CGbMS).*

## 3. Fixed point results

**Theorem 3.1.** *Let $(\Omega, \Theta)$ be a complete controlled generalized b-metric space which satisfies the hypothesis $(H_1)$ and $\Gamma$ a self-mapping on $\Omega$. suppose that there exists $k \in (0, 1)$ such that*

$$\Theta(\Gamma\delta, \Gamma\nu) \leq k\Theta(\delta, \nu)$$

*Also, assume that there exists $\delta_0 \in \Omega$ such that*

$$\sup \Theta(\Gamma^i\delta_0, \Psi^j\delta_0) < \infty \ forall\ i, j \in \mathbb{N}$$

*Then, $\Gamma$ has a fixed point in $\Omega$. Inaddition, if every two fixed points u,v of $\Gamma$ have*

$$\Theta(u, v) < \infty$$

*it results in a unique fixed point*

*Proof.* Let $\delta_0 \in \Omega$, consider the sequence $\{\delta_n\}$ defined by

$$\delta_1 = \Gamma\delta_0, \delta_2 = \Gamma\delta_1 = \Gamma^2\delta_0, \delta_3 = \Gamma\delta_2 = \Gamma^3\delta_0, \cdots, \delta_n = \Gamma\delta_{n-1} = \Gamma^n\delta_0, \cdots$$

$$\begin{aligned}\Theta(\delta_n, \delta_{n+1}) &= \Theta(\Gamma_{\delta_{n-1}}, \Gamma_{\delta_n}) \\ &\leq k\Theta(\delta_{n-1}, \delta_n) \\ &= k\Theta(\Gamma_{\delta_{n-2}}, \Gamma_{\delta_{n-1}}) \\ &\leq k^2\Theta(\delta_{n-2}, \delta_{n-1}) \\ &\vdots \\ &\leq k^n\Theta(\delta_0, \delta_1)\end{aligned}$$

when $n \to \infty, k \in (0,1)$, and $\Theta(\delta_n, \delta_{n+1}) \to 0$ without loss of generality supposing that $n < m$ where $m = n + p$ for some natural number $p$.

$$\Theta(\delta_n, \delta_{n+p}) \leq k\Theta(\delta_{n-1}, \delta_{n-1+p}) \leq k^2\Theta(\delta_{n-2}, \delta_{n-2+p}) \leq k^n\Theta(\delta_0, \delta_p)$$

$k \in (0,1)$, and since $\sup \Theta(\delta_0, \delta_p) < \infty$ as $n, p \to \infty$ we deduce that $\Theta(\delta_n, \delta_{n+p}) \to 0$, as $n \to \infty$ Thus $\delta_n$ is Cauchy sequence. Considering , $(\Omega, \Theta)$ is a complete controlled generalized b-metric space we deduce that $\delta_n$ converges to the same $u \in \Omega$

$$\Theta(\delta_{n+1}, \Gamma u) = \Theta(\Gamma_{\delta_n}, \Gamma u) \leq k\Theta(\delta_n, u)$$

as $n \to \infty$ , we have, $\Theta(\delta_{n+1}, \Gamma u) = 0$
This implies that $\delta_n$ converges to $\Gamma u$ as $n \to \infty$ But, $\delta_n$ converges to $u$

So, by the uniqueness of the limit, we conclude that $\Gamma u = u$. Hence, $\Gamma$ has a fixed point in $\Omega$. Now assume that there exists another fixed point of $\Gamma$ in $\Omega$, say $v$, by our assumption $\Theta(u, v) < \infty$. Hence,

$$\Theta(u, v) = \Theta(\Gamma u, \Gamma v) \leq k\Theta(u, v) < \Theta(u, v)$$

$$\Theta(u, v) = 0.$$

Therefore, $u = v$ □

Next, we subtitle Ciric's fixed point theorem for quasicontraction-type mappings in the occurrence of controlled generalized b-metric spaces. Let$(\Omega, \Theta)$ be a controlled generalized b-metric space, and $T : \Omega \longrightarrow \Omega$, be a mapping.

