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Bipolar Fuzzy Quasi-Ideals in Γ-Semirings: A Study

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Abstract. This paper investigates bipolar fuzzy quasi-ideals in the context of Γ-semirings, offering new insights into their structural properties. Our results reveal that bipolar fuzzy quasi-ideals serve as a generalization of bipolar fuzzy ideals, while bipolar fuzzy bi-ideals extend this framework further. We also establish that in regular Γ-semirings, the two concepts coincide, leading to a unified interpretation. Notably, the intersection of a bipolar fuzzy right ideal and a bipolar fuzzy left ideal forms a bipolar fuzzy quasi-ideal, highlighting key properties that deepen our understanding of ideal structures in Γ-semirings.

1. Introduction

The concept of Γ-rings, introduced by Nobusawa [10], represents a key generalization of classical ring theory, marking a significant development in algebraic structures. Semirings, another essential algebraic framework, were rigorously studied by Vandiver [12], who established foundational principles for their exploration. Building on these contributions, Rao [9] proposed the concept of Γ-semirings, a more expansive and versatile structure that unifies the characteristics of rings, Γ-rings, and semirings, offering a more comprehensive algebraic model. This progression illustrates the dynamic evolution of algebraic theory, encouraging further research and broader applications.

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In 1965, Zadeh introduced the concept of fuzzy sets [14], which has since inspired numerous extensions, including intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, and neutrosophic sets. Mandal [8] contributed to this area by investigating fuzzy ideals and fuzzy interior ideals in ordered semirings. Zhang [15], in 1994, introduced bipolar-valued fuzzy sets, extending the membership degree interval from $[0,1]$ to $[-1,1]$, thus expanding the fuzzy set framework. Further development of fuzzy concepts within Γ-semirings has been pursued by scholars such as Bhargavi [1] and Eswarlal, who explored fuzzy notions in this extended algebraic context.

Parvatham and colleagues [6,7,11] advanced the study of bipolar fuzzy Γ-semirings (BFGSRs), bipolar fuzzy ideals (BFIs), and bipolar fuzzy bi-ideals (BFBIs). Vijay Kumar et al. [13] introduced the concepts of bipolar fuzzy quasi-ideals (BFQIs) and bipolar N subgroups in near rings. Bhargavi et al. [2–5] explored vague bi-ideals, vague quasi-ideals, vague interior ideals, and various hybrid fuzzy structures in Γ-semirings, further enriching the field.

In this paper, we introduce the concept of bipolar fuzzy quasi-ideals (BFQIs) within the framework of Γ-semirings and examine their key properties. Our analysis shows that in regular Γ-semirings, the notions of BFQIs and bipolar fuzzy bi-ideals (BFBIs) coincide. Furthermore, we establish that the intersection of a bipolar fuzzy right ideal (BFRI) and a bipolar fuzzy left ideal (BFLI) in a Γ-semiring always results in a BFQI, reinforcing the structural coherence of these concepts.

2. Preliminaries

In this section, we revisit key concepts and foundational definitions that are essential for the subsequent analysis. To provide context, we begin with an overview of Γ-semirings, which serve as an extension of classical ring and semiring structures. These algebraic systems incorporate the operations of both rings and semirings, offering a more versatile framework for studying various generalizations. By establishing these preliminary notions, we ensure a comprehensive understanding of the algebraic foundation necessary for the detailed exploration of bipolar fuzzy quasi-ideals and bi-ideals in the following sections.

Definition 2.1. *[1] Let $\vee$ and Γ be two additive commutative semigroups. Then $\vee$ is called a Γ-semiring if there exists a mapping $\vee \times \Gamma \times \vee \to \vee$ image denoted by $\ddot{c}\alpha\ddot{p}$ for $\ddot{c}, \ddot{p} \in \vee$ and $\alpha \in \Gamma$, satisfying the following conditions: for all $\ddot{c}, \ddot{p}, \ddot{u} \in \vee$ and $\alpha, \beta \in \Gamma$,*
(i) $\ddot{c}\alpha(\ddot{p} + \ddot{u}) = \ddot{c}\alpha\ddot{p} + \ddot{c}\alpha\ddot{u}$,
(ii) $(\ddot{c} + \ddot{p})\alpha\ddot{u} = \ddot{c}\alpha\ddot{u} + \ddot{p}\alpha\ddot{u}$,
(iii) $\ddot{c}(\alpha + \beta)\ddot{u} = \ddot{c}\alpha\ddot{u} + \ddot{c}\beta\ddot{u}$,
(iv) $\ddot{c}\alpha(\ddot{p}\beta\ddot{u}) = (\ddot{c}\alpha\ddot{p})\beta\ddot{u}$.

Definition 2.2. *[3] An element v of a Γ-semiring $\vee$ is said to be regular if $v \in v\Gamma \vee \Gamma v$. If all the elements of a Γ-semiring $\vee$ are regular, then $\vee$ is known as a regular Γ-semiring.*

Definition 2.3. *[3] An element v of a Γ-semiring $\vee$ is said to be intra-regular if $v \in \vee\Gamma v\Gamma v\Gamma\vee$. If all the elements of a Γ-semiring $\vee$ are regular, then $\vee$ is known as an intra-regular Γ-semiring.*

Definition 2.4. *[1] A non-empty subset I of a Γ-semiring $\vee$ is called idempotent if I is an additive subsemigroup of $\vee$ and $I\Gamma I = I$.*

Definition 2.5. *[1] A non-empty subset I of a Γ-semiring $\vee$ is called a quasi-ideal (QI) of $\vee$ if I is a Γ-subsemiring of $\vee$ and $I\Gamma \vee \cap \vee \Gamma I \subseteq I$.*

Definition 2.6. *[14] Let $\vee$ be any non-empty set. A mapping $\xi : \vee \to [0,1]$ is called a fuzzy set of $\vee$.*

Definition 2.7. *[15] Let $\vee$ be the universe of discourse. A bipolar fuzzy set (BFS) ξ in $\vee$ is an object having the form $\xi := \{(\ddot{v}, \xi^-(\ddot{v}), \xi^+(\ddot{v})) : \ddot{v} \in \vee\}$, where $\xi^- : \vee \to [-1,0]$ and $\xi^+ : \vee \to [0,1]$ are mappings.*

For the sake of simplicity, we shall use the symbol $\xi = (\vee; \xi^-, \xi^+)$ for the BFS $\xi := \{(\ddot{v}, \xi^-(\ddot{v}), \xi^+(\ddot{v})) : \ddot{v} \in \vee\}$.

