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A New Class of Generalized Starlike Bi-Univalent Functions Subordinated to Legendre Polynomials

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Abstract. A new class of generalized starlike bi-univalent functions is introduced in this paper, which is defined using Legendre Polynomials within the open unit disk $\mathbb{D}$. This paper sheds a light on the properties and behaviors of these starlike bi-univalent functions, providing estimations for the modulus of the initial Taylor series coefficients of the functions falling under this particular class and one of its various subclasses. Additionally, this paper also investigates the classical Fekete-Szegö functional problem for functions f that are part of the aforementioned class. Moreover, we obtain the classical Fekete-Szegö inequalities of functions belonging to this class and to one of its various subclasses.

1. Introduction

Consider the set $\mathcal{H}$, which consists of all functions $f(\zeta)$ that are analytic within the open unit disk denoted as $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and normalized by the conditions $f(0) = 0 = 1 - f'(0)$. The exploration of such functions contributes to a deeper comprehension of complex analysis and its applications. Moreover, any function f belongs to the set $\mathcal{H}$ can be written as

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \quad \text{where } \zeta \in \mathbb{D}. \tag{1.1}$$

Let the functions f and g be analytic in the open unit disk $\mathbb{D}$. We say that f is subordinated by g in $\mathbb{D}$, denoted as $f(z) \prec g(\zeta)$ for all $\zeta \in \mathbb{D}$, if there exists a Schwarz function w satisfying $w(0) = 0$ and $|w(\zeta)| < 1$ for all $\zeta \in \mathbb{D}$, such that $f(\zeta) = g(w(\zeta))$ for all $\zeta \in \mathbb{D}$. This relationship between f and g is a fundamental concept in complex analysis, providing a way to compare the behavior of two analytic functions within the unit disk. Notably, when the function g is univalent over $\mathbb{D}$, the condition $f(\zeta) \prec g(\zeta)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. This equivalence highlights the significance of the subordination principle in understanding the relationship between analytic

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functions. For further insights and detailed discussions on the Subordination Principle, interested readers are encouraged to explore the monographs [10], [11], [22] and [24]. These sources provide comprehensive explanations and applications of this principle in the context of complex analysis and geometric function theory.

In this paper, $\mathcal{S}$ represents the set of functions that are univalent in the open unit disk $\mathbb{D}$ and belong to the set $\mathcal{H}$. As known univalent functions are injective functions. Hence, they are invertible and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, according to Koebe one-quarter Theorem [10], the image of $\mathbb{D}$ under any function $f \in \mathcal{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius 1/4. Accordingly, every function $f \in \mathcal{S}$ has an inverse $f^{-1} = g$ which is defined as

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \;\; r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{1.2}$$

For this reason, we define the class Σ as follows. A function $f \in \mathcal{H}$ is said to be bi-univalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denote the class of all bi-univalent functions in $\mathcal{H}$ which are given by equation (1.1). For example, the following functions belong to the class Σ:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, Koebe function, $\dfrac{2z - z^2}{2}$ and $\dfrac{z}{1-z^2}$ do not belong to the class Σ. For more information about univalent and bi-univalent functions we refer the readers to the articles [18], [20], [25] the monograph [10], [13] and the references provided therein.

The research conducted in geometric function theory sheds a light on the intricate relationships between coefficients and the geometric properties of functions. By examining the bounds placed on the modulus of a function's coefficients, researchers can gain a deeper understanding of how these functions behave and interact within the mathematical framework. This analytical approach not only enhances our comprehension of the underlying principles governing geometric function theory but also paves the way for further exploration and discovery in this dynamic field of study. For example, within the class $\mathcal{S}$, it is established that the modulus of the coefficient a_n is bounded by the value of n. These bounds on the modulus of coefficients provide valuable insights into the geometric characteristics of these functions. Specifically, the restriction on the second coefficients of functions belonging to the class $\mathcal{S}$ offers crucial details regarding the growth and distortion bounds within this class.