**Definition 3.1.** *Let $k \in (0,1)$. we denoted that $\Gamma$ is a k-quasicontraction if*

$$\Theta(\Gamma_\delta, \Gamma\nu) \leq k \max\{\Theta(\delta, \nu), \Theta(\delta, \Gamma\delta), \Theta(\nu, \Gamma\nu), \Theta(\delta, \Gamma\nu), \Theta(\nu, \Gamma\delta)\},$$

*for all $(\delta, \nu) \in \Omega^2$*

**Proposition 3.1.** *Assume that $\Gamma$ is a k-quasicontraction for $k \in (0,1)$.So, any fixed point $\vartheta \in \Omega$ of $\Gamma$ satisfies*

$$\Theta(\vartheta, \vartheta) < \infty \Longrightarrow \Theta(\vartheta, \vartheta) = 0$$

*Proof.* Let $\vartheta \in \Omega$ be a fixed point of $\Gamma$ such that $\Theta(\vartheta, \vartheta) < \infty$. Since $\Gamma$ is a k-quasicontraction.Hence, we have

$$\Theta(\vartheta, \vartheta) = \Theta(\Gamma\vartheta, \Gamma\vartheta) \leq k\Theta(\vartheta, \vartheta),$$

Which implies that $\Theta(\vartheta, \vartheta) = 0$, since $k \in (0, 1)$. □

Next, we prove the following theorem, that is a generalization of many results in the literature.

**Theorem 3.2.** *Let $(\Omega, \Theta)$ be a complete controlled generalized b-metric space which satisfies the hypothesis $(H_1)$ and $\Gamma$ a self-mapping on $\Omega$. Assume that the following conditions satisfied:*

(N1) $\Theta(\delta, \nu)$ *is complete;*

(N2) $\Gamma$ *is a k-quasicontraction of some $k \in (0, 1)$;*

(N3) *there exist $\delta_0 \in \Omega$ such that $\Psi(\Theta, \Gamma, \delta_0) < \infty$; for every $(\delta, \nu) \in \Omega^2$, let $\theta(\delta_0, \nu) < 1/k$, $k \in (0, 1)$.*

*Then $\{\Gamma^n \delta_0\}$ converges to some $\vartheta \in \Omega$, if $\Theta(\delta_0, \Gamma\vartheta) < \infty$, then $\vartheta$ is a unique fixed point of $\Gamma$.*

*Proof.* Let $n \in \mathbb{N}$ $(n \geq 1)$. Considering $\Gamma$ is a k-quasicontraction, for all $i, j \in \mathbb{N}$, we get

$$\Theta(\Gamma^{n+i}_{\delta 0}, \Gamma^{n+j}_{\delta 0}) \leq kmax\{\Theta(\Gamma^{n-1+i}_{\delta 0}, \Gamma^{n-1+j}_{\delta 0}), \Theta(\Gamma^{n-1+i}_{\delta 0}, \Gamma^{n+i}_{\delta 0}),$$
$$\Theta(\Gamma^{n-1+i}_{\delta 0}, \Gamma^{n+j}_{\delta 0}), \Theta(\Gamma^{n-1+j}_{\delta 0}, \Gamma^{n+j}_{\delta 0}), \Theta(\Gamma^{n-1+j}_{\delta 0}, \Gamma^{n+i}_{\delta 0}))\}$$

If we consider each case from the above equation we will reach to the following inequality

$$\Psi(\Theta, \Gamma, \Gamma^n_{\delta_0}) \leq k\Psi(\Theta, \Gamma, \Gamma^{n-1}_{\delta_0})$$

Hence, for any $n \geq 1$, we can conclude that

$$\Psi(\Theta, \Gamma, \Gamma^n_{\delta_0}) \leq k^n \Psi(\Theta, \Gamma, \delta_0) \quad \{*\}$$

Using this results in $\{*\}$, for every $n, m \in \mathbb{N}$, we have

$$\Theta(\Gamma^n_{\delta_0}, \Gamma^{n+m}_{\delta_0}) \leq \Psi(\Theta, \Gamma, \Gamma^n_{\delta_0}) \leq k^n \Psi(\Theta, \Gamma, \delta_0).$$

