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Estimation of Parameters of Lomax Exponential Distribution

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ABSTRACT. The concept of p-th upper record value was introduced by Dziubdziela and Kopocinski [5]. Estimation of the location and scale parameters of the Lomax exponential (LE) distribution are obtained based on generalized record values. The best linear unbiased methods of estimation are used for this purposes.

1. Introduction

The record values have been effectively applied in numerous real-world scenarios as well as in scientific domains such as weather, education, economics, sports data, meteorology, epidemiology, and Olympic and COVID-19 records. Chandler [3] defined record values as a model of the next extremes in a series of independent, identical random variables. The literature has reviewed the theory of record values and its distributional features in great detail. Asgharzed and Abdi [2], Minimal and Thomas [11,12], Khan *et al.* [7], Khan and Khan [8], Khan *et al.* [9], Kumar and Dey [10], Singh *et.al* [13] are just a few of the authors who have recently used the idea of "record values" in their writing.

Let $W_1, W_{2,} \dots$ to an independent and identically sequence of random variables (rv) with a continuous distribution function $F(w)$ and probability distribution function (pdf) $f(w)$ over the support $(-\infty, \infty)$. Denote the p-th upper record times by $U_1^{(p)} = 1$ and $a > 1$

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$$U_{a+1}^{(p)} = \min\left\{ j > U_a^{(p)} : W_{j:j+p-1} > W_{U_a^{(p)}:U_a^{(p)}+p+1} \right\},\ a = 1,2,\dots$$

The sequence $\{V_a^{(p)}, a \geq 1\}$, where $V_a^{(p)} = W_{U_a^{(p)}}$ is called the sequence of generalized upper values (p-th upper record values) of $\{W_a, a \geq 1\}$. Note that for $p = 1$, we have $V_a^{(1)} = W_{U_a}$, $a \geq 1$, which are record values of $\{W_a, a \geq 1\}$ as define in Ahsanullah [1].

The marginal densities of $V_a^{(p)}$ and joint marginal densities of $V_b^{(p)}$ and $V_a^{(p)}$ are given by (Dziubdziela and Kopocionski [5], Grudziaen [6]):

$$f_{V_a^{(p)}}(w) = \frac{p^a}{(a-1)!}[R(w)]^{a-1}[1-F(w)]^{p-1} f(w)$$

(1.1)

and

$$f_{V_b^{(p)}, V_a^{(p)}}(w,v) = \frac{p^a}{(b-1)!(a-b-1)!}[R(V)-R(w)]^{a-b-1}$$
$$\times [R(w)]^{b-1} r(w)[1-F(v)]^{p-1} f(v),\quad w < v \tag{1.2}$$

where $R(w) = -l\,n\,[1-F(w)]$, $l\,n$ is the natural logarithm and $r(w) = \frac{d}{dw}R(w)$

The joint density of the p-th upper record values, $V_1^{(p)}, V_2^{(p)}, \dots, V_a^{(p)}$ is given below as

$$f_{V_1^{(p)}, V_2^{(p)}, \dots V_a^{(p)}}(v_1, v_2, \cdots, v_a) = p^n \prod_{i=1}^{a-1} \frac{f(v_i)}{1-F(v_i)}(1-F(v_a))^{p-1} f(v_a),$$
$$v_1 < v_2 < \dots\dots < v_a. \tag{1.3}$$

The *pdf* and *cdf* of the Lomax-G family for any continuous probability distribution are as follows (Cordeiro *et al.* [4])

$$f(w) = \lambda\beta^\lambda i(w) \frac{i(w)}{[1-I(w)][\beta - \log\{1-I(w)\}]^{-\lambda}},\quad 0 < w < \infty, \lambda, \beta > 0 \tag{1.4}$$

and it cdf is given by

$$\bar{F}(w) = 1 - \left(\frac{\beta}{\beta - \log[1-I(w)]} \right)^\lambda,\quad 0 < w < \infty,\ \lambda, \beta > 0 \tag{1.5}$$

where $\beta > 0$ is a scale parameter, $\lambda > 0$ is a shape parameter respectively.