The exploration of coefficient-related properties of functions within the bi-univalent class Σ commenced in the 1970s. Notably, Lewin's work, in 1967 [18], marked a significant milestone as he examined the bi-univalent function class and established a bound for the coefficient $|a_2|$. Following this, Netanyahu's research, in 1969 [25], determined that the maximum value of $|a_2|$ is

$\frac{4}{3}$ for functions categorized under Σ. Furthermore, Brannan and Clunie, in 1979 [5], demonstrated that for functions in this class, the inequality $|a_2| \leq \sqrt{2}$ holds true. This foundational work has spurred numerous investigations into the coefficient bounds for various subclasses of bi-univalent functions. Despite the extensive research conducted on the coefficient bounds for bi-univalent functions, there remains a significant gap in knowledge regarding the general coefficients $|a_2|$ for cases where $n \geq 4$. The challenge of estimating the coefficients, particularly the general coefficient $|a_n|$, continues to be an unresolved issue in the field. This ongoing inquiry highlights the complexity and richness of the bi-univalent function class, suggesting that further exploration is necessary to fully understand the behavior of these coefficients in higher dimensions.

Fekete and Szegö, in 1933 [17], determined the maximum value of $|a_3 - \lambda a_2^2|$ for a univalent function f, with the real parameter $0 \leq \lambda \leq 1$. This result led to the establishment of the Fekete-Szegö problem, which involves maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathcal{H}$ with any complex number λ. Numerous researchers have delved into the Fekete-Szegö functional and other coefficient estimates problems. For instance, relevant articles include [2], [3], [4], [6], [8], [14], [15], [17], [20], [30], and the references provided therein. These studies have contributed to a deeper understanding of the Fekete-Szegö problem and its implications in the field of geometric function theory.

2. Preliminaries

The information presented in this section is indispensable for understanding the principal outcomes of the research. Legendre polynomials belong to a well-established family of classical orthogonal polynomials. They are defined by their compliance with a second-order linear differential equation, which emerges naturally in the context of solving initial value problems in three-dimensional spaces exhibiting spherical symmetry. The equation associated with Legendre polynomials is classified as a Legendre second-order differential equation:

$$(1-x^2)y'' - 2xy' + \lambda y = 0, \ \ -1 \leq x \leq 1. \tag{2.1}$$

The process of identifying the parameters $\lambda \in \mathbb{R}$ that allow Equation (2.1) to possess a bounded solution is referred to as a singular Sturm-Liouville problem. The significance of these eigenvalues λ lies in their role in determining the nature of the solutions to the differential equation. In this context, the necessity for boundary conditions is eliminated, as the boundedness of the solution itself serves as a substitute for these conditions. It has been established that the only permissible values of λ that yield bounded solutions are of the form $\lambda = n(n+1)$, where n is a natural number. These values of λ are called the eigenvalues of the Sturm-Liouville problem.

The polynomial solutions of Legendre's differential equation can be explicitly expressed as follows. These solutions play a significant role in various applications, particularly in mathematical physics and engineering, where they are utilized to solve problems involving spherical symmetry.

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}, \tag{2.2}$$

where $\lfloor z \rfloor$ is the floor of z, i.e. the greatest integer $m \le z$. The polynomial functions denoted as $P_n(x)$ represent the distinct polynomial solutions to Legendre's Equation (2.1) under the condition $y(1) = 1$, and these are recognized as Legendre's polynomials. It is important to note that when n is even, the polynomial $P_n(x)$ exclusively comprises even powers of x, while for odd n it contains only odd powers. Consequently, $P_n(x)$ is classified as an even function for even n and as an odd function for odd n. Their unique properties, such as orthogonality and recurrence relations, further enhance their utility in both theoretical and applied mathematics. The first few of them are: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$.

It is important to highlight that the Legendre polynomials can be represented in a more concise manner. Specifically, the n^{th} Legendre polynomial, denoted as P_n, can be formulated using Rodrigues' formula (2.3), which serves as a foundational tool in the study of these mathematical functions.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \tag{2.3}$$

As a result of Rodrigue's formula, one can observe a specific connection that exists between three consecutive Legendre polynomials. This relationship plays a crucial role in understanding the properties and behaviors of these polynomials and their applications in mathematical physics.

$$(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x).$$

It can be demonstrated that the Legendre polynomials are produced by the generating function

$$g(x, t) = \frac{1}{\sqrt{t^2 - 2xt + 1}}.$$

This relationship highlights the significance of this function in generating these important polynomials, which have numerous applications in physics and engineering. Additionally, when the function $g(x, t)$ is expanded as a Taylor series in terms of t, the coefficient corresponding to t^n is the Legendre polynomial $P_n(x)$:

$$g(x, t) = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{2.4}$$

In this paper, the symbol $\mathcal{P}$ denotes the Caratheodory class, which is formally defined as

$$\mathcal{P} = \{\Omega \in \mathcal{H} : \Omega(0) = 1, \ \mathcal{R}(\Omega(z)) > 0, \ z \in \mathbb{D}\}.$$