Definition 2.8. *[15] Let $\xi = (\vee; \xi^-, \xi^+)$ be a BFS and $s \times t \in [-1,0] \times [0,1]$, the sets $\xi_s^- = \{\ddot{v} \in \vee : \xi^-(\ddot{v}) \leq s\}$ and $\xi_t^+ = \{\ddot{v} \in \vee : \xi^+(\ddot{v}) \geq t\}$ are called negative s-cut and positive t-cut, respectively. For $s \times t \in [-1,0] \times [0,1]$, the set $\xi_{(s,t)} = \xi_s^- \cap \xi_t^+$ is called the (s,t)-set of $\xi = (\vee; \xi^-, \xi^+)$.*

Definition 2.9. *[15] Let $\xi = (\vee; \xi^-, \xi^+)$ and $\eta = (\vee; \eta^-, \eta^+)$ be two BFSs in a universe of discourse $\vee$. The intersection of ξ and η is defined as*

$$(\xi^- \cap \eta^-)(\ddot{v}) = \min\{\xi^-(\ddot{v}), \eta^-(\ddot{v})\} \text{ and } (\xi^+ \cap \eta^+)(\ddot{v}) = \min\{\xi^+(\ddot{v}), \eta^+(\ddot{v})\}, \forall \ddot{v} \in \vee.$$

The union of ξ and η is defined as

$$(\xi^- \cup \eta^-)(\ddot{v}) = \max\{\xi^-(\ddot{v}), \eta^-(\ddot{v})\} \text{ and } (\xi^+ \cup \eta^+)(\ddot{v}) = \max\{\xi^+(\ddot{v}), \eta^+(\ddot{v})\}, \forall \ddot{v} \in \vee.$$

A BFS ξ is contained in another bipolar fuzzy set η, written with $\xi \subseteq \eta$ if

$$\xi^-(\ddot{v}) \geq \eta^-(\ddot{v}) \text{ and } \xi^+(\ddot{v}) \leq \eta^+(\ddot{v}), \forall \ddot{v} \in \vee.$$

Definition 2.10. *[6] Let D be a subset of a Γ-semiring $\vee$. The bipolar fuzzy characteristic function δ_D of D is given by*

$$\delta_D^+(\ddot{v}) = \begin{cases} 1 \text{ if } \ddot{v} \in D \\ 0 \text{ otherwise} \end{cases} \text{ and } \delta_D^-(\ddot{v}) = \begin{cases} -1 \text{ if } \ddot{v} \in D \\ 0 \text{ otherwise.} \end{cases}$$

Definition 2.11. *[1] A BFS $\xi = (\vee; \xi^-, \xi^+)$ in a Γ-semiring $\vee$ is called a bipolar fuzzy Γ-semiring (BFGSR) of $\vee$ if it satisfies the following properties: for all $\ddot{c}, \ddot{p} \in \vee$ and $\gamma \in \Gamma$,*
(i) $\xi^-(\ddot{c} + \ddot{p}) \leq \max\{\xi^-(\ddot{c}), \xi^-(\ddot{p})\}$,
(ii) $\xi^-(\ddot{c}\gamma\ddot{p}) \leq \max\{\xi^-(\ddot{c}), \xi^-(\ddot{p})\}$,
(iii) $\xi^+(\ddot{c} + \ddot{p}) \geq \min\{\xi^+(\ddot{c}), \xi^+(\ddot{p})\}$,
(iv) $\xi^+(\ddot{c}\gamma\ddot{p}) \geq \min\{\xi^+(\ddot{c}), \xi^+(\ddot{p})\}$.

Definition 2.12. *[3] A BFS $\xi = (\vee; \xi^-, \xi^+)$ in a Γ-semiring $\vee$ is called a bipolar fuzzy left (resp., right) ideal (BFL(R)I) of $\vee$ if it satisfies the following properties: for any $\ddot{e}, \ddot{o} \in \vee$ and $\varrho \in \Gamma$,*
(i) $\xi^-(\ddot{e} + \ddot{o}) \leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\}$,
(ii) $\xi^-(\ddot{e}\varrho\ddot{o}) \leq \xi^-(\ddot{o})$ *(resp.,* $\leq \xi^-(\ddot{e})$*)*,
(iii) $\xi^+(\ddot{e} + \ddot{o}) \geq \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}$,

(iv) $\xi^+(\ddot{e}\varrho\ddot{o}) \geq \xi^+(\ddot{o})$ *(resp.,* $\geq \xi^+(\ddot{e})$*).*
Also, ξ *is a bipolar fuzzy ideal (BFI) of* $\vee$ *if it is both a BFLI and a BFRI of* $\vee$*.*

Definition 2.13. *[11] A BFS* $\xi = (\vee; \xi^-, \xi^+)$ *in a* Γ*-semiring* $\vee$ *is called a bipolar fuzzy bi-ideal (BFBI) of* $\vee$ *if it satisfies the following properties: for any* $\ddot{c}, \ddot{p}, \ddot{u} \in \vee$ *and* $\alpha, \beta \in \Gamma$*,*
(i) $\xi^-(\ddot{c} + \ddot{p}) \leq \max\{\xi^-(\ddot{c}), \xi^-(\ddot{p})\}$*,*
(ii) $\xi^-(\ddot{c}\alpha\ddot{p}\beta\ddot{u}) \leq \max\{\xi^-(\ddot{c}), \xi^-(\ddot{u})\}$*,*
(iii) $\xi^+(\ddot{c} + \ddot{p}) \geq \min\{\xi^+(\ddot{c}), \xi^+(\ddot{p})\}$*,*
(iv) $\xi^+(\ddot{c}\alpha\ddot{p}\beta\ddot{u}) \geq \min\{\xi^+(\ddot{c}), \xi^+(\ddot{u})\}$*.*

3. Main Results

In this section, we introduce and thoroughly examine the concept of bipolar fuzzy quasi-ideals, focusing on their unique properties and defining characteristics. Building on this foundation, we further investigate the impact of replacing the join operation $\vee$ with a Γ-semiring operation, analyzing how this modification affects the structure and behavior of these quasi-ideals within the broader algebraic framework. This approach provides deeper insight into the versatility and adaptability of bipolar fuzzy quasi-ideals in the context of Γ-semirings.

