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Uncertainty Principles for the Fractional Dunkl Transform

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Abstract. In this paper, we examine the Donoho–Stark uncertainty principle in the context of the fractional Dunkl transform. We rigorously derive a formulation of the Donoho–Stark uncertainty principle for the fractional Dunkl transform and provide an application that illustrates its practical significance. Furthermore, we introduce a signal restoration algorithm tailored for the fractional Dunkl transform.

1. Introduction

The Fractional Fourier Transform (FrFT) has emerged as a powerful generalization of the classical Fourier Transform, extending its capabilities by introducing an additional degree of freedom, as a tool in harmonic analysis, building upon the works of Wiener [20] and Condon [2]. Namias was the first to formally introduce the term "Fractional Fourier Transform" in 1982 [12]. Later, in 1987, McBride refined Namias' fractional operators [10], establishing new theorems for these modified operators and developing an operational calculus.. This extension allows the FrFT to interpolate between the time and frequency domains, providing a more flexible framework for signal analysis, particularly in areas such as optics, quantum mechanics, and signal processing . Over the years, the FrFT has evolved from a theoretical concept into a versatile tool for solving practical problems in various scientific and engineering fields.

The evolution of the FrFT has sparked interest in further generalizations, aimed at broadening its applicability and improving its adaptability to specific domains. These generalizations include the fractional Hankel transform [9], fractional Dunkl transform [6], fractional Jacobi-Dunkl transform [7], fractional Opdam-Cherednik transform [1], Hardy type theorems for fractional Dunkl transform [16] and fractional Bessel-Fourier wavelet transform as aprticular case in [11]. Each generalization carries distinct properties and offers unique advantages for specialized applications.

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An important aspect of the fractional Fourier transform (FrFT) and its generalizations is their connection to the uncertainty principle, a foundational concept in harmonic analysis and quantum mechanics. The classical Fourier transform inherently expresses the uncertainty principle, which states that a signal cannot be simultaneously localized in both time and frequency domains. The FrFT extends this principle by offering a continuous spectrum between time and frequency representations, allowing for a more flexible distribution of uncertainty. This property makes the FrFT a valuable tool for analyzing signals with varying levels of localization, offering new insights into time-frequency analysis and paving the way for advanced applications in fields such as signal processing, quantum mechanics, and optics. Through its generalizations, the uncertainty principle is further adapted to non-Euclidean spaces and systems with additional symmetries, expanding the range of problems that can be addressed.

As a continuation of our investigation into uncertainty principles in the behavior of various operators, such as the Weinstein transform [14, 17], the Weinstein wavelet transform [15, 18], and the Weinstein-Gabor transform [8], our main objective in the present paper is to study several uncertainty principles for the fractional Dunkl transform.

The layout of this article is as follows. Section 2 is dedicated to providing an overview of the fractional Dunkl transform and its basic properties. In section 3, we explore a formulation of the Donoho–Stark uncertainty principle for the fractional Dunkl transform and provide an application. Finally, we introduce a signal restoration algorithm tailored for the fractional Dunkl transform, in Section 4.

2. Preliminaires

Let us consider the following functional spaces along this paper:

- $C_0(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$ vanish at infinity.
- $L^p_\nu(\mathbb{R})$ denote the space of all measurable functions on $\mathbb{R}$ such that:

$$||h||_{\nu,p} = \left(\int_{\mathbb{R}} |h(\xi)|^p |\xi|^{2\nu+1} d\xi\right)^{\frac{1}{p}} < +\infty, \quad \text{if} \quad p \in [1, \infty),$$
$$||h||_{\nu,\infty} = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}} |h(\xi)| < \infty.$$

Dunkl operators are differential-difference operators linked to finite reflection groups in Euclidean space. C.F. Dunkl first introduced these operators in his works [3–5], where he established the foundation for a theory of special functions and integral transforms in multiple variables within the context of reflection groups. The fractional Dunkl operator Λ^θ_ν introduced by Ghazouani and Bouzeffour [6] is defined for all $f \in C^1(\mathbb{R})$ as below:

$$\begin{aligned}\Lambda^\theta_\nu h(\xi) &:= \frac{d}{d\xi} h(\xi) + \frac{2\nu+1}{\xi}\left[\frac{h(\xi) - h(-\xi)}{2}\right] + i\cot(\theta)\xi h(\xi)\\ &= \Lambda_\nu h(\xi) + i\cot(\theta)\xi h(\xi),\end{aligned}$$

where $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$, $\nu \geq -1/2$ and

$$\Lambda_\nu h(\xi) = \frac{d}{d\xi}h(\xi) + \frac{2\nu+1}{\xi}\left[\frac{h(\xi)-h(-\xi)}{2}\right]$$

denoted the classical Dunkl operator of parameter ν related to the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ (see [3]).