It is established in the literature (for example, see [13], page 102) that the function $\psi(z)$ is a member of the class $\mathcal{P}$ for any real number α, with ψ expressed as

$$\psi(z) = \frac{1-z}{\sqrt{1-(2\cos\alpha)z+z^2}}.$$

Notably, the function $\psi(z)$ transforms the open unit disk $\mathbb{D}$ onto the right half-plane $\mathcal{R}(w) > 0$, with the exception of the slit along the positive real axis extending from $\left|\cos(\alpha/2)\right|^{-1}$ to infinity. Consequently, ψ exhibits starlikeness with respect to the point 1. By consulting Equation (2.4), it is straightforward to verify the following equation, for any z within the open unit disk $\mathbb{D}$.

$$\psi(z) = 1 + \sum_{n=1}^{\infty} \left[P_n(\cos\alpha) - P_{n-1}(\cos\alpha)\right] z^n \tag{2.5}$$

$$= 1 + \sum_{n=1}^{\infty} \delta_n(\alpha) z^n. \tag{2.6}$$

Using the Rodregue's formula (2.3), we easily obtain the following initial values of $\delta_n(\alpha) = P_n(\cos\alpha) - P_{n-1}(\cos\alpha)$ which are listed below:

$$\delta_1(\alpha) = \cos\alpha - 1,\ \delta_2(\alpha) = \frac{1}{2}(\cos\alpha - 1)(1 + 3\cos\alpha).$$

Additional information regarding the Legengre polynomials readers are encouraged to consult the articles referenced as [1], [7], [23] and [26], as well as the monographs [10], [13], [27], [29], and the related sources.

The characterization of starlike functions is essential in complex analysis, particularly in the geometric function theory. A function f is said to be starlike if for any $\mu \in f(\mathbb{D})$ and $k \in [0,1]$, then $k\mu$ belongs to $f(\mathbb{D})$. Furthermore, the class of starlike functions of order α represented by $\mathcal{S}^*(\oslash)$, where the parameter α satisfies the condition $0 \leq \alpha < 1$. A function $f \in \mathcal{S}$ is considered to be a member of the class $\mathcal{S}^*(\oslash)$ if the following condition holds for all $\zeta \in \mathbb{D}$:

$$\mathcal{R}\rceil \left\{ \frac{\zeta f'(\zeta)}{f(\zeta)} \right\} > \alpha.$$

When considering the case where $\alpha = 0$, it follows that $\mathcal{S}^*(0)$ coincides with the classical class of starlike functions, denoted as $\mathcal{S}^*$. The class $\mathcal{S}^*(\prec)$ was introduced by Ma and Minda [19] and is defined as

$$\mathcal{S}^*(\prec) = \left\{ f \in \mathcal{S} : \frac{\zeta f'(\zeta)}{f(\zeta)} \prec \phi(\zeta) \right\},$$

where ϕ is a univalent function characterized by having a positive real part within the open unit disk $\mathbb{D}$. Moreover, the image $\phi(\mathbb{D})$ is symmetric with respect to the real axis, and it satisfies the conditions $\phi'(0) > 0$ and is starlike with respect to the value $\phi(0) = 1$.

Notably, substituting ϕ with specific functions yields various recognized subclasses of starlike functions. For instance, Sokól and Stankiewicz [28] defined the class $\mathcal{S}^*(\sqrt{1+z})$. Furthermore, Mendiratta et al. [21] introduced the class $\mathcal{S}_e^*$, which arises from the choice of $\phi(z) = e^z$. Another example is the class $\mathcal{S}^*_{SG}$, which is derived from the function $\phi(z)\dfrac{2}{(1+e^{-z})}$, as presented by Goel and Kumar [12]. When $\phi(z)$ is taken to be $1+\sin z$, the resulting class is $\mathcal{S}^*_{Sin}$, introduced by Cho et al. [9].

Furthermore, Kumar and Bango [16] defined the class $\mathcal{S}^*_{log}$, which is obtained by selecting $\phi(z) = 1 - \log(1+z)$. Building upon these foundations, we aim to establish a new class of generalized starlike bi-univalent functions that are associated with Legendre polynomials, which we denote as $\mathcal{S}^*(\lambda, \alpha, \psi)$, which we define as follows.