In this paper, we have considered LE distribution with *pdf*

$$f(w)=\frac{\lambda}{\beta}\left(1+\frac{w-\alpha}{\beta}\right)^{-(\lambda+1)}, \quad 0<w<\infty,\ \beta,\lambda>0 \tag{1.6}$$

and corresponding *df*

$$F(w)=1-\left(1+\frac{w-\alpha}{\beta}\right)^{-\lambda},\ 0<w<\infty,\ \beta,\lambda>0 \tag{1.7}$$

The precise expressions for the mean, variance, and covariance of record statistics obtained from the LE distribution are presented in this study. Additionally, these results are used to derive their BLUEs and maximum likelihood estimates of the parameters.

2. Single Moments

Theorem 2.1: Fix a positive integer, $p\geq 1$ and $\alpha\geq 1$.

$$E(V_a^{(p)}) = \alpha+\beta(W_1^a-1) \tag{2.1}$$

Where $W_1=\left(\frac{\lambda p}{\lambda p-1}\right)$

Proof: It can easily be shown, on using (1.6), (1.7) and (1.1), that

$$E(V_a^{(p)}) = \frac{(\lambda p)^a}{\beta(a-1)!}\int_\alpha^\infty w\left[ln\left(1+\frac{w-\alpha}{\beta}\right)\right]^{a-1}\left(1+\frac{w-\alpha}{\beta}\right)^{-(\lambda p+1)}dw$$

Set,

$$ln\left(1+\frac{w-\alpha}{\beta}\right)=t$$

$$E(V_a^{(p)}) = \frac{(\lambda p)^a}{(a-1)!}\int_0^\infty[(\alpha-\beta)+\beta e^t]\,t^{a-1}e^{-(\lambda p)t}dt$$

after simplification, we get the result.

3. Product Moments:

Theorem 3.1: For $1\leq b\leq a-2$

$$E(V_b^{(p)}V_a^{(p)}) = (\alpha-\beta)^2+\beta(\alpha-\beta)W_1^a+\beta(\alpha-\beta)W_1^b+\beta^2W_1^{a-b}\ W_2^b \tag{3.1}$$

Where $W_2=\left(\frac{\lambda p}{\lambda p-2}\right)$

Proof: It can easily be shown, on using (1.6), (1.7) and (1.2), that

$$E(V_b^{(p)}V_a^{(p)}) = \frac{(\lambda\, p)^a}{\beta^2 (b-1)!(a-b-1)!}\int_\alpha^\infty w\left[l\,n\left(1+\frac{w-\alpha}{\beta}\right)\right]^{b-1}\left(1+\frac{w-\alpha}{\beta}\right)^{-1} l(w)\,dw \qquad (3.2)$$

Where

$$l(w) = \int_w^\infty y\left[l\,n\left(1+\frac{y-\alpha}{\beta}\right)-l\,n\left(1+\frac{w-\alpha}{\beta}\right)\right]^{a-b-1}\left(1+\frac{y-\alpha}{\beta}\right)^{-\lambda\, p-1} dy$$

Set,

$$l\,n\left(1+\frac{y-\alpha}{\beta}\right)-l\,n\left(1+\frac{w-\alpha}{\beta}\right)=t$$

after simplification, we get

$$I(w)=\left(1+\frac{w-\alpha}{\beta}\right)\left[\frac{(\alpha-\beta)}{(\lambda\, p)^{a-b}}+\beta\frac{\left(1+\frac{w-\alpha}{\beta}\right)}{(\lambda\, p-1)^{a-b}}\right]$$

We obtain the outcome by substituting the value of $l(w)$ in (3.2) and simplifying.

4. Estimation of the Parameters α and β When Shape Parameter λ is Known

In view of (2.1), we have

$$E(V_a^{(p)}) = \alpha+\beta\,(W_1^a-1)$$

In the same way, you can get

$$E(V_a^{(p)})^2 = (\alpha-\beta)^2+2\beta(\alpha-\beta)W_1^a+\beta^2 W_2^a$$

(4.1)

Hence,

$$Variance\,(V_a^{(p)}) = \beta^2(W_2^a - W_1^{2a}) \qquad (4.2)$$

In view of (3.1), we have

$$E(V_b^{(p)}V_a^{(p)}) = (\alpha-\beta)^2+\beta(\alpha-\beta)W_1^a+\beta(\alpha-\beta)W_1^b+\beta^2 W_1^{n-m}\ W_2^m$$