The fractional Dunkl operator Λ_ν^θ extends a broad range of integral transforms, depending on the choice of the parameter θ, the multiplicity function ν, and the specific function spaces, including:

(1) If $\nu = -1/2$, then the fractional Dunkl operator Λ_ν^θ is reduced to:
   (a) the operator $\frac{d}{d\xi}$ (when $\theta = \pi/2$) which is closely associated with the classical Fourier transform.
   (b) the operator $\frac{d}{d\xi} + i\cot(\theta)\xi$, which is closely related to the classical fractional Fourier transform [10].

(2) If $\nu > -1/2$ then the fractional Dunkl operator Λ_ν^θ is reduced to:
   (a) Λ_ν^θ coincides, for $\theta = \pi/2$, with the Dunkl operator Λ_ν which is closely associated with the Dunkl transform [3–5].
   (b) The fractional Hankel operator

$$\mathcal{B}_{\nu,\theta} = \frac{d^2}{d\xi^2} + \left(\frac{2\nu+1}{\xi} + 2i\cot(\theta)\xi\right)\frac{d}{d\xi} + 2i(\nu+1)\cot(\theta) - \cot^2(\theta)\xi^2,$$

   on the even subspace $C^2(\mathbb{R})^e = \left\{h \in C^2(\mathbb{R}) : h(\xi) = h(-\xi)\right\}$ of the square $\left(\Lambda_\nu^\theta\right)^2$, which is closely associated with the fractional Hankel transform [9].
   (c) The Bessel operator (when $\theta = \pi/2$)

$$\mathcal{B}_\nu = \frac{d^2}{d\xi^2} + \frac{2\nu+1}{\xi}\frac{d}{d\xi},$$

   on the even subspace $C^2(\mathbb{R})^e = \left\{h \in C^2(\mathbb{R}) : h(\xi) = h(-\xi)\right\}$ of the square $\left(\Lambda_\nu^\theta\right)^2$, which is closely associated with the Hankel transform.

Definition 2.1. *For $n \in \mathbb{Z}$ and h a function in $L^1_\nu(\mathbb{R})$, the fractional Dunkl transform $\mathcal{D}_\nu^\theta$ is defined as below [6]:*

(1) $\mathcal{D}_\nu^{2n\pi} = h(\xi)$
(2) $\mathcal{D}_\nu^{(2n+1)\pi}h(\xi) = h(-\xi)$
(3) $\mathcal{D}_\nu^{(\theta+2n)\pi}h(\xi) = \mathcal{D}_\nu^\theta h(\xi)$, *with* $\theta \in \mathbb{R}$
(4) *If* $\theta \in ((2n-1)\pi, (2n+1))\pi$, *then*

$$D_\nu^\theta h(\lambda) = c_\nu^\theta \int_{\mathbb{R}} h(\xi)\Psi_\nu^\theta(\lambda,\xi)|\xi|^{2\nu+1}d\xi,$$

where

$$c_\nu^\theta = \frac{e^{i(\nu+1)(\hat{\theta}\pi/2-\theta)}\alpha_\nu}{|\sin(\theta)|^{\nu+1}}, \qquad \alpha_\nu = \frac{1}{2^{\nu+1}\Gamma(\nu+1)}, \qquad \hat{\theta} := \mathrm{sgn}(\sin(\theta)),$$

and $\Psi_\nu^\theta(\lambda,\xi)$ *denote the fractional Dunkl kenrel defined by*

$$\Psi_\nu^\theta(\lambda,\xi)=e^{-\frac{i}{2}(\lambda^2+\xi^2)cot(\theta)}E_\nu\left(\frac{i\lambda\xi}{sin(\theta)}\right), \tag{2.1}$$

which is the unique analytic solution of the following differential–difference system, for all $\xi\in\mathbb{R}$*:*

$$\begin{cases}\Lambda_\nu^\theta h=\frac{i\xi}{\sin(\theta)}h\\ f(0)=e^{-\frac{i}{2}\xi^2\cot(\theta)}.\end{cases}$$

Note that $E_\nu(z)$ *is the Dunkl kernel of type* A_2 *given by (see [13])*

$$E_\nu(z)=j_\nu(iz)+\frac{z}{2(\nu+1)}j_{\nu+1}(iz),$$

and j_ν *is the normalized spherical Bessel function*

$$j_\nu(\lambda):=2^\nu\Gamma(\nu+1)\frac{J_\nu(\lambda)}{\lambda^\nu}=\Gamma(\nu+1)\sum_{n=0}^{+\infty}\frac{(-1)^n(\lambda/2)^{2n}}{n!\Gamma(n+\nu+1)}. \tag{2.2}$$

Note that J_ν *is the classical Bessel function (see [19]).*

In the next proposition, we highlight several important properties of the fractional Dunkl kernel that will be beneficial for the rest of the paper (see [6]).