Definition 3.1. *A BFS* $\xi = (\vee; \xi^-, \xi^+)$ *in* $\vee$ *is called a bipolar fuzzy quasi-ideal (BFQI) of* $\vee$ *if it satisfies the following properties: for any* $\ddot{e}, \ddot{o} \in \vee$*,*
(i) $\xi^-(\ddot{e} + \ddot{o}) \leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\}$*,*
(ii) $(\xi^-\Gamma\delta^-) \cup (\delta^-\Gamma\xi^-) \supseteq \xi^-$*,*
(iii) $\xi^+(\ddot{e} + \ddot{o}) \geq \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}$*,*
(iv) $(\xi^+\Gamma\delta^+) \cap (\delta^+\Gamma\xi^+) \subseteq \xi^+$*.*

Example 3.1. *Let* $\vee$ *be the set of all natural numbers with zero and* Γ *be the set of all negative even integers. Then* $\vee$ *and* Γ *are additive commutative semigroups. Define a mapping* $\vee \times \Gamma \times \vee$ *by* $\ddot{e}\varrho\ddot{o}$ *as usual product of* $\ddot{e}, \ddot{o} \in \vee$ *and* $\varrho \in \Gamma$*. Then* $\vee$ *is a* Γ*-semiring. Define a BFS* $\xi = (\vee; \xi^-, \xi^+)$ *in* $\vee$ *as follows:*

$$\xi^-(\psi) = \begin{cases} -0.8 \text{ if } \psi \text{ is even or zero} \\ -0.5 \text{ otherwise} \end{cases} \quad \text{and } \xi^+(\psi) = \begin{cases} 0.8 \text{ if } \psi \text{ is even or zero} \\ 0.5 \text{ otherwise.} \end{cases}$$

Then ξ *is a BFQI of* $\vee$*.*

Theorem 3.1. *A BFS* $\xi = (\vee; \xi^-, \xi^+)$ *in* $\vee$ *is a BFLI of* $\vee$ *if and only if for all* $\ddot{e}, \ddot{o} \in \vee$*,*
(i) $\xi^-(\ddot{e} + \ddot{o}) \leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\}$*,*
(ii) $\delta^-\Gamma\xi^- \supseteq \xi^-$*,*
(iii) $\xi^+(\ddot{e} + \ddot{o}) \geq \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}$*,*
(iv) $\delta^+\Gamma\xi^+ \subseteq \xi^+$*.*

Proof. Suppose ξ is a BFLI of $\vee$. Then (i) and (iii) hold. Let $\ddot{v} \in \vee$.
Then $(\delta^-\Gamma\xi^-)(\ddot{v}) = \inf\{\max\{\delta^-(\ddot{e}), \xi^-(\ddot{o}) : \ddot{v} = \ddot{e}\varrho\ddot{o}\}\} = \xi^-(\ddot{o}) \geq \xi^-(\ddot{v})$, so $\delta^-\Gamma\xi^- \supseteq \xi^-$.
Also, $(\delta^+\Gamma\xi^+)(\ddot{v}) = \sup\{\min\{\delta^+(\ddot{e}), \xi^+(\ddot{o}) : \ddot{v} = \ddot{e}\varrho\ddot{o}\}\} = \xi^+(\ddot{o}) \leq \xi^+(\ddot{v})$, so $\delta^+\Gamma\xi^+ \subseteq \xi^+$.

Conversely, suppose that all four conditions hold. Let $\ddot{e}, \ddot{o} \in \vee$. Then $\xi^-(\ddot{e}\varrho\ddot{o}) \leq (\delta^-\Gamma\xi^-)(\ddot{e}\varrho\ddot{o}) = \inf\{\max\{\delta^-(\ddot{e}), \xi^-(\ddot{o})\}\} = \xi^-(\ddot{o})$ and $\xi^+(\ddot{e}\varrho\ddot{o}) \geq (\delta^+\Gamma\xi^+)(\ddot{e}\varrho\ddot{o}) = \sup\{\min\{\delta^+(\ddot{e}), \xi^+(\ddot{o})\}\} = \xi^+(\ddot{o})$. Hence, ξ is a BFLI of $\vee$. □

Theorem 3.2. *Any BFI of $\vee$ is a BFQI of $\vee$ and any BFQI of $\vee$ is a BFBI of $\vee$.*

Proof. Suppose ξ is a BFI of $\vee$. Then for any $\ddot{e}, \ddot{o} \in \vee$ and $\varrho \in \Gamma$,
(i) $\xi^-(\ddot{e}+\ddot{o}) \leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\}$,
(ii) $\xi^-(\ddot{e}\varrho\ddot{o}) \leq \xi^-(\ddot{o})$ and $\xi^-(\ddot{e}\varrho\ddot{o}) \leq \xi^-(\ddot{e})$,
(iii) $\xi^+(\ddot{e}+\ddot{o}) \geq \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}$,
(iv) $\xi^+(\ddot{e}\varrho\ddot{o}) \geq \xi^+(\ddot{o})$ and $\xi^+(\ddot{e}\varrho\ddot{o}) \geq \xi^+(\ddot{e})$.
Thus, $(\xi^-\Gamma\delta^-) \cup (\delta^-\Gamma\xi^-) \supseteq \delta^-\Gamma\xi^- \supseteq \xi^-$ and $(\xi^+\Gamma\delta^+) \cap (\delta^+\Gamma\xi^+) \subseteq \delta^+\Gamma\xi^+ \subseteq \xi^+$. Also, $(\xi^-\Gamma\delta^-) \cup (\delta^-\Gamma\xi^-) \supseteq \xi^-\Gamma\delta^- \supseteq \xi^-$ and $(\xi^+\Gamma\delta^+) \cap (\delta^+\Gamma\xi^+) \subseteq \xi^+\Gamma\delta^+ \subseteq \xi^+$. Hence, ξ is a BFQI of $\vee$.

Now, suppose ξ is a BFQI of $\vee$. Let $\ddot{e}, \ddot{o}, \ddot{v} \in \vee$ and $\varrho, \tau \in \Gamma$. Then
(i) $\xi^-(\ddot{e}+\ddot{o}) \leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\}$,
(ii) $\xi^+(\ddot{e}+\ddot{o}) \geq \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}$.
Also,

$$\begin{aligned}
\xi^-(\ddot{e}\varrho\ddot{o}) &\leq [(\xi^-\Gamma\delta^-) \cup (\delta^-\Gamma\xi^-)](\ddot{e}\varrho\ddot{o}) \\
&= \max\{(\xi^-\Gamma\delta^-)(\ddot{e}\varrho\ddot{o}), (\delta^-\Gamma\xi^-)(\ddot{e}\varrho\ddot{o})\} \\
&= \max\{\inf\{\max\{\xi^-(\ddot{e}), \delta^-(\ddot{o})\}\}, \inf\{\max\{\delta^-(\ddot{e}), \xi^-(\ddot{o})\}\}\} \\
&= \max\{\xi^-(\ddot{e}), \xi^-(\ddot{o})\},
\end{aligned}$$

$$\begin{aligned}
\xi^+(\ddot{e}\varrho\ddot{o}) &\geq [(\xi^+\Gamma\delta^+) \cap (\delta^+\Gamma\xi^+)](\ddot{e}\varrho\ddot{o}) \\
&= \min\{(\xi^+\Gamma\delta^+)(\ddot{e}\varrho\ddot{o}), (\delta^+\Gamma\xi^+)(\ddot{e}\varrho\ddot{o})\} \\
&= \min\{\sup\{\min\{\xi^+(\ddot{e}), \delta^+(\ddot{o})\}\}, \sup\{\min\{\delta^+(\ddot{e}), \xi^+(\ddot{o})\}\}\} \\
&= \min\{\xi^+(\ddot{e}), \xi^+(\ddot{o})\}.
\end{aligned}$$