Definition 2.1. *A function $f(z)$ belongs to the family Σ is considered to be part of the class $\mathcal{S}^*(\lambda, \alpha, \psi)$ if it obeys the following subordination conditions:*

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{1/\lambda}\right] \prec \psi(z)$$

and

$$\frac{1}{2}\left[\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{1/\lambda}\right] \prec \psi(w)$$

where the function $g(w) = f^{-1}(w)$ is given by the Equation (1.2), the parameter $0 < \lambda \leq 1$ and α any real number.

The following lemmas, which are thoroughly detailed in literature (see, for instance, [15]), are widely recognized principles that are of considerable relevance to the research we are presenting.

Lemma 2.1. *if Ω belongs to the Caratheodory class, then for $z \in \mathbb{D}$ the function Ω can be written as*

$$\Omega(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

Moreover, $|c_n| \leq 2$, for each natural number n.

The lemma outlined below is well-documented in the literature, is considered a fundamental principle that plays a crucial role in the research we are undertaking.

Lemma 2.2. *Let K and L be real numbers. Let p and q be complex numbers. If $|p| < r$ and $|q| < r$,*

$$\left|(K+L)p + (K-L)q\right| \leq \begin{cases} 2r|K|, & \text{if } |K| \geq |L| \\ 2r|L|, & \text{if } |K| \leq |L|. \end{cases}$$

This article aims to investigate a particular category of starlike bi-univalent functions within the open unit disk $\mathbb{D}$ defined using the Legendre polynomials, which we denote as $\mathcal{S}^*(\lambda, \alpha, \psi)$. The primary focus is on deriving estimates for the modulus of the initial coefficients $|a_2|$ and $|a_3|$ associated with Taylor representation of the functions in this class. Additionally, the article examines the Fekete-Szegö functional problem associated with functions in this specific class, providing a deeper understanding of their properties.

3. Main Results

This section of the paper is devoted to explore the bounds for the modulus of the initial coefficients of functions that are part of the class $\mathcal{S}^*(\lambda, \alpha, \psi)$, as denoted by Equation (1.1). Then, we examine the classical Fekete-Szegö functional applied to functions that are members of our specified class.

Theorem 3.1. *Let a function f be in the family Σ. If the function f belongs to the class $\mathcal{S}^*(\lambda, \alpha, \psi)$ and is represented by the equation (1.1), then the following inequalities hold:*

$$|a_2| \leq \frac{2\sqrt{2}|1-\cos\alpha|}{\sqrt{|(4\lambda^2+2\lambda+2)(\cos\alpha-1)+(1+\lambda)^2(1-3\cos\alpha)|}}, \tag{3.1}$$

$$|a_3| \leq \frac{\lambda|1-\cos\alpha|}{1+\lambda}\left(1+\frac{4\lambda|1-\cos\alpha|}{1+\lambda}\right). \tag{3.2}$$

Proof. Suppose a function f belongs to the class $\mathcal{S}^*(\lambda, \alpha, \psi)$. According to the Definition 2.1 and Subordination Principle, we can find two Schwarz functions $p(z)$ and $q(w)$ defined on the open unit disk $\mathbb{D}$ such that

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{1/\lambda}\right] = \psi(p(z)), \tag{3.3}$$

and

$$\frac{1}{2}\left[\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{1/\lambda}\right] = \psi(q(w)). \tag{3.4}$$

Now, using those Schwarz functions, we define new analytic functions $u(z)$ and $v(w)$ as follows:

$$u(z) = \frac{1+p(z)}{1-p(z)} \quad \text{and} \quad v(w) = \frac{1+q(w)}{1-q(w)}.$$

It is clear that, these functions u and v are analytic in the open unit disk $\mathbb{D}$ and belong to the Caratheodory class. Thus, we can write them as

$$u(z) = \frac{1+p(z)}{1-p(z)} = 1+u_1z+u_2z^2+\cdots$$

and

$$v(w) = \frac{1+q(w)}{1-q(w)} = 1+v_1w+v_2w^2+\cdots$$

Moreover, $u(0) = 1 = v(0)$, they have positive real parts, $|u_j| \leq 2$ and $|v_j| \leq 2$ for all $j \in \mathbb{N}$.