Proposition 2.1. *Let* $\theta\in\mathbb{R}\backslash\pi\mathbb{Z}$*.*

(1) *For each* $\lambda\in\mathbb{R}$ *and* $\xi\in\mathbb{C}$*, the fractional Dunkl kenrel have the integral representation*

$$\Psi_\nu^\theta(\lambda,\xi)=\frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}e^{-\frac{i}{2}(\lambda^2+\xi^2)cot(\theta)}\int_{-1}^{1}e^{\frac{i\lambda\xi t}{sin(\theta)}}(1-t^2)^{\nu-\frac{1}{2}}(1+t)dt. \tag{2.3}$$

As particular case, we have the following inequality:

$$\forall\lambda\in\mathbb{R},\forall\xi\in\mathbb{R};\qquad|\Psi_\nu^\theta(\lambda,\xi)|\le 1. \tag{2.4}$$

(2) *There exists a positive constant* $K(\nu,\theta)>0$ *such that for all* λ *and* $\xi\in\mathbb{R}$*, we have the following inequality:*

$$|\Psi_\nu^\theta(\lambda,\xi)|\le K(\nu,\theta)min(1,|\lambda\xi|^{-(\nu+\frac{1}{2})}). \tag{2.5}$$

In the next proposition, we highlight several important properties of the fractional Dunkl transform that will be beneficial for the rest of the paper (see [6]).

Proposition 2.2. (1) *Suppose that* $\theta\notin\pi\mathbb{Z}$*, for all* $h\in L_\nu^1(\mathbb{R})$*, its Fractional Dunkl transform* $\mathcal{D}_\nu^\theta h$ *belongs to* $C_0(\mathbb{R})$ *and verifies:*

$$\|\mathcal{D}_\nu^\theta h\|_{\nu,\infty}\le\frac{1}{\Gamma(\nu+1)(2|sin(\theta)|)^{\nu+1}}\|h\|_{\nu,1}. \tag{2.6}$$

(2) *Let* θ,β *be in* $\mathbb{R}$ *and let* $h\in L_\nu^1(\mathbb{R})$ *with* $\mathcal{D}_\nu^\beta(h)\in L_\nu^1(\mathbb{R})$*, then:*

$$\mathcal{D}_\nu^\theta\circ\mathcal{D}_\nu^\beta(h)=\mathcal{D}_\nu^{\theta+\beta}(h). \tag{2.7}$$

with equality almost everywhere when $\theta+\beta\in\pi\mathbb{Z}$*.*

(3) *Let $\theta \in \mathbb{R}$. If $h \in L^1_\nu(\mathbb{R}) \cap L^2_\nu(\mathbb{R})$, then $\mathcal{D}^\theta_\nu h \in L^2_\nu(\mathbb{R})$ and*

$$\|\mathcal{D}^\theta_\nu h\|_{2,\nu} = \|h\|_{2,\nu}. \tag{2.8}$$

(4) *Let $\theta \in \mathbb{R}$. The fractional Dunkl transform $\mathcal{D}^\theta_\nu$ has a unique extension to an unitary operator on $L^2_\nu(\mathbb{R})$, with inverse $(\mathcal{D}^\theta_\nu)^{-1} = \mathcal{D}^{-\theta}_\nu$.*

3. Donoho–stark Uncertainty Principle

Let Ω and Σ be two measurable subsets of $\mathbb{R}$. Let us define the time-limiting operator P_Ω as below

$$P_\Omega h = h\mathcal{X}_\Omega,$$

and, we define the partial sum operator S_Σ by

$$\mathcal{D}^\theta_\nu\left(S_\Sigma h\right) = \mathcal{X}_\Sigma \mathcal{D}^\theta_\nu h.$$

We denote by $\mathcal{B}^p_\nu(\Sigma)$, with $1 \le p \le 2$, the set of all functions $h \in L^p_\nu(\mathbb{R})$ that are bandlimited to Σ,

$$h \in \mathcal{B}^p_\nu(E) \Leftrightarrow S_\Sigma h = h.$$

Definition 3.1. *Let $\varepsilon \in (0,1)$ and (Σ,Ω) be a pair of measurable subsets of $\mathbb{R}$. For all $h \in L^1_\nu(\mathbb{R}) \cap L^2_\nu(\mathbb{R})$, we say that*

(1) *h is ε-timelimited on Σ, if*

$$\int_{\mathbb{R}\setminus\Sigma} |h(\xi)||\xi|^{2\nu+1} d\xi \le \varepsilon \int_{\mathbb{R}} |h(\xi)||\xi|^{2\nu+1} d\xi. \tag{3.1}$$

(2) *h is ε-bandlimited on Ω, if*

$$\left(\int_{\mathbb{R}\setminus\Omega} \left|\mathcal{D}^\theta_\nu h(\xi)\right|^2 |\xi|^{2\nu+1} d\xi\right)^{\frac{1}{2}} \le \varepsilon \left(\int_{\mathbb{R}} |h(\xi)|^2 |\xi|^{2\nu+1} d\xi\right)^{\frac{1}{2}}. \tag{3.2}$$