Therefore, ξ is a BFGSR of $\vee$. Also,

$$\begin{aligned}
\xi^-(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &\leq [(\xi^-\Gamma\delta^-) \cup (\delta^-\Gamma\xi^-)](\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \\
&= \max\{(\xi^-\Gamma\delta^-)(\ddot{e}\varrho\ddot{o}\tau\ddot{v}), (\delta^-\Gamma\xi^-)(\ddot{e}\varrho\ddot{o}\tau\ddot{v})\} \\
&= \max\{\inf\{\max\{\xi^-(\ddot{e}), \delta^-(\ddot{o}\tau\ddot{v})\}\}, \inf\{\max\{\delta^-(\ddot{e}\varrho\ddot{o}), \xi^-(\ddot{v})\}\}\} \\
&= \max\{\xi^-(\ddot{e}), \xi^-(\ddot{v})\},
\end{aligned}$$

$$\begin{aligned}
\xi^+(\ddot{e}\varrho\ddot{o}\tau\ddot{v}) &\geq [(\xi^+\Gamma\delta^+) \cap (\delta^+\Gamma\xi^+)](\ddot{e}\varrho\ddot{o}\tau\ddot{v}) \\
&= \min\{(\xi^+\Gamma\delta^+)(\ddot{e}\varrho\ddot{o}\tau\ddot{v}), (\delta^+\Gamma\xi^+)(\ddot{e}\varrho\ddot{o}\tau\ddot{v})\} \\
&= \min\{\sup\{\min\{\xi^+(\ddot{e}), \delta^+(\ddot{o}\tau\ddot{v})\}\}, \sup\{\min\{\delta^+(\ddot{e}\varrho\ddot{o}), \xi^+(\ddot{v})\}\}\} \\
&= \min\{\xi^+(\ddot{e}), \xi^+(\ddot{v})\}.
\end{aligned}$$

Hence, ξ is a BFBI of $\vee$. □

Theorem 3.3. *If $\vee$ is regular, then every BFBI of $\vee$ is a BFQI of $\vee$.*

Proof. Given $\vee$ is regular. Suppose ξ is a BFBI of $\vee$. Then for any $\ddot{e},\ddot{o}\in\vee$,
(i) $\xi^-(\ddot{e}+\ddot{o})\leq\max\{\xi^-(\ddot{e}),\xi^-(\ddot{o})\}$,
(ii) $\xi^+(\ddot{e}+\ddot{o})\geq\min\{\xi^+(\ddot{e}),\xi^+(\ddot{o})\}$.

We shall show that $(\xi^+\Gamma\delta^+)\cap(\delta^+\Gamma\xi^+)\subseteq\xi^-$ and $(\xi^-\Gamma\delta^-)\cup(\delta^-\Gamma\xi^-)\supseteq\xi^-$.
Case 1: Suppose $(\xi^+\Gamma\delta^+)(\ddot{p})\leq\xi^+(\ddot{p})$. Then

$$\begin{aligned}(\xi^+\Gamma\delta^+)\cap(\delta^+\Gamma\xi^+)(\ddot{p})&=\min\{(\xi^+\Gamma\delta^+)(\ddot{p}),(\delta^+\Gamma\xi^+)(\ddot{p}):\ddot{p}=\ddot{e}\varrho\ddot{o}\tau\ddot{v}\}\\&\leq(\xi^+\Gamma\delta^+)(\ddot{p})\\&\leq\xi^+(\ddot{p}).\end{aligned}$$

Case 2: Suppose $(\xi^+\Gamma\delta^+)(\ddot{p})>\xi^+(\ddot{p})$. Then $\xi^+(\ddot{p})<(\xi^+\Gamma\delta^+)(\ddot{p})=\sup\{\{\min\{(\xi^+(\ddot{e}),\delta^+(\ddot{o}):\ddot{p}=\ddot{e}\varrho\ddot{o}\}\}=\xi^+(\ddot{e})$. Also, $(\delta^+\Gamma\xi^+)(\ddot{p})=\sup\{\{\min\{(\delta^+(\ddot{f}),\xi^+(\ddot{m}):\ddot{p}=\ddot{f}\tau\ddot{m}\}\}=\xi^+(\ddot{m})$. Since $\vee$ is regular, there exist $\ddot{s}\in\vee$ and $\varphi,\eta\in\Gamma$ such that $\ddot{p}=\ddot{p}\varphi\ddot{s}\eta\ddot{p}$. Since ξ is a BFBI of $\vee$, we have

$$\begin{aligned}\xi^+(\ddot{p})&=\xi^+(\ddot{p}\varphi\ddot{s}\eta\ddot{p})\\&=\xi^+((\ddot{e}\varrho\ddot{o})\varphi\ddot{s}\eta(\ddot{f}\tau\ddot{m}))\\&=\xi^+(\ddot{e}\varrho(\ddot{o}\varphi\ddot{s}\eta\ddot{f})\tau\ddot{m})\\&\geq\min\{\xi^+(\ddot{e}),\xi^+(\ddot{m})\}.\end{aligned}$$

If $\min\{\xi^+(\ddot{e}),\xi^+(\ddot{m})\}=\xi^+(\ddot{e})$, then $\xi^+(\ddot{p})\geq\xi^+(\ddot{e})$ which is a contradiction. Thus, $\min\{\xi^+(\ddot{e}),\xi^+(\ddot{m})\}=\xi^+(\ddot{m})$, so $\xi^+(\ddot{p})\geq\xi^+(\ddot{m})$. Now,

$$\begin{aligned}[(\xi^+\Gamma\delta^+)\cap(\delta^+\Gamma\xi^+)](\ddot{p})&=\min\{(\xi^+\Gamma\delta^+)(\ddot{p}),(\delta^+\Gamma\xi^+)(\ddot{p})\}\\&\leq(\delta^+\Gamma\xi^+)(\ddot{p})\\&=\xi^+(\ddot{m})\\&\leq\xi^+(\ddot{p}).\end{aligned}$$