Now, Since $\Psi(\Theta, \Gamma, \delta_0) < \infty$ and $k \in (0, 1)$, we deduce to the following

$$\lim_{n,m \to \infty} \Theta(\Gamma^n \delta_0, \Gamma^{n+m} \delta_0) = 0$$

That will implies $\{\Gamma^n_{\delta_0}\}$ is a $\Theta$-Cauchy sequence.
Since $(\Omega, \Theta)$ is a $\Theta$-complete, there exist some $\vartheta \in \Omega$ where $\{\Gamma^n_{\delta_0}\}$ is $\Theta$-convergent to $\vartheta$.
So, by our assumption, $\Theta(\delta_0, \Gamma\vartheta) < \infty$, using the inequality we will get

$$\Theta(\Gamma^n_{\delta_0}, \Gamma^{n+m}_{\delta_0}) \leq k^n \Psi(\Theta, \Gamma, \delta_0)$$

for every $m, n \in \mathbb{N}$, by recall the property from the definition of CGbMS.

$$\Theta(\vartheta, \Gamma^n \delta_0) \leq \theta(\vartheta, \Gamma\delta_0) \limsup_{m \to \infty} \Theta(\Gamma^n \delta_0, \Gamma^{n+m} \delta_0). \leq \theta(\vartheta, \Gamma\delta_0) k^n \delta(\Theta, \Gamma, \delta_0).$$

for each $n \in \mathbb{N}$.
Side by side, we have

$$\Theta(\Gamma\delta_0, \Gamma\vartheta) \leq kmax\{\Theta(\delta_0, \vartheta), \Theta(\delta_0, \Gamma\delta_0), \Theta(\vartheta, \Gamma\vartheta), \Theta(\Gamma\delta_0, \vartheta), \Theta(\delta_0, \Gamma\vartheta\}.$$

Using the above two inequality {∗}, we get

$$\Theta(\Gamma\delta_0,\Gamma\vartheta) \leq max\{k\theta(\vartheta,\Gamma\delta_0)\Psi(\Theta,\Gamma,\delta_0), k\Psi(\Theta,\Gamma,\delta_0), k\Theta(\vartheta,\Gamma\vartheta), k\Theta(\delta_0,\Gamma\vartheta)\}.$$

Once again, recall the above inequality

$$\Theta(\Gamma^2\delta_0,\Gamma\vartheta) \leq max\{k^2\theta(\vartheta,T\delta_0)\Psi(\Theta,\Gamma,\delta_0), k^2\Psi(\Theta,\Gamma,\delta_0), k\Theta(\vartheta,\Gamma\vartheta), k^2\Theta(\delta_0,\Gamma\vartheta)\}.$$

If we continue in this process , we will get,

$$\Theta(\Gamma^n\delta_0,\Gamma\vartheta) \leq max\{k^n\theta(\vartheta,\Gamma\delta_0)\Psi(\Theta,\Gamma,\delta_0), k^n\Psi(\Theta,\Gamma,\delta_0), k\Theta(\vartheta,\Gamma\vartheta), k^n\Theta(\delta_0,\Gamma\vartheta)\}.$$

For every $n \geq 1$.
Also by using our assumption of $\theta(\vartheta,T\delta_0) < 1/k$ and $k \in (0,1)$

$$\Theta(\Gamma^n\delta_0,\Gamma\vartheta) \leq max\{k^{n-1}\Psi(\Theta,\Gamma,\delta_0), k^n\Psi(\Theta,\Gamma,\delta_0), k\Theta(\vartheta,T\vartheta), k^n\Theta(\delta_0,\Gamma\vartheta)\}.$$

and for $n \geq 1$

$$\limsup_{n\to\infty} \Theta(\Gamma^n\delta_0,\Gamma\vartheta) \leq k\Theta(\vartheta,\Gamma\vartheta)$$