Equivalently, we get the following representations of p and q

$$p(z) = \frac{u(z)-1}{u(z)+1} = \frac{1}{2}\left[u_1z+\left(u_2-\frac{u_1^2}{2}\right)z^2+\cdots\right], \tag{3.5}$$

and

$$q(w) = \frac{v(w)-1}{v(w)+1} = \frac{1}{2}\left[v_1w+\left(v_2-\frac{v_1^2}{2}\right)w^2+\cdots\right]. \tag{3.6}$$

Therefore, by consulting Equation (2.5), Equation (3.5) and Equation (3.6) we obtain the following:

$$\psi(p(z)) = 1 + \frac{1}{2}\delta_1 u_1 z + \left[\frac{1}{2}\delta_1\left(u_2 - \frac{u_1^2}{2}\right) + \frac{1}{4}\delta_2 u_1^2\right] z^2 + \cdots \tag{3.7}$$

and

$$\psi(q(w)) = 1 + \frac{1}{2}\delta_1 v_1 w + \left[\frac{1}{2}\delta_1\left(v_2 - \frac{v_1^2}{2}\right) + \frac{1}{4}\delta_2 v_1^2\right] w^2 + \cdots \tag{3.8}$$

Now, by equating the coefficients on bothsides of Equations (3.3) and (3.7), and on bothsides of Equations (3.4) and (3.8), a set of equations can be derived as follows:

$$\frac{1+\lambda}{2\lambda} a_2 = \frac{1}{2}\delta_1 u_1, \tag{3.9}$$

$$\frac{1+\lambda}{2\lambda}(2a_3 - a_2^2) + \frac{1-\lambda}{4\lambda^2} a_2^2 = \frac{1}{2}\delta_1\left(u_2 - \frac{u_1^2}{2}\right) + \frac{1}{4}\delta_2 u_1^2, \tag{3.10}$$

$$\frac{-(1+\lambda)}{2\lambda} a_2 = \frac{1}{2}\delta_1 v_1, \tag{3.11}$$

and

$$\frac{1+\lambda}{2\lambda}(3a_2^2 - 2a_3) + \frac{1-\lambda}{4\lambda^2} a_2^2 = \frac{1}{2}\delta_1\left(v_2 - \frac{v_1^2}{2}\right) + \frac{1}{4}\delta_2 v_1^2. \tag{3.12}$$

Hence, using Equation (3.9) and Equation (3.11), we easily get the following equations:

$$u_1 = -v_1, \tag{3.13}$$

and

$$2(1+\lambda)^2 a_2^2 = \lambda^2(\delta_1)^2(u_1^2 + u_2^2). \tag{3.14}$$

On the other hand, adding Equation (3.10) to Equation (3.12), we obtain

$$\left[4\lambda(1+\lambda) + 2(1-\lambda)\right] a_2^2 = 2\delta_1\lambda^2(u_2 + v_2) + \lambda^2(\delta_2 - \delta_1)(u_1^2 + v_1^2).$$

Thus, substituting $u_1^2 + v_1^2$ from Equation (3.14), in the last equation, we obtain the following equation:

$$a_2^2 = \frac{2(\delta_1)^3\lambda^2(u_2+v_2)}{(4\lambda^2 + 2\lambda + 2)(\delta_1)^2 - 2(1+\lambda)^2(\delta_2 - \delta_1)}. \tag{3.15}$$

Therefore, considering the initial values $\delta_1 = \cos\alpha - 1$ and $\delta_2 = \frac{1}{2}(\cos\alpha - 1)(1 + 3\cos\alpha)$, simple calculations give the following conclusion:

$$a_2^2 = \frac{2(1-\cos\alpha)^2\lambda^2(u_2+v_2)}{(4\lambda^2 + 2\lambda + 2)(\cos\alpha - 1) + (1+\lambda)^2(1 - 3\cos\alpha)}.$$

Hence, using the constraints $|u_j| \leq 2$ and $|v_j| \leq 2$ for all $j \in \mathbb{N}$, we get the sought-after estimation of $|a_2|$ presented in Equation (3.1).