Hence, $(\xi^+\Gamma\delta^+)\cap(\delta^+\Gamma\xi^+)\subseteq\xi^+$.
Case 3: Suppose $(\xi^-\Gamma\delta^-)(\ddot{p})\geq\xi^(\ddot{p})$. Then

$$\begin{aligned}[(\xi^-\Gamma\delta^-)\cup(\delta^-\Gamma\xi^-)](\ddot{p})&=\max\{(\xi^-\Gamma\delta^-)(\ddot{p}),(\delta^-\Gamma\xi^-)(\ddot{p})\}\\&\geq(\xi^-\Gamma\delta^-)(\ddot{p})\\&\geq\xi^-(\ddot{p}).\end{aligned}$$

Thus, $(\xi^-\Gamma\delta^-)\cup(\delta^-\Gamma\xi^-)\supseteq\xi^-$.
Case 4: Suppose $(\xi^-\Gamma\delta^-)(\ddot{p})<\xi^-(\ddot{p})$. Then $\xi^-(\ddot{p})>(\xi^-\Gamma\delta^-)(\ddot{p})=\inf\{\{\max\{(\xi^-(\ddot{e}),\delta^-(\ddot{o}):\ddot{p}=\ddot{e}\varrho\ddot{o}\}\}=\xi^-(\ddot{e})$. Also, $(\delta^-\Gamma\xi^-)(\ddot{p})=\inf\{\{\max\{(\delta^-(\ddot{f}),\xi^-(\ddot{m}):\ddot{p}=\ddot{f}\tau\ddot{m}\}\}=\xi^-(\ddot{m})$. Since $\vee$ is regular,

there exist $\ddot{s} \in \vee$ and $\varphi, \eta \in \Gamma$ such that $\ddot{p} = \ddot{p}\varphi\ddot{s}\eta\ddot{p}$. Since ξ is a BFBI of $\vee$, we have

$$\begin{aligned}\xi^-(\ddot{p}) &= \xi^-(\ddot{p}\varphi\ddot{s}\eta\ddot{p})\\ &= \xi^-((\ddot{e}\varrho\ddot{o})\varphi\ddot{s}\eta(\ddot{f}\tau\ddot{m}))\\ &= \xi^-(\ddot{e}\varrho(\ddot{o}\varphi\ddot{s}\eta\ddot{f})\tau\ddot{m})\\ &\leq \max\{\xi^-(\ddot{e}), \xi^-(\ddot{m})\}.\end{aligned}$$

If $\max\{\xi^-(\ddot{e}), \xi^-(\ddot{m})\} = \xi^-(\ddot{e})$, then $\xi^-(\ddot{p}) \leq \xi^-(\ddot{e})$ which is a contradiction. Thus, $\max\{\xi^-(\ddot{e}), \xi^-(\ddot{m})\} = \xi^-(\ddot{m})$, so $\xi^-(\ddot{p}) \geq \xi^-(\ddot{m})$. Now,

$$\begin{aligned}[(\xi^-\Gamma\delta^-) \cup (\delta^-\Gamma\xi^-)](\ddot{p}) &= \max\{(\xi^-\Gamma\delta^-)(\ddot{p}), (\delta^-\Gamma\xi^-)(\ddot{p})\}\\ &\geq (\delta^-\Gamma\xi^-)(\ddot{p})\\ &= \xi^-(\ddot{m})\\ &= ge\xi^-(\ddot{p}).\end{aligned}$$

Hence, $(\xi^-\Gamma\delta^-) \cup (\delta^-\Gamma\xi^-) \supseteq \xi^-$. Therefore, ξ is a BFQI of $\vee$. □

Theorem 3.4. *The intersection of a BFRI and a BFLI of* $\vee$ *is a BFQI of* $\vee$.

Proof. Let ξ be a BFRI and $\varkappa$ be a BFLI of $\vee$. Then for any $\ddot{e}, \ddot{o} \in \vee$,
(i) $(\xi^+ \cap \varkappa^+)(\ddot{e} + \ddot{o}) \geq \min\{(\xi^+ \cap \varkappa^+)(\ddot{e}), (\xi^+ \cap \varkappa^+)(\ddot{o})\}$,
(ii) $(\xi^- \cap \varkappa^-)(\ddot{e} + \ddot{o}) \geq \max\{(\xi^- \cap \varkappa^-)(\ddot{e}), (\xi^- \cap \varkappa^-)(\ddot{o})\}$.
Since ξ is a BFRI of $\vee$, we have $\xi^+\Gamma\delta^+ \subseteq \xi^+$ and $\xi^-\Gamma\delta^- \supseteq \xi^-$. Since $\varkappa$ is a BFLI of $\vee$, we have $\delta^+\Gamma\varkappa^+ \subseteq \varkappa^+$ and $\delta^-\Gamma\varkappa^- \supseteq \varkappa^-$. Now,

$$\begin{aligned}[(\xi^+ \cap \varkappa^+)\Gamma\delta^+] \cap [\delta^+\Gamma(\xi^+ \cap \varkappa^+)] &\subseteq (\xi^+\Gamma\delta^+) \cap (\varkappa^+\Gamma\delta^+)\\ &\subseteq \xi^+ \cap \varkappa^+,\end{aligned}$$

$$\begin{aligned}[(\xi^- \cap \varkappa^-)\Gamma\delta^-] \cup [\delta^-\Gamma(\xi^+ \cap \varkappa^-)] &\supseteq (\xi^-\Gamma\delta^-) \cup (\varkappa^-\Gamma\delta^-)\\ &\supseteq \xi^- \cup \varkappa^-.\end{aligned}$$

Hence, $\xi \cap \varkappa$ is a BFQI of $\vee$. □

Theorem 3.5. *Let* κ *be a nonempty subset of* $\vee$. *Then* δ_κ *is a BFQI of* $\vee$ *if and only if* κ *is a QI of* $\vee$.