Definition 3.2. *Let Ω and Σ be two measurable subsets of $\mathbb{R}$. Let $1 < p \le 2$, such that $q = \frac{p}{p-1}$ and h in $L^p_\nu(\mathbb{R})$. We say that:*

(1) *h is ε-concentrated to Ω in $L^p_\nu(\mathbb{R})$, if there exists a measurable function g vanishing outside Ω such that*

$$\|h - g\|_{\nu,p} \le \varepsilon \|h\|_{\nu,p}.$$

(2) *$\mathcal{D}^\theta_\nu h$ is ε-concentrated to Σ in $L^q_\nu(\mathbb{R})$, if there exists a measurable function φ vanishing outside Σ such that*

$$\left\|\mathcal{D}^\theta_\nu h - \varphi\right\|_{\nu,q} \le \varepsilon \left\|\mathcal{D}^\theta_\nu h\right\|_{\nu,q}.$$

Remark 3.1. (1) *If h is $\varepsilon-$concentrated to Ω in $L^p_\nu(\mathbb{R})$, then we have*

$$\begin{aligned} \|h - P_\Omega h\|_{\nu,p} &= \left(\int_{\mathbb{R}\setminus\Omega} |h(\xi)|^p |\xi|^{2\nu+1} d\xi\right)^{\frac{1}{p}} \\ &\le \|h - g\|_{\nu,p} \le \varepsilon_\Omega \|h\|_{\nu,p}. \end{aligned} \tag{3.3}$$

(2) *If $\mathcal{D}_\nu^\theta h$ is ε_Σ-concentrated to Σ in $L_\nu^q(\mathbb{R})$, then we have*

$$\begin{aligned}\left\|\mathcal{D}_\nu^\theta h-\mathcal{D}_\nu^\theta\left(S_\Sigma f\right)\right\|_{\nu,q}&=\left(\int_{\mathbb{R}\setminus\Sigma}\left|\mathcal{D}_\nu^\theta h(\xi)\right|^q|\xi|^{2\nu+1}d\xi\right)^{\frac{1}{q}}\\&\leq\left\|\mathcal{D}_\nu^\theta h-\varphi\right\|_{\nu,q}\leq\varepsilon_\Sigma\left\|\mathcal{D}_\nu^\theta h\right\|_{\nu,q}.\end{aligned}\tag{3.4}$$

Now, we state a Donoho-Stark uncertainty principle for the fractional Dunkl transform.

Theorem 3.1. *Let Σ and Ω be two measurable subsets of $\mathbb{R}$ satisfies $0<|\Sigma|<\infty$ and $0<|\Omega|<\infty$. We consider that h belongs to $L_\nu^1(\mathbb{R})\cap L_\nu^2(\mathbb{R})$. If f is ε_Σ-timelimited on Σ and ε_Ω-bandlimited on Ω, then we have*

$$(1-\varepsilon_\Sigma)^2\left(1-\varepsilon_\Omega^2\right)|sin(\theta)|^{2\nu+2}\leq\alpha_\nu^2|\Sigma||\Omega|.\tag{3.5}$$

Proof. Let h be a function belongs to $L_\nu^1(\mathbb{R})\cap L_\nu^2(\mathbb{R})$. Then, we have

$$\int_\Sigma|h(\xi)||\xi|^{2\nu+1}d\xi=\int_{\mathbb{R}}|h(\xi)||\xi|^{2\nu+1}d\xi-\int_{\mathbb{R}\setminus\Sigma}|h(\xi)||\xi|^{2\nu+1}d\xi.$$

Since h is ε-timelimited on Σ, hence according to inequality (3.1), it follows that

$$\int_\Sigma|h(\xi)||\xi|^{2\nu+1}d\xi\geq(1-\varepsilon_\Sigma)\int_{\mathbb{R}}|h(\xi)||\xi|^{2\nu+1}d\xi.$$

By squaring the last inequality, we get

$$\left(\int_\Sigma|h(\xi)||\xi|^{2\nu+1}d\xi\right)^2\geq(1-\varepsilon_\Sigma)^2\left(\int_{\mathbb{R}}|h(\xi)||\xi|^{2\nu+1}d\xi\right)^2.$$

Furthermore, according to the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left(\int_\Sigma|h(\xi)||\xi|^{2\nu+1}d\xi\right)^2&\leq\left(\int_\Sigma|h(\xi)|^2|\xi|^{2\nu+1}d\xi\right)\left(\int_\Sigma|\xi|^{2\nu+1}d\xi\right)\\&\leq|\Sigma|\int_{\mathbb{R}}|h(\xi)|^2|\xi|^{2\nu+1}d\xi.\end{aligned}\tag{3.6}$$