Secondly, we seek to determine the coefficient estimate for $|a_3|$. By substituting Equation (3.12) from Equation (3.10), we can derive the following equation:

$$8\lambda(1+\lambda)(a_3-a_2^2) = 2\lambda^2\delta_1\left[(u_2-v_2)-\left(\frac{u_1^2-u_2^2}{2}\right)\right]+\lambda^2\delta_2(u_1^2-u_2^2).$$

By utilizing the Equation (3.13), the last equation is transformed into

$$a_3 = \frac{\lambda\delta_1(u_2-v_2)}{4(1+\lambda)}+a_2^2. \tag{3.16}$$

Consequently, by employing Equation (3.14) and using the constraints $|u_j| \leq 2$ and $|v_j| \leq 2$ for all $j \in \mathbb{N}$, we obtain

$$|a_3| \leq \frac{\lambda|1-\cos\alpha|}{1+\lambda}+\frac{4\lambda^2(1-\cos\alpha)^2}{(1+\lambda)^2}.$$

This gives the required estimation of $|a_3|$. Consequently, the proof of Theorem 3.1 is now concluded. □

Remark 3.1. *By choosing specific values of λ in Definition 2.1, we can derive various established classes. Among these, the classical category of starlike bi-univalent functions is the most extensively examined, which is obtained by setting $\lambda = 1$. Numerous researchers have explored the class of classical starlike bi-univalent functions that are subordinated to orthogonal polynomials. In this paper when $\lambda = 1$, we get the class $\mathcal{S}^*(1,\alpha,\psi)$ where any function belongs to it satisfies the following subordination conditions:*

$$\frac{zf'(z)}{f(z)} \prec \frac{1-z}{\sqrt{1-(2\cos\alpha)z+z^2}}, \tag{3.17}$$

and

$$\frac{wg'(w)}{g(w)} \prec \frac{1-w}{\sqrt{1-(2\cos\alpha)w+w^2}}. \tag{3.18}$$

The following corollary arises directly from Theorem 3.1 under the condition that $\lambda = 1$. The methods utilized in establishing this corollary bear a strong resemblance to those used in the proof of the previous Theorem 3.1, which is why we have opted to omit the comprehensive proof.

Corollary 3.1. *If a function $f \in \Sigma$ is represented by (1.1) and is obeying the Subordination conditions (3.17) and (3.18), then it can be concluded that*

$$|a_2| \leq \frac{2|1-\cos\alpha|}{\sqrt{2|1+\cos\alpha|}},$$

and

$$|a_3| \leq \frac{|1-\cos\alpha|}{2}+(1-\cos\alpha)^2.$$

In the following part of this section, we will establish the Fekete-Szegö inequalities for functions that are members of our specified class $\mathcal{S}^*(\lambda,\alpha,\psi)$.

Theorem 3.2. *If a function f is a member of the class $\mathcal{S}^*(\lambda,\alpha,\psi)$ and is represented by equation (1.1), then for a real number ζ the following inequality holds*

$$|a_3-\zeta a_2^2|\leq\begin{cases}\frac{\lambda|1-\cos\alpha|}{1+\lambda}, & \textit{if } |1-\zeta|\leq|B|\\ 4\lambda|A||1-\cos\alpha|, & \textit{if } |1-\zeta|\geq|B|,\end{cases}\tag{3.19}$$

where

$$A=\frac{2(\lambda\delta_1)^2(1-\zeta)}{(4\lambda^2+2\lambda+2)(\delta_1)^2-2(1+\lambda)^2(\delta_2-\delta_1)},$$

and

$$B=\frac{(4\lambda^2+2\lambda+2)(\cos\alpha-1)+(1+\lambda)^2(1-3\cos\alpha)}{8\lambda^2(1+\lambda)(\cos\alpha-1)}.$$

Proof. For any real number ζ, the utilization of Equation (3.16) and Equation (3.15) leads to the following outcome

$$\begin{aligned}a_3-\zeta a_2^2&=\frac{\lambda\delta_1(u_2-v_2)}{4(1+\lambda)}+(1-\zeta)a_2^2\\&=\frac{\lambda\delta_1(u_2-v_2)}{4(1+\lambda)}+\frac{2(1-\zeta)(\delta_1)^3\lambda^2(u_2+v_2)}{(4\lambda^2+2\lambda+2)(\delta_1)^2-2(1+\lambda)^2(\delta_2-\delta_1)}.\end{aligned}$$

The last equation can be written as follows

$$a_3-\zeta a_2^2=(\lambda\delta_1)\left\{\left[A+\frac{1}{4(1+\lambda)}\right]u_2+\left[A-\frac{1}{4(1+\lambda)}\right]v_2\right\}.$$

Therefore, with the assistance of Lemma 2.2, we are able to achieve the following inequality

$$|a_3-\zeta a_2^2|\leq\begin{cases}\frac{\lambda|\delta_1|}{1+\lambda}, & \text{if } |A|\leq\frac{1}{4(1+\lambda)}\\ 4\lambda|\delta_1||A|, & \text{if } |A|\geq\frac{1}{4(1+\lambda)}\end{cases}$$