Since $\Theta(\delta_0,\Gamma\vartheta) < \infty$ and $\Psi(\Theta,\Gamma,\delta_0) < \infty$. we will get

$$\Theta(\Gamma\vartheta,\vartheta) \leq \theta(\vartheta,\Gamma\vartheta)\limsup_{n\to\infty} \Theta(\Gamma^n_{\delta_0},\Gamma\vartheta)$$

$$\leq 1/k \limsup_{n\to\infty} \Theta(\Gamma^n_{\delta_0},\Gamma\vartheta)$$

$$< (1/k)k\Theta(\vartheta,\Gamma\vartheta)$$

This implies that $\Theta(\Gamma\vartheta,\vartheta) = 0$,
and since we have $\Theta(\vartheta,\Gamma\vartheta) < \infty$ and $k \in (0,1)$. This means a is $\vartheta$ fixed point of $\Gamma$. Now by proposition we have $\Theta(\Gamma\vartheta,\Theta) = 0$.

Assume that there is another fixed point for $\Gamma$, let say $\varpi \in \Omega$ such that $\Theta(\vartheta,\varpi) < \infty$ and $\Theta(\varpi,\varpi) < \infty$ so we will get that $\Theta(\varpi,\varpi) = 0$ and since $\Gamma$ is a k-quasicontraction, that means

$$\Theta(\vartheta,\varpi) = \Theta(\Gamma\vartheta,\Gamma\varpi) \leq k\Theta(\vartheta,\varpi)$$

Which is mean $\vartheta = \varpi$ □

In the next section, we introduce an application about results to a system of linear equations.

## 4. Linear system of equations

Consider the set $\Omega = [1, \infty)^n$ where $n$ is a natural number. Now, consider controlled generalized b-metric space $(\Omega, \Theta)$ in Example 2.1, and that is

$$\Theta(\delta, \nu) = \max_{1 \leq i \leq n} |\delta_i| + |\nu_i|$$

for all $\delta = (\delta_1, ..., \delta_n), \nu = (\nu_1, ..., \nu_n) \in \Omega$.

**Theorem 4.1.** *Recognize the following system*

$$\begin{cases} s_{11}\delta_1 + s_{12}\delta_2 + s_{13}\delta_3 + s_{1n}\delta_n = r_1 \\ s_{21}\delta_1 + s_{22}\delta_2 + s_{23}\delta_3 + s_{2n}\delta_n = r_2 \\ \vdots \\ s_{n1}\delta_1 + s_{n2}\delta_2 + s_{n3}\delta_3 + s_{nn}\delta_n = r_n \end{cases}$$

*if* $k = \max_{1 \leq i \leq n} \Big( \sum_{j=1, j \neq i}^{n} |s_{ij}| + |1 + s_{ii}| \Big) < 1$, *then the aforementioned linear system has a unique solution.*

*Proof.* Understand the map $\Gamma : \Omega \to \Omega$ identified by $\Gamma\delta = (B + I_n)\delta - r$ in which

$$B = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix}$$

$\delta = (\delta_1, \delta_2, \cdots, \delta_n)$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in [1, \infty)^n$, $I_n$ is the identity matrix of $n \times n$ matrices then $r = (r_1, r_2, \cdots, r_n) \in \mathbb{C}^n$. to prove that $\Theta(\Gamma\delta, \Gamma\nu) \leq k\Theta(\delta, \nu)$, $\forall \delta, \nu \in [1, \infty)^n$.

We recall by

$$\tilde{B} = B + I_n = (\tilde{b}_{ij}), \;\; i, j = 1, ..., n,$$

with $\tilde{b}_{ij} = \begin{cases} s_{ij}, j \neq i \\ 1 + s_{ii}, j = i \end{cases}$ Hence,

$$\max_{1 \leq i \leq n} \sum_{j=1}^{n} |\tilde{b}_{ij}| = \max_{1 \leq i \leq n} \Big( \sum_{j=1, j \neq i}^{n} |s_{ij}| + |1 + s_{ii}| \Big) = k < 1.$$

Therefore,

$$\begin{aligned} \Theta(\Gamma\delta, \Gamma\nu) &= \max_{1 \leq i \leq n} \Big( |(\Gamma\delta)_i| + |(\Gamma\nu)_i| \Big) \\ &\leq \max_{1 \leq i \leq n} \Big( \sum_{j=1}^{n} |\tilde{b}_{ij}||\delta_j| + |\nu_j| \Big) \\ &\leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |\tilde{b}_{ij}| \max_{1 \leq k \leq n} \Big( |\delta_k| + |\nu_k| \Big) = k\Theta(\delta, \nu), \end{aligned}$$