Proof. Let δ_κ is a BFQI of $\vee$. Then for any $\ddot{e}, \ddot{o} \in \vee$,
(i) $\delta^-_\kappa(\ddot{e} + \ddot{o}) \leq \max\{\delta^-_\kappa(\ddot{e}), \delta^-_\kappa(\ddot{o})\} = \max\{-1, -1\} = -1$,
(ii) $\delta^+_\kappa(\ddot{e} + \ddot{o}) \geq \min\{\delta^+_\kappa(\ddot{e}), \delta^+_\kappa(\ddot{o})\} = \min\{1, 1\} = 1$.
Thus, $\ddot{e} + \ddot{o} \in \kappa$. Let $\ddot{p} \in (\vee\Gamma\kappa) \cap (\kappa\Gamma\vee)$. Then $\ddot{p} \in \vee\Gamma\kappa$ and $\ddot{p} \in \kappa\Gamma\vee$. Thus, $\ddot{p} = \ddot{e}\varrho\ddot{o}$ and $\ddot{p} = \ddot{f}\tau\ddot{m}$ for some $\ddot{e}, \ddot{m} \in \vee, \ddot{o}, \ddot{f} \in \kappa$, and $\varrho, \tau \in \Gamma$. Also,

$$\begin{aligned}\delta^-_\kappa(\ddot{p}) &\leq [(\delta^-_\kappa\Gamma\delta^-)\cup(\delta^-\Gamma\delta^-_\kappa)](\ddot{p})\\ &= \max\{(\delta^-_\kappa\Gamma\delta^-)(\ddot{p}),(\delta^-\Gamma\delta^-_\kappa)(\ddot{p})\}\\ &= \max\{\inf\{\max\{\delta^-_\kappa(\ddot{e}),\delta^-(\ddot{o})\}\},\inf\{\max\{\delta^-(\ddot{f}),\delta^-_\kappa(\ddot{m})\}\}\}\\ &= \max\{-1,-1\}\\ &= -1,\end{aligned}$$

$$\begin{aligned}\delta^+_\kappa(\ddot{p}) &\geq [(\delta^+_\kappa\Gamma\delta^+)\cap(\delta^+\Gamma\delta^+_\kappa)](\ddot{p})\\ &= \min\{(\delta^+_\kappa\Gamma\delta^+)(\ddot{p}),(\delta^+\Gamma\delta^+_\kappa)(\ddot{p})\}\\ &= \min\{\sup\{\min\{\delta^+_\kappa(\ddot{e}),\delta^+(\ddot{o})\}\},\sup\{\min\{\delta^+(\ddot{f}),\delta^+_\kappa(\ddot{m})\}\}\}\\ &= \min\{1,1\}\\ &= 1.\end{aligned}$$

Thus, $\ddot{p}\in\kappa$. Hence, κ is a QI of $\vee$.

Conversely, suppose that κ is a QI of $\vee$. Let $\ddot{e},\ddot{o}\in\vee$.

If $\ddot{e},\ddot{o}\in\kappa$, then $\ddot{e}+\ddot{o}\in\kappa$. Thus, $\delta^-_\kappa(\ddot{e}+\ddot{o})=-1=[-1,-1]=\max\{\delta^-_\kappa(\ddot{e}),\delta^-_\kappa(\ddot{o})\}$ and $\delta^+_\kappa(\ddot{e}+\ddot{o})=1=[1,1]=\min\{\delta^+_\kappa(\ddot{e}),\delta^+_\kappa(\ddot{o})\}$.

If $\ddot{e},\ddot{o}\notin\kappa$, then $\delta^-_\kappa(\ddot{e})=0=\delta^-_\kappa(\ddot{o})$ and $\delta^+_\kappa(\ddot{e})=0=\delta^+_\kappa(\ddot{o})$. Thus, $\delta^-_\kappa(\ddot{e}+\ddot{o})=0\leq\max\{\delta^-_\kappa(\ddot{e}),\delta^-_\kappa(\ddot{o})\}$ and $\delta^+_\kappa(\ddot{e}+\ddot{o})=0\geq\min\{\delta^+_\kappa(\ddot{e}),\delta^+_\kappa(\ddot{o})\}$.

If $\ddot{e}\notin\kappa,\ddot{o}\in\kappa$, then $\delta^-_\kappa(\ddot{e})=0=\delta^+_\kappa(\ddot{e}),\delta^-_\kappa(\ddot{o})=-1$, and $\delta^+_\kappa(\ddot{o})=1$. Thus, $\delta^-_\kappa(\ddot{e}+\ddot{o})\leq\max\{\delta^-_\kappa(\ddot{e}),\delta^-_\kappa(\ddot{o})\}$ and $\delta^+_\kappa(\ddot{e}+\ddot{o})\geq\min\{\delta^+_\kappa(\ddot{e}),\delta^+_\kappa(\ddot{o})\}$.

If $\ddot{e}\in\kappa$, then $\delta^-_\kappa(\ddot{e})=-1$ and $\delta^+_\kappa(\ddot{e})=1$. Also, $[(\delta^-_\kappa\Gamma\delta^-)\cup(\delta^-\Gamma\delta^-_\kappa)](\ddot{e})\geq(\delta^-_\kappa\Gamma\delta^-)(\ddot{e})=\delta^-_\kappa(\ddot{e})=-1$ and $[(\delta^+_\kappa\Gamma\delta^+)\cap(\delta^+\Gamma\delta^+_\kappa)](\ddot{e})\leq(\delta^+_\kappa\Gamma\delta^+)(\ddot{e})=\delta^+_\kappa(\ddot{e})=1$.

If $\ddot{e}\notin\kappa$, then $\delta^-_\kappa(\ddot{e})=0,\delta^+_\kappa(\ddot{e})=0$, and $\ddot{e}\notin\kappa$. Thus, $\ddot{e}\notin(\vee\Gamma\kappa)\cap(\kappa\Gamma\vee)$. Then the following three cases arise:

Case 1: Suppose $\ddot{e}\notin\vee\Gamma\kappa$ and $\ddot{e}\notin\kappa\Gamma\vee$. Then $\ddot{e}=\ddot{p}\varrho\ddot{o}$ and $\ddot{f}\tau\ddot{m}$ for some $\ddot{p},\ddot{m}\in\vee,\ddot{o},\ddot{f}\notin\kappa$, and $\varrho,\tau\in\Gamma$. Now,

$$\begin{aligned}[(\delta^-_\kappa\Gamma\delta^-)\cup(\delta^-\Gamma\delta^-_\kappa)](\ddot{e}) &= \max\{(\delta^-_\kappa\Gamma\delta^-)(\ddot{e}),(\delta^-\Gamma\delta^-_\kappa)(\ddot{e})\}\\ &= \max\{\inf\{\max\{\delta^-_\kappa(\ddot{f}),\delta^-(\ddot{m})\}\},\inf\{\max\{\delta^-(\ddot{p}),\delta^-_\kappa(\ddot{o})\}\}\}\\ &= \max\{0,0\}\\ &= 0\\ &= \delta^-_\kappa(\ddot{e}),\end{aligned}$$