Therefore,

$$|\Sigma|\int_{\mathbb{R}}|h(\xi)|^2|\xi|^{2\nu+1}d\xi\geq(1-\varepsilon_\Sigma)^2\left(\int_{\mathbb{R}}|h(\xi)||\xi|^{2\nu+1}d\xi\right)^2.$$

Since h is ε-bandlimited on Ω, then according to inequality (3.2) and Plancherel theorem for the fractional Dunkl transform, we obtain

$$\begin{aligned}\int_\Omega\left|\mathcal{D}_\nu^\theta h(\xi)\right|^2|\xi|^{2\nu+1}d\xi&=\int_{\mathbb{R}}\left|\mathcal{D}_\nu^\theta h(\xi)\right|^2|\xi|^{2\nu+1}d\xi-\int_{\mathbb{R}\setminus\Omega}\left|\mathcal{D}_\nu^\theta h(\xi)\right|^2|\xi|^{2\nu+1}d\xi\\&\geq\int_{\mathbb{R}}\left|\mathcal{D}_\nu^\theta h(\xi)\right|^2|\xi|^{2\nu+1}d\xi-\varepsilon_\Omega^2\int_{\mathbb{R}}|h(\xi)|^2|\xi|^{2\nu+1}d\xi\\&=\left(1-\varepsilon_\Omega^2\right)\int_{\mathbb{R}}|h(\xi)|^2|\xi|^{2\nu+1}d\xi.\end{aligned}\tag{3.7}$$

After that, according to Riemann-Lebesgue lemma (2.6), it becomes that

$$\begin{aligned}\int_\Omega \left|\mathcal{D}_\nu^\theta h(\xi)\right|^2 |\xi|^{2\nu+1} d\xi &\leq \left\|\mathcal{D}_\nu^\theta h\right\|_\infty^2 \int_\Omega |\xi|^{2\nu+1} d\xi \\ &\leq \frac{\alpha_\nu^2}{|sin(\theta)|^{2\nu+2}} |\Omega| \left(\int_{\mathbb{R}} |h(\xi)||\xi|^{2\nu+1} d\xi\right)^2.\end{aligned} \tag{3.8}$$

Which means that

$$\frac{\alpha_\nu^2}{|sin(\theta)|^{2\nu+2}} |\Omega| \left(\int_{\mathbb{R}} |h(\xi)||\xi|^{2\nu+1} d\xi\right)^2 \geq \left(1-\varepsilon_\Omega^2\right) \int_{\mathbb{R}} |h(\xi)|^2 |\xi|^{2\nu+1} d\xi. \tag{3.9}$$

Finally, combining the inequalities (3.7) and (3.9), we get the desired result. □

Proposition 3.1. *Let Ω and Σ be two measurable subsets of $\mathbb{R}$ such that $0 < |\Sigma| < \infty$ and $0 < |\Sigma| < \infty$. Let $h \in L_\nu^2(\mathbb{R}) \cap L_\nu^2(\mathbb{R})$. Assume that h is ε_Ω-concentrated to Ω and its fractional Dunkl transform $\mathcal{D}_\nu^\theta h$ is ε_Σ-concentrated to Σ, then we have*

$$(1-\varepsilon_\Omega)(1-\varepsilon_\Sigma) |sin(\theta)|^{2(\nu+1)} \|h\|_{\nu,2} \leq \alpha_\nu^2 |\Sigma||\Omega|^{\frac{1}{2}} \|h\|_{\nu,1}.$$

Proof. Let h be a function belongs to $L_\nu^2(\mathbb{R}) \cap L_\nu^2(\mathbb{R})$. Then, we have

$$\begin{aligned}\left\|\mathcal{X}_\Sigma \mathcal{D}_\nu^\theta h\right\|_{\nu,2} &\leq \left\|\mathcal{D}_\nu^\theta h\right\|_\infty \left(\int_\Sigma |\xi|^{2\nu+1} d\xi\right)^{\frac{1}{2}} \\ &\leq |\Sigma|^{\frac{1}{2}} \frac{\alpha_\nu}{|sin(\theta)|^{\nu+1}} \|h\|_{\nu,1}.\end{aligned}$$

Since the fractional Dunkl transform $\mathcal{D}_\nu^\theta h$ is ε_Σ-concentrated to Σ, it becomes that

$$\begin{aligned}\left\|\mathcal{X}_\Sigma \mathcal{D}_\nu^\theta h\right\|_{\nu,2} &\geq \left\|\mathcal{D}_\nu^\theta h\right\|_{\nu,2} - \left\|\mathcal{X}_{\mathbb{R}\setminus\Sigma} \mathcal{D}_\nu^\theta h\right\|_{\nu,2} \\ &\geq \left\|\mathcal{D}_\nu^\theta h\right\|_{\nu,2} - \varepsilon_\Sigma \left\|\mathcal{D}_\nu^\theta h\right\|_{\nu,2}.\end{aligned}$$