Bering in mind all Theorem 3.1 hypotheses are accomplished . However, $\Gamma$ has fixed point that is unique. Furthermore, the aforementioned linear system has a desired unique solution as expected. □

## 5. Conclusions

In the culmination of this scholarly discourse, we have unveiled a novel iteration of metric spaces, wherein we have rigorously established fixed-point results for self-mapping. Our endeavor extends beyond the theoretical framework, as we also introduce a practical application within the context of solving linear equations, thereby contributing to the burgeoning field of applied mathematics. Furthermore, our study serves as a catalyst for the generalization of several established findings within the existing literature, underscoring the versatility and adaptability of the proposed metric space.

As we bring this article to a close, it is incumbent upon us to offer not only a summary of our contributions but also to lay the groundwork for future inquiries. One of the noteworthy contributions lies in the formulation and proof of fixed-point theorems for self-mappings within our introduced metric space. This theoretical advance has implications beyond the confines of our specific study and holds the potential for applications in various mathematical and scientific domains.

Moreover, the application we introduce for solving systems of linear equations adds a pragmatic dimension to our theoretical findings. By providing a real-world context for the utility of our metric space, we aim to bridge the gap between theoretical abstraction and practical relevance. This application not only enriches our understanding of the proposed metric space but also opens avenues for interdisciplinary collaborations and practical implementations.

In a broader context, our work contributes to the ongoing dialogue within the academic community, presenting an innovative perspective on metric spaces and their applicability. We extend a metaphorical bridge to connect our findings with the existing body of literature, thereby contributing to the collective knowledge in this domain. The generalization of established results further attests to the robustness and adaptability of our proposed metric space, reinforcing its potential significance in various mathematical inquiries.

As we look ahead, it is customary to conclude with an invitation for further exploration. We posit an open question that beckons as a beacon for future research endeavors: Can we identify a set of weaker hypotheses to supplant hypothesis (H1) in Theorem (3.1) and (3.4)? This question serves as a call to action, prompting researchers to delve into the intricacies of the proposed metric space and its underlying assumptions. Addressing this query may lead to refinements in the theory, widening its scope and applicability, and fostering a continued trajectory of scholarly investigation. In this way, our work not only contributes to the current state of knowledge but also sets the stage for ongoing discourse and exploration within the dynamic realm of metric spaces and fixed-point theorems.

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## References


[1] M. Jleli, B. Samet, A Generalized Metric Space and Related Fixed Point Theorems, Fixed Point Theory Appl. 2015 (2015), 61. https://doi.org/10.1186/s13663-015-0312-7.

[2] N. Souayah, N. Mlaiki, S. Haque, D. Rizk, A.S. Baazeem, W. Shatanawi, A New Type of Three Dimensional Metric Spaces with Applications to Fractional Differential Equations, AIMS Math. 7 (2022), 17802–17814. https://doi.org/10.3934/math.2022980.

[3] S. Czerwik, Contraction Mappings in $b$-Metric Spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5–11. https://eudml.org/doc/23748.

[4] P. Hitzler, A.K. Seda, Dislocated Topologies, J. Electr. Eng. 51 (2000), 3–7.

[5] S. Banach, Sur les Opérations dans les Ensembles Abstraits et leur Application aux Équations Intégrales, Fundamenta Mathematicae 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[6] A.J. Gnanaprakasam, G. Mani, O. Ege, A. Aloqaily, N. Mlaiki, New Fixed Point Results in Orthogonal B-Metric Spaces with Related Applications, Mathematics 11 (2023), 677. https://doi.org/10.3390/math11030677.

[7] N. Savanović, I.D. Aranđelović, Z.D. Mitrović, The Results on Coincidence and Common Fixed Points for a New Type Multivalued Mappings in $b$-Metric Spaces, Mathematics 10 (2022), 856. https://doi.org/10.3390/math10060856.