Finally, simplifying the last equation then utilizing the initial values $\delta_1=\cos\alpha-1$ and $\delta_2=\frac{1}{2}(\cos\alpha-1)(1+3\cos\alpha)$, we obtain the anticipated outcome as indicated in the inequality (3.19). This marks the conclusion of the proof for the Theorem. □

The following corollary is a natural outcome of the previously stated Theorem 3.2. When λ is assigned a value of 1, the resulting Fekete-Szegö inequality becomes associated with the classical class of the starlike bi-univalent functions. The approach used to establish this corollary is quite similar to that of the earlier theorem; hence, we have chosen to omit the comprehensive proof for this corollary.

Corollary 3.2. *If a function $f\in\Sigma$ is represented by (1.1) and is obeying the Subordination conditions (3.17) and (3.18), then for a real number ζ the following holds*

$$|a_3-\zeta a_2^2|\leq\begin{cases}\frac{|1-\cos\alpha|}{2}, & \textit{if } |1-\zeta|\leq\frac{1+\cos\alpha}{4|1-\cos\alpha|}\\ \frac{|1-\cos\alpha||1-\zeta|}{2|1+\cos\alpha|}, & \textit{if } |1-\zeta|\geq\frac{1+\cos\alpha}{4|1-\cos\alpha|}.\end{cases}$$

4. Conclusion

This research paper explored a novel class of generalized starlike bi-univalent functions that are associated with Legendre polynomials. The author has established estimates for the initial coefficients and investigated the Fekete-Szegö functional problem for functions within this class. The results of this investigation are projected to produce a range of outcomes for subclasses related to Gegenbauer and Horadam polynomials, along with their specific variations, including Fibonacci, Lucas, Pell polynomials, and both the first and second kinds of Chebyshev polynomials. Additionally, the findings presented in this paper are expected to inspire researchers to expand these ideas to include harmonic functions and symmetric q-calculus.

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References

[1] V. Aboites, Easy Route to Tchebycheff Plynomials, Rev. Mex. Fís. E. 65 (2019), 12–14. https://doi.org/10.31349/RevMexFisE.65.12.

[2] W. Al-Rawashdeh, Fekete-Szegö Functional of a Subclass of Bi-Univalent Functions Associated with Gegenbauer Polynomials, Eur. J. Pure Appl. Math. 17 (2024), 105–115. https://doi.org/10.29020/nybg.ejpam.v17i1.5004.

[3] W. Al-Rawashdeh, Applications of Gegenbauer Polynomials to a Certain Subclass of P-Valent Functions, WSEAS Trans. Math. 22 (2023), 1025–1030. https://doi.org/10.37394/23206.2023.22.111.

[4] W. Al-Rawashdeh, Horadam Polynomials and a Class of Biunivalent Functions Defined by Ruscheweyh Operator, Int. J. Math. Math. Sci. 2023 (2023), 2573044. https://doi.org/10.1155/2023/2573044.

[5] D.A. Brannan, J.G. Clunie, Aspects of Contemporary Complex Analysis, in: Proceedings of the NATO Advanced Study Institute, University of Durham, Durham, Academic Press, 1979. https://lccn.loc.gov/80040887.

[6] M. Çağlar, H. Orhan, M. Kamali, Fekete-Szegö Problem for a Subclass of Analytic Functions Associated with Chebyshev Polynomials, Bol. Soc. Paran. Mat. 40 (2022), 1–6. https://doi.org/10.5269/bspm.51024.

[7] Y. Cheng, R. Srivastava, J.L. Liu, Applications of the Q-Derivative Operator to New Families of Bi-Univalent Functions Related to the Legendre Polynomials, Axioms 11 (2022), 595. https://doi.org/10.3390/axioms11110595.

[8] J.H. Choi, Y.C. Kim, T. Sugawa, A General Approach to the Fekete-Szegö Problem, J. Math. Soc. Japan 59 (2007), 707–727. https://doi.org/10.2969/jmsj/05930707.

[9] N.E. Cho, V. Kumar, S.S. Kumar, V. Ravichandran, Radius Problems for Starlike Functions Associated with the Sine Function, Bull. Iran. Math. Soc. 45 (2019), 213–232. https://doi.org/10.1007/s41980-018-0127-5.

[10] P.L. Duren, Univalent Functions, Springer, New York, 1983.