Therefore, we obtain

$$(1-\varepsilon_\Sigma) \left\|\mathcal{D}_\nu^\theta h\right\|_{\nu,2} \leq \left\|\mathcal{X}_\Sigma \mathcal{D}_\nu^\theta h\right\|_{\nu,2} \leq |\Sigma|^{\frac{1}{2}} \frac{\alpha_\nu}{|sin(\theta)|^{\nu+1}} \|h\|_{\nu,1}. \tag{3.10}$$

On the other hand, as the function h is ε_Ω-concentrated to Ω, it becomes that

$$(1-\varepsilon_\Omega) \leq \|P_\Omega h\|_{\nu,2} \leq |\Sigma|^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \frac{\alpha_\nu}{|sin(\theta)|^{\nu+1}}. \tag{3.11}$$

Finally, according to the inequalities (3.10) and (3.11), we get the desired result. □

Theorem 3.2. *Let Σ and Ω be two measurable subsets of $\mathbb{R}$ satisfies $0 < |\Sigma| < \infty$ and $0 < |\Omega| < \infty$. Suppose that $\varepsilon_\Omega + \varepsilon_\Sigma < 1$. If h is ε_Ω-concentrated to Ω in $L_\nu^2(\mathbb{R})$ and its fractional Dunkl transform $\mathcal{D}_\nu^\theta h$ is ε_Σ-concentrated to Σ in $L_\nu^2(\mathbb{R})$, then we have*

$$(1-\varepsilon_\Omega-\varepsilon_\Sigma) |sin(\theta)|^{\nu+1} \leq \alpha_\nu \sqrt{|\Sigma||\Omega|}.$$

Proof. Consider that h belongs $L^2_\nu(\mathbb{R})$. We assume that $\|h\|_{\nu,2} = \|\mathcal{D}^\theta_\nu h\|_{\nu,2} = 1$. As though h is ε_Ω-concentrated to Ω, it becomes from the inequality (3.11):

$$\|P_\Omega h\|_{\nu,2} \leq |\Sigma|^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \frac{\alpha_\nu}{|sin(\theta)|^{\nu+1}}.$$

Since $\mathcal{D}^\theta_\nu h$ is ε_Σ-concentrated to Σ in $L^2_\nu(\mathbb{R})$, it becomes that

$$\begin{aligned}\left\|\mathcal{D}^\theta_\nu h - \mathcal{D}^\theta_\nu \left(S_\Sigma P_\Omega h\right)\right\|_{\nu,2} &\leq \left\|\mathcal{D}^\theta_\nu h - \mathcal{D}^\theta_\nu \left(S_\Sigma f\right)\right\|_{\nu,2} + \left\|\mathcal{D}^\theta_\nu \left(S_\Sigma f\right) - \mathcal{D}^\theta_\nu \left(S_\Sigma P_\Omega h\right)\right\|_{\nu,2} \\ &\leq \varepsilon_\Sigma + \left\|\mathcal{D}^\theta_\nu S_\Sigma\right\|_{\nu,2} \|h - P_\Omega h\|_{\nu,2} \\ &\leq \varepsilon_\Sigma + \varepsilon_\Omega.\end{aligned}$$

Therefore,

$$\left\|\mathcal{D}^\theta_\nu \left(S_\Sigma P_\Omega h\right)\right\|_{\nu,2} \geq \left\|\mathcal{D}^\theta_\nu h\right\|_{\nu,2} - \left\|\mathcal{D}^\theta_\nu h - \mathcal{D}^\theta_\nu \left(S_\Sigma P_\Omega h\right)\right\|_{\nu,2} \geq 1 - \varepsilon_\Sigma - \varepsilon_\Omega. \tag{3.12}$$

On the other hand, since we have

$$\left\|\mathcal{D}^\theta_\nu \left(S_\Sigma P_\Omega h\right)\right\|_{\nu,2} = \left\|\left(S_\Sigma P_\Omega h\right)\right\|_{\nu,2} \leq \left\|\left(P_\Omega h\right)\right\|_{\nu,2},$$

hence, we obtain

$$\left\|\mathcal{D}^\theta_\nu \left(S_\Sigma P_\Omega h\right)\right\|_{\nu,2} \leq |\Sigma|^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \frac{\alpha_\nu}{|sin(\theta)|^{\nu+1}}. \tag{3.13}$$

Finally, according to the inequalities (3.12) and (3.13), we obtain the desired result. □

Applications. As mentioned in beginning of this paper that the fractional Dunkl operator Λ^θ_ν extends a broad range of integral transforms, depending on the choice of the parameter θ, the multiplicity function ν, and the specific function spaces. We obtain the Donoho–Stark uncertainty principle for some particular cases including:

(1) If $\nu = -1/2$, then the fractional Dunkl operator Λ^θ_ν is reduced to:
   (a) If $\theta = \pi/2$ we find the Donoho–Stark uncertainty principle associated with the classical Fourier transform.
   (b) If $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$, we find the Donoho–Stark uncertainty principle associated with the classical fractional Fourier transform.