$$\begin{aligned}[(\delta^+_\kappa\Gamma\delta^+)\cap(\delta^+\Gamma\delta^+_\kappa)](\ddot{e}) &= \min\{(\delta^+_\kappa\Gamma\delta^+)(\ddot{e}),(\delta^+\Gamma\delta^+_\kappa)(\ddot{e})\}\\ &= \min\{\sup\{\min\{\delta^+_\kappa(\ddot{f}),\delta^+(\ddot{m})\}\},\sup\{\min\{\delta^+(\ddot{p}),\delta^+_\kappa(\ddot{o})\}\}\}\end{aligned}$$

$$= \min\{0,0\}$$
$$= 0$$
$$= \delta^+_\kappa(\ddot{e}).$$

Case 2: Suppose $\ddot{e} \in \vee\Gamma\kappa$ and $\ddot{e} \notin \kappa\Gamma\vee$. Then $\ddot{e} = \ddot{p}\varrho\ddot{o}$ and $\ddot{f}\tau\ddot{m}$ for some $\ddot{p}, \ddot{m} \in \vee, \ddot{o} \in \kappa, \ddot{f} \notin \kappa$, and $\varrho, \tau \in \Gamma$. Now,

$$\begin{aligned}[(\delta^-_\kappa\Gamma\delta^-) \cup (\delta^-\Gamma\delta^-_\kappa)](\ddot{e}) &= \max\{(\delta^-_\kappa\Gamma\delta^-)(\ddot{e}), (\delta^-\Gamma\delta^-_\kappa)(\ddot{e})\} \\ &= \max\{\inf\{\max\{\delta^-_\kappa(\ddot{f}), \delta^-(\ddot{m})\}\}, \inf\{\max\{\delta^-(\ddot{p}), \delta^-_\kappa(\ddot{o})\}\}\} \\ &= \max\{0,0\} \\ &= 0 \\ &= \delta^-_\kappa(\ddot{e}),\end{aligned}$$

$$\begin{aligned}[(\delta^+_\kappa\Gamma\delta^+) \cap (\delta^+\Gamma\delta^+_\kappa)](\ddot{e}) &= \min\{(\delta^+_\kappa\Gamma\delta^+)(\ddot{e}), (\delta^+\Gamma\delta^+_\kappa)(\ddot{e})\} \\ &= \min\{\sup\{\min\{\delta^+_\kappa(\ddot{f}), \delta^+(\ddot{m})\}\}, \sup\{\min\{\delta^+(\ddot{p}), \delta^+_\kappa(\ddot{o})\}\}\} \\ &= \min\{0,0\} \\ &= 0 \\ &= \delta^+_\kappa(\ddot{e}).\end{aligned}$$

Case 3: Suppose $\ddot{e} \notin \vee\Gamma\kappa$ and $\ddot{e} \in \kappa\Gamma\vee$. Then $\ddot{e} = \ddot{p}\varrho\ddot{o}$ and $\ddot{f}\tau\ddot{m}$ for some $\ddot{p}, \ddot{m} \in \vee, \ddot{o} \notin \kappa, \ddot{f} \in \kappa$, and $\varrho, \tau \in \Gamma$. Now,

$$\begin{aligned}[(\delta^-_\kappa\Gamma\delta^-) \cup (\delta^-\Gamma\delta^-_\kappa)](\ddot{e}) &= \max\{(\delta^-_\kappa\Gamma\delta^-)(\ddot{e}), (\delta^-\Gamma\delta^-_\kappa)(\ddot{e})\} \\ &= \max\{\inf\{\max\{\delta^-_\kappa(\ddot{f}), \delta^-(\ddot{m})\}\}, \inf\{\max\{\delta^-(\ddot{p}), \delta^-_\kappa(\ddot{o})\}\}\} \\ &= \max\{0,0\} \\ &= 0 \\ &= \delta^-_\kappa(\ddot{e}),\end{aligned}$$

$$\begin{aligned}[(\delta^+_\kappa\Gamma\delta^+) \cap (\delta^+\Gamma\delta^+_\kappa)](\ddot{e}) &= \min\{(\delta^+_\kappa\Gamma\delta^+)(\ddot{e}), (\delta^+\Gamma\delta^+_\kappa)(\ddot{e})\} \\ &= \min\{\sup\{\min\{\delta^+_\kappa(\ddot{f}), \delta^+(\ddot{m})\}\}, \sup\{\min\{\delta^+(\ddot{p}), \delta^+_\kappa(\ddot{o})\}\}\} \\ &= \min\{0,0\} \\ &= 0 \\ &= \delta^+_\kappa(\ddot{e}).\end{aligned}$$

Hence, in all the above three cases, $(\delta^+_\kappa\Gamma\delta^+) \cap (\delta^+\Gamma\delta^+_\kappa) \subseteq \delta^+_\kappa$ and $(\delta^-_\kappa\Gamma\delta^-) \cup (\delta^-\Gamma\delta^-_\kappa) \supseteq \delta^-_\kappa$. Therefore, δ_κ is a BFQI of $\vee$. □

Theorem 3.6. *Every QI of $\vee$ is idempotent if and only if every BFQI of $\vee$ is idempotent.*

Proof. Suppose every QI of $\vee$ is idempotent. Let ξ be a BFQI of $\vee$. Then $\xi^+\Gamma\xi^+ \subseteq (\xi^+\Gamma\delta^+) \cap (\delta^+\Gamma\xi^+) \subseteq \xi^+$ and $\xi^-\Gamma\xi^- \supseteq (\xi^-\Gamma\delta^-) \cup (\delta^-\Gamma\xi^-) \supseteq \xi^-$. By theorem 4.8 of [2], $\vee$ is regular and intra-regular. Let $\ddot{p} \in \vee$. Then $\ddot{p} = \ddot{p}\varrho\ddot{m}\tau\ddot{p}$ and $\ddot{p} = \ddot{e}\varphi\ddot{p}\phi\ddot{p}\eta\ddot{o}$ for some $\ddot{m}, \ddot{e}, \ddot{o} \in \vee$ and $\varrho, \tau, \varphi, \phi, \eta \in \Gamma$. Now, $\ddot{p} = \ddot{p}\varrho\ddot{m}\tau\ddot{p} = \ddot{p}\varrho\ddot{m}\tau\ddot{p}\varrho\ddot{m}\tau\ddot{p} = \ddot{p}\varrho\ddot{m}\tau(\ddot{e}\varphi\ddot{p}\phi\ddot{p}\eta\ddot{o})\varrho\ddot{m}\tau\ddot{p} = (\ddot{p}\varrho\ddot{m}\tau\ddot{e}\varphi\ddot{p})\phi(\ddot{p}\eta\ddot{o}\varrho\ddot{m}\tau\ddot{p})$. Since any BFQI of $\vee$ is a BFBI of $\vee$, we have