In view of (3.1) and (2.1), we get

$$Cov\,(V_b^{(p)}V_a^{(p)}) = E(V_b^{(p)}V_a^{(p)})-E(V_b^{(p)})E(V_a^{(p)}),$$

we get

$$Cov(V_b^{(p)}V_a^{(p)}) = W_1\,Cov(V_b^{(p)}V_{a-1}^{(p)})\ ,\ a>b \qquad (4.3)$$

By employing it recursively, it can be achieved to confirm that

$$Cov(V_b^{(p)} V_a^{(p)}) = W_1^{a-b} \; Var(V_b^{(p)}), \quad a > b \tag{4.4}$$

Let us consider the following transformation

$$Z_1^{(p)} = V_1^{(p)}$$

$$Z_i^{(p)} = \left(\frac{1}{W_2}\right)^{\frac{i-1}{2}} \left[V_i^{(p)} - W_1 \; V_{i-1}^{(p)} + \alpha \frac{W_1}{\lambda \, p} \right] \quad , \qquad i = 2,3,\ldots,a$$

Then, applying (2.1), we get

$$E(Z_1^{(p)}) = \alpha + \beta \frac{W_1}{\lambda \, p} \tag{4.5}$$

$$E(Z_i^{(p)}) = \left(\frac{1}{W_2}\right)^{\frac{i-1}{2}} \beta \frac{W_1}{\lambda p} \quad , \qquad i = 2,3,\ldots,a \tag{4.6}$$

Similarly, on using (4.2), we obtain

$$Var(Z_i^{(p)}) = \beta^2 \, W_2 \left(\frac{W_1}{\lambda \, p}\right)^2 \quad , \quad i = 2,3,\ldots,a \tag{4.7}$$

$$Cov(Z_i^{(p)}, Z_j^{(p)}) = 0, \quad i \neq j \; , 1 \leq i < j \leq a \tag{4.8}$$

Let $Z' = (Z_1^{(p)}, Z_2^{(p)}, \ldots\ldots, Z_a^{(p)})$. Then

$$E(Z) = B\theta \tag{4.9}$$

where

$$B = \begin{pmatrix} 1 & \frac{W_1}{\lambda \, p} \\ 0 & \left(\frac{1}{W_2}\right)^{\frac{1}{2}} \frac{W_1}{\lambda \, p} \\ 0 & \left(\frac{1}{W_2}\right) \frac{W_1}{\lambda \, p} \\ \vdots & \vdots \\ 0 & \left(\frac{1}{W_2}\right)^{\frac{a-1}{2}} \frac{W_1}{\lambda \, p} \end{pmatrix} , \quad \theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The best linear unbiased estimates $\hat{\alpha}, \hat{\beta}$ of α and β, respectively, based on $V_1^{(p)}, V_2^{(p)}, \ldots, V_a^{(p)}$ are given by

$$\hat{\theta} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = (B'B)^{-1} \; B'Z \tag{4.10}$$

We have

$$(B'B) = \begin{pmatrix} 1 & \dfrac{W_1}{\lambda\, p} \\ \dfrac{W_1}{\lambda\, p} & S \end{pmatrix},$$

where

$$S = \left(\frac{W_1}{\lambda P}\right)^2 \left[1 + \left(\frac{\lambda\, p}{2W_2}\right)\left\{1 - \left(\frac{1}{W_2}\right)^{a-1}\right\}\right].$$

Now

$$(B'B)^{-1} = \frac{2\,(\lambda\, p)}{(W_1)^2} \frac{W_2}{\left[1 - \left(\dfrac{1}{W_2}\right)^{a-1}\right]} \begin{pmatrix} S & -\dfrac{W_1}{\lambda\, p} \\ -\dfrac{W_1}{\lambda\, p} & 1 \end{pmatrix}$$

Substituting for $(B'B)^{-1}$ in (4.10) and simplifying the resulting expression, we obtain

$$\hat{\theta} = \frac{2\,(\lambda\, p)}{(W_1)^2} \frac{W_