(2) If $\nu > -1/2$ then the fractional Dunkl operator Λ^θ_ν is reduced to:
   (a) If$\theta = \pi/2$, we find the Donoho–Stark uncertainty principle associated with the Dunkl transform.
   (b) If $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$, we find the Donoho–Stark uncertainty principle related to the fractional Hankel transform, on the even subspace

$$C^2(\mathbb{R})^e = \left\{h \in C^2(\mathbb{R}) : h(\xi) = h(-\xi)\right\},$$

   of the square $\left(\Lambda^\theta_\nu\right)^2$.

(c) If$\theta = \pi/2$, we find the Donoho–Stark uncertainty principle related to the Hankel transform, on the even subspace

$$C^2(\mathbb{R})^e = \left\{h \in C^2(\mathbb{R}) : h(\xi) = h(-\xi)\right\},$$

of the square $\left(\Lambda_\nu^\theta\right)^2$.

4. Signal Restoration Algorithm

In this section, we present an algorithm for signal recovery. Consider a signal $h \in L^2_\nu(\mathbb{R})$ that is transmitted to a receiver and concentrated on Ω. Now, assume that the receiver cannot observe the complete data of h, and the signal is not detected on Ω. Additionally, the observed signal is affected by observational noise $\varrho \in L^2_\nu(\mathbb{R})$. Consequently, the received signal ρ is expressed as follows:

$$\rho(\xi) = \begin{cases} h(\xi) + \varrho(\xi), & x \in \Omega^c \\ 0, & x \in \Omega, \end{cases}$$

where Ω^c is the complement of Ω. We suppose that, without loss of data, $\varrho = 0$ on Ω. Which implies,

$$\rho(\xi) = (I - P_\Omega)\, h(\xi) + \varrho(\xi).$$

Theorem 4.1. *Let f be a function in $L^2_\nu(\mathbb{R})$ such that $h = S_\Sigma h$ and $h_n = \sum_{k=0}^n (P_\Omega S_\Sigma)^k \rho$. If Σ and Ω be two measurable subsets of $\mathbb{R}$ satisfying:*

$$\alpha_\nu |\Omega|^{\frac{1}{2}} |\Sigma|^{\frac{1}{2}} < |sin(\theta)|^{\nu+1}, \tag{4.1}$$

then, the information of a function h over $\xi \in \Omega$ can then be retrieved using the following algorithm

$$\begin{cases} h_0 = \rho \\ h_{n+1} = \rho + P_\Omega S_\Sigma h_n. \end{cases}$$

Then, h_n converges to h as $n \to \infty$ in $L^2_\nu(\mathbb{R})$-norm.

Proof. We put $A = (I - P_\Omega S_\Sigma)^{-1}$. Firstly, we show that the operator A is bounded. According to the inequality (3.13) and the Plancherel formula for the fractional Dunkl transform, we obtain

$$\|P_\Omega S_\Sigma\| = \sup \frac{\left\|P_\Omega S_\Sigma f\right\|_{\nu,2}}{\|h\|_{\nu,2}} \leq |\Sigma|^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \frac{\alpha_\nu}{|sin(\theta)|^{\nu+1}} < 1.$$

Therefore, the operator $(I - P_\Omega S_\Sigma)$ is invertible and the series $\sum_{k=0}^\infty (P_\Omega S_\Sigma)^k$ converges for the operator norm $\|\cdot\|$.

Next, we propose an algorithm for calculating $A\rho$. We put

$$h_n = \sum_{k=0}^n (P_\Omega S_\Sigma)^k \rho, \quad \text{and} \quad \mathcal{R}_n = \sum_{k=n}^\infty (P_\Omega S_\Sigma)^k.$$

As $\mathcal{R}_n$ is the remainder of a converging series, therefore, $\|\mathcal{R}_n\| \to 0$ as $n \to \infty$. Furthermore, we obtain

$$\lim_{n\to\infty} \left\|h_n - A\rho\right\|_{\nu,2} = \lim_{n\to\infty} \left\|\sum_{k=n}^{\infty} \left(P_\Omega S_\Sigma\right)^k \rho\right\|_{\nu,2} = \lim_{n\to\infty} \left\|\mathcal{R}_n \rho\right\|_{\nu,2} \leq \lim_{n\to\infty} \|\mathcal{R}_n\| \, \|\rho\|_{\nu,2} = 0.$$

Hence, h_n converges to $A\rho$ as $n \to \infty$ in $L^2_\nu(\mathbb{R})$-norm. After that, we prove that $A\rho = h$ for $\xi \in \Omega$. Since $h = S_\Sigma h$, we obtain that

$$\rho = \left(I - P_\Omega\right) h + \varrho = \left(I - P_\Omega S_\Sigma\right) h + \varrho.$$

On other hand, according to the hypothesis that the noise $\varrho = 0$ on Ω, we get

$$\text{Aæ}(\xi) = A\left(I - P_\Omega S_\Sigma\right) h(\xi) = \left(I - P_\Omega S_\Sigma\right)^{-1} \left(I - P_\Omega S_\Sigma\right) h(\xi) = h(\xi), \quad \xi \in \Omega.$$

Therefore $h_n \to h$ as $n \to \infty$ in $L^2_\nu(\mathbb{R})$-norm and the loss information of h over $\xi \in \Omega$ can be recovered by the above algorithm. □

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References

[1] F. Bahba, A. Saoudi, On the Fractional Opdam–Cherednik Transform, J. Anal. 32 (2024), 3047–3063. https://doi.org/10.1007/s41478-024-00772-x.