[11] P. Duren, Subordination, in: J.D. Buckholtz, T.J. Suffridge (Eds.), Complex Analysis, Springer, Berlin, 1977: pp. 22–29. https://doi.org/10.1007/BFb0096821.

[12] P. Goel, S.S. Kumar, Certain Class of Starlike Functions Associated with Modified Sigmoid Function, Bull. Malays. Math. Sci. Soc. 43 (2020), 957–991. https://doi.org/10.1007/s40840-019-00784-y.

[13] A.W. Goodman, Univalent Functions, Mariner Publishing Co. Inc., 1983.

[14] M. Kamali, M. Cağlar, E. Deniz and M. Turabaev, Fekete-Szegö Problem for a New Subclass of Analytic Functions Satisfyingsubordinate Condition Associated with Chebyshev Polynomials, Turk. J. Math. 45 (2021), 1195–1208. https://doi.org/10.3906/mat-2101-20.

[15] F.R. Keogh, E.P. Merkes, A Coefficient Inequality for Certain Classes of Analytic Functions, Proc. Amer. Math. Soc. 20 (1969), 8–12. https://doi.org/10.1090/S0002-9939-1969-0232926-9.

[16] S.S. Kumar, S. Banga, On A Special Type Of Ma-Minda Function, arXiv:2006.02111 [math.CV] (2022). http://arxiv.org/abs/2006.02111.

[17] M. Fekete, G. Szegö, Eine Bemerkung Über Ungerade Schlichte Funktionen, J. Lond. Math. Soc. s1-8 (1933), 85–89. https://doi.org/10.1112/jlms/s1-8.2.85.

[18] M. Lewin, On a Coefficient Problem for Bi-Univalent Functions, Proc. Amer. Math. Soc. 18 (1967), 63–68. https://doi.org/10.1090/S0002-9939-1967-0206255-1.

[19] W. Ma, D. Minda, A Unified Treatment of Some Special Classes of Univalent Functions, in: Proceedings of the Conference on Complex Analysis, Tianjin, 1992.

[20] N. Magesh, S. Bulut, Chebyshev Polynomial Coefficient Estimates for a Class of Analytic Bi-Univalent Functions Related to Pseudo-Starlike Functions, Afr. Mat. 29 (2018), 203–209. https://doi.org/10.1007/s13370-017-0535-3.

[21] R. Mendiratta, S. Nagpal, V. Ravichandran, On a Subclass of Strongly Starlike Functions Associated with Exponential Function, Bull. Malays. Math. Sci. Soc. 38 (2015), 365–386. https://doi.org/10.1007/s40840-014-0026-8.

[22] S. Miller, P. Mocabu, Differential Subordination: Theory and Applications, CRC Press, New York, 2000.

[23] E. Muthaiyan, A. Wanas, On Some Coefficient Inqualities Involving Legendre Polynomials in the Class of Bi-Univalent Functions, Turk. J. Ineq. 7 (2023), 39–46.

[24] Z. Nehari, Conformal Mappings, McGraw-Hill, New York, 1952.

[25] E. Netanyahu, The Minimal Distance of the Image Boundary from the Origin and the Second Coefficient of a Univalent Function in $|z| < 1$, Arch. Ration. Mech. Anal. 32 (1969), 100–112. https://doi.org/10.1007/BF00247676.

[26] A. Raposo, H. Weber, D. Alvarez-Castillo, M. Kirchbach, Romanovski Polynomials in Selected Physics Problems, Centr. Eur. J. Phys. 5 (2007), 253–284. https://doi.org/10.2478/s11534-007-0018-5.

[27] K. Rilely, M. Hobson, S. Bence, Mathematical Methods for Physics and Engineering, Cambridge University Press, Cambridge, 2006.

[28] J. Sokól, J. Stankiewicz, Radius of Convexity of Some Subclasses of Strongly Starlike Functions, Zesz. Nauk. Politech. Rzesz., Mat. 19 (1996), 101–105.

[29] H.M. Srivastava, H.L. Manocha, Ellis Horwood, Chichester and Halstead Press, New York, 1984

[30] H.M. Srivastava, M. Kamalı, A. Urdaletova, A Study of the Fekete-Szegö Functional and Coefficient Estimates for Subclasses of Analytic Functions Satisfying a Certain Subordination Condition and Associated with the Gegenbauer Polynomials, AIMS Math. 7 (2022), 2568–2584. https://doi.org/10.3934/math.2022144.