[8] W. Shatanawi, A. Pitea, Best Proximity Point and Best Proximity Coupled Point in a Complete Metric Space with (P)-Property, Filomat 29 (2015), 63–74. https://doi.org/10.2298/FIL1501063S.

[9] T. Van An, N. Van Dung, Z. Kadelburg, S. Radenović, Various Generalizations of Metric Spaces and Fixed Point Theorems, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 109 (2015), 175–198. https://doi.org/10.1007/s13398-014-0173-7.

[10] M. Wang, N. Saleem, X. Liu, A.H. Ansari, M. Zhou, Fixed Point of $(\alpha,\beta)$-Admissible Generalized Geraghty F-Contraction with Application, Symmetry 14 (2022), 1016. https://doi.org/10.3390/sym14051016.

[11] W. Shatanawi, Z. Mustafa, N. Tahat, Some Coincidence Point Theorems for Nonlinear Contraction in Ordered Metric Spaces, Fixed Point Theory Appl. 2011 (2011), 68. https://doi.org/10.1186/1687-1812-2011-68.

[12] A. Al-Rawashdeh, H. Aydi, A. Felhi, S. Sahmim, W. Shatanawi, On Common Fixed Points for $\alpha - F$-Contractions and Applications, J. Nonlinear Sci. Appl. 09 (2016), 3445–3458. https://doi.org/10.22436/jnsa.009.05.128.

[13] S.G. Matthews, Partial Metric Topology, Ann. N. Y. Acad. Sci. 728 (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x.

[14] R. Pant, R. Shukla, H.K. Nashine, R. Panicker, Some New Fixed Point Theorems in Partial Metric Spaces with Applications, J. Funct. Spaces 2017 (2017), 1072750. https://doi.org/10.1155/2017/1072750.

[15] V. Berinde, M. Păcurar, 2022. The Early Developments in Fixed Point Theory on $b$-Metric Spaces, Carpathian J. Math. 38 (2022), 523–538. https://www.jstor.org/stable/27150504.

[16] D. Santina, W.A. Mior Othman, K.B. Wong, N. Mlaiki, New Generalization of Metric-Type Spaces-Strong Controlled, Symmetry 15 (2023), 416. https://doi.org/10.3390/sym15020416.

[17] A. Aloqaily, D.S. Sagheer, I. Urooj, S. Batul, N. Mlaiki, Solving Integral Equations via Hybrid Interpolative $\mathfrak{R}$ $\mathfrak{I}$-Type Contractions in $b$-Metric Spaces, Symmetry 15 (2023), 465. https://doi.org/10.3390/sym15020465.

[18] I.A. Bakhtin, The Contraction Mapping Principle in Quasi-Metric Spaces, Funct. Anal. 30 (1989), 26–37.

[19] L.C. Ceng, C.F. Wen, Y.C. Liou, J.C. Yao, A General Class of Differential Hemivariational Inequalities Systems in Reflexive Banach Spaces, Mathematics 9 (2021), 3173. https://doi.org/10.3390/math9243173.

[20] M. Jleli, E. Karapınar, B. Samet, Further Generalizations of the Banach Contraction Principle, J. Inequal. Appl. 2014 (2014), 439. https://doi.org/10.1186/1029-242X-2014-439.

[21] A. Aloqaily, N. Souayah, K. Matawie, N. Mlaiki, W. Shatanawi, A New Best Proximity Point Results in Partial Metric Spaces Endowed with a Graph, Symmetry 15 (2023), 611. https://doi.org/10.3390/sym15030611.

[22] T. Abdeljawad, E. Karapınar, K. Taş, Existence and Uniqueness of a Common Fixed Point on Partial Metric Spaces, Appl. Math. Lett. 24 (2011), 1900–1904. https://doi.org/10.1016/j.aml.2011.05.014.

[23] K. Shah, T. Abdeljawad, B. Abdalla, M.S. Abualrub, Utilizing Fixed Point Approach to Investigate Piecewise Equations with Non-Singular Type Derivative, AIMS Math. 7 (2022), 14614–14630. https://doi.org/10.3934/math.2022804.