[2] E.U. Condon, Immersion of the Fourier Transform in a Continuous Group of Functional Transformations, Proc. Natl. Acad. Sci. 23 (1937), 158–164. https://doi.org/10.1073/pnas.23.3.158.

[3] C.F. Dunkl, Differential-Difference Operators Associated to Reflection Groups, Trans. Am. Math. Soc. 311 (1989), 167–183. https://doi.org/10.1090/S0002-9947-1989-0951883-8.

[4] C.F. Dunkl, Integral Kernels with Reflection Group Invariance, Can. J. Math. 43 (1991), 1213–1227. https://doi.org/10.4153/CJM-1991-069-8.

[5] C.F. Dunkl, Hankel Transforms Associated to Finite Reflection Groups, Contemp. Math. 138 (1992), 123–138. https://doi.org/10.1090/conm/138/1199124.

[6] S. Ghazouani, F. Bouzeffour, Heisenberg Uncertainty Principle for a Fractional Power of the Dunkl Transform on the Real Line, J. Comput. Appl. Math. 294 (2016), 151–176. https://doi.org/10.1016/j.cam.2015.06.013.

[7] I. Haouala, A. Saoudi, Fractional Jacobi-Dunkl Transform: Properties and Application, Fract. Calc. Appl. Anal. 25 (2022), 2313–2331. https://doi.org/10.1007/s13540-022-00102-7.

[8] I. Kallel, A. Saoudi, Uncertainty Principle for the Weinstein-Gabor Transforms, Int. J. Anal. Appl. 22 (2024), 94. https://doi.org/10.28924/2291-8639-22-2024-94.

[9] F.H. Kerr, A fractional power theory for Hankel transform in $L^2(\mathbb{R}^+)$, J. Math. Anal. Appl. 158 (1991), 114–123. https://doi.org/10.1016/0022-247X(91)90271-Z.

[10] A.C. McBRIDE, F.H. Kerr, On Namias's Fractional Fourier Transforms, IMA J. Appl. Math. 39 (1987), 159–175. https://doi.org/10.1093/imamat/39.2.159.

[11] H.B. Mohamed, A. Saoudi, Linear Canonical Fourier–Bessel Wavelet Transform: Properties and Inequalities, Integr. Transf. Spec. Funct. 35 (2024), 270–290. https://doi.org/10.1080/10652469.2024.2317724.

[12] V. Namias, The Fractional Order Fourier Transform and Its Application to Quantum Mechanics, IMA J. Appl. Math. 25 (1980), 241–265. https://doi.org/10.1093/imamat/25.3.241.

[13] M. Rösler, Dunkl Operators: Theory and Applications, in: E. Koelink, W. Van Assche (Eds.), Orthogonal Polynomials and Special Functions, Springer, Berlin, Heidelberg, 2003: pp. 93–135. https://doi.org/10.1007/3-540-44945-0_3.

[14] A. Saoudi, A Variation of L^p Uncertainty Principles in Weinstein Setting, Indian J. Pure Appl. Math. 51 (2020), 1697–1712. https://doi.org/10.1007/s13226-020-0490-9.

[15] A. Saoudi, Time-Scale Localization Operators in the Weinstein Setting, Results Math. 78 (2023), 14. https://doi.org/10.1007/s00025-022-01792-4.

[16] A. Saoudi, Hardy Type Theorems for Linear Canonical Dunkl Transform, Complex Anal. Oper. Theory 18 (2024), 57. https://doi.org/10.1007/s11785-023-01478-x.

[17] A. Saoudi and I.A. Kallel, A Variation of L^p Local Uncertainty Principles for Weinstein Transform, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. 25 (2024), 3–10. https://doi.org/10.59277/PRA-SER.A.25.1.01.

[18] A. Touati, I. Kallel, A. Saoudi, Uncertainty Principles for the Weinstein Wavelet Transform, International Journal of Analysis and Applications 22 (2024), 211. https://doi.org/10.28924/2291-8639-22-2024-211.

[19] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 2006.

[20] N. Wiener, Hermitian Polynomials and Fourier Analysis, J. Math. Phys. 8 (1929), 70–73. https://doi.org/10.1002/sapm19298170.