$$\begin{aligned}(\xi^+\Gamma\xi^+)(\ddot{p}) &= (\xi^+\Gamma\xi^+)((\ddot{p}\varrho\ddot{m}\tau\ddot{e}\varphi\ddot{p})\phi(\ddot{p}\eta\ddot{o}\varrho\ddot{m}\tau\ddot{p}))\\ &= \sup\{\{\min\{(\xi^+(\ddot{p}\varrho\ddot{m}\tau\ddot{e}\varphi\ddot{p}), \xi^+(\ddot{p}\eta\ddot{o}\varrho\ddot{m}\tau\ddot{p})\}\}\\ &\geq \sup\{\{\min\{\min\{\xi^+(\ddot{p}), \xi^+(\ddot{p})\}, \min\{\xi^+(\ddot{p}), \xi^+(\ddot{p})\}\}\}\\ &= \xi^+(\ddot{p}).\end{aligned}$$

Thus, $\xi^+\Gamma\xi^+ \supseteq \xi^+$, so $\xi^+\Gamma\xi^+ = \xi$. Also,

$$\begin{aligned}(\xi^-\Gamma\xi^-)(\ddot{p}) &= (\xi^-\Gamma\xi^-)((\ddot{p}\varrho\ddot{m}\tau\ddot{e}\varphi\ddot{p})\phi(\ddot{p}\eta\ddot{o}\varrho\ddot{m}\tau\ddot{p}))\\ &= \inf\{\{\max\{(\xi^-(\ddot{p}\varrho\ddot{m}\tau\ddot{e}\varphi\ddot{p}), \xi^-(\ddot{p}\eta\ddot{o}\varrho\ddot{m}\tau\ddot{p})\}\}\\ &\leq \inf\{\{\max\{\max\{\xi^-(\ddot{p}), \xi^-(\ddot{p})\}, \max\{\xi^-(\ddot{p}), \xi^-(\ddot{p})\}\}\}\\ &= \xi^-(\ddot{p}).\end{aligned}$$

Thus, $\xi^-\Gamma\xi^- \subseteq \xi^+$, so $\xi^-\Gamma\xi^- = \xi$. Hence, every BFQI of $\vee$ is idempotent.

Conversely, suppose that every BFQI of $\vee$ is idempotent. Let κ be a QI of $\vee$. By Theorem 3.5, we have δ_κ is a BFQI of $\vee$. Thus, $\delta^+_\kappa = \delta^+_\kappa\Gamma\delta^+$ and $\delta^-_\kappa = \delta^-_\kappa\Gamma\delta^-$. Therefore, $\kappa\Gamma\kappa = \kappa$. Hence, every QI of $\vee$ is idempotent. □

4. Conclusion

This research provides a comprehensive analysis of BFQIs in Γ-semirings, highlighting their generalization of BFIs. We establish that BFBIs extend BFQIs, broadening their applicability within the Γ-semiring structure. In regular Γ-semirings, we show that the two concepts coincide, offering a unified perspective. Additionally, the study identifies that the intersection of a BFRI and a BFLI always forms a BFQI. These findings contribute significantly to the understanding of fuzzy ideal theory in Γ-semirings.

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References

[1] Y. Bhargavi, T. Eswarlal, Fuzzy Γ-Semirings, Int. J. Pure Appl. Math. 98 (2015), 339-349. https://doi.org/10.12732/ijpam.v98i3.6.

[2] Y. Bhargavi, T. Eswarlal, Vague Bi-Ideals and Vague Quasi-Ideals of a Γ-Semiring, Int. J. Sci. Res. 4 (2015), 2694-2699.

[3] Y. Bhargavi, A. Iampan, Vague Interior Ideals of Γ-Semirings, ICIC Express Lett. 18 (2024), 1023-1030. https://doi.org/10.24507/icicel.18.10.1023.

[4] Y. Bhargavi, A. Iampan, B. Nageswararao, Vague Bi-Quasi-Interior Ideals of Γ-Semirings, Int. J. Anal. Appl. 21 (2023), 98. https://doi.org/10.28924/2291-8639-21-2023-98.

[5] Y. Bhargavi, S. Ragamayi, T. Eswarlal, G. Jayalalitha, Vague Bi-Interior Ideals of a Γ-Semiring, Adv. Math.: Sci. J. 9 (2020), 4425-4435. https://doi.org/10.37418/amsj.9.7.13.

[6] P. Madhulatha, Y. Bhargavi, Bipolar Fuzzy Γ-Semirings, AIP Conf. Proc. 2707 (2023), 020006. https://doi.org/10.1063/5.0143364.

[7] P. Madhulatha, Y. Bhargavi, A. Iampan, Bipolar Fuzzy Ideals of Γ-Semirings, Asia Pac. J. Math. 10 (2023), 38. https://doi.org/10.28924/APJM/10-38.

[8] D. Mandal, Fuzzy Ideals and Fuzzy Interior Ideals in Ordered Semirings, Fuzzy Inf. Eng. 6 (2014), 101–114. https://doi.org/10.1016/j.fiae.2014.06.008.

[9] M.M.K. Rao, Γ-Semiring, Southeast Asian Bull. Math. 19 (1995), 49-54.

[10] N. Nobusawa, On a Generalization of the Ring Theory, Osaka J. Math. 1 (1964), 81-89.

[11] M. Parvatham, Y. Bhargavi, A. Iampan, Bipolar Fuzzy Bi-Ideals of Γ-Semirings, Asia Pac. J. Math. 11 (2024), 69. https://doi.org/10.28924/APJM/11-69.

[12] H.S. Vandiver, Note on a Simple Type of Algebra in Which the Cancellation Law of Addition Does Not Hold, Bull. Amer. Math. Soc. 40 (1934), 916-920.

[13] K.V. Kumar, B. Jyothi, P.N. Swamy, T. Nagaiah, G. Omprakasham, Characterization on Bipolar Fuzzy Quasi Ideals and Bipolar N-Subgroups of near Rings, AIP Conf. Proc. 2246 (2020), 020053. https://doi.org/10.1063/5.0014444.

[14] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[15] W.R. Zhang, Bipolar Fuzzy Sets and Relations: A Computational Framework for Cognitive Modeling and Multiagent Decision Analysis, in: Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference, IEEE, San Antonio, TX, USA, 1994: pp. 305–309. https://doi.org/10.1109/IJCF.1994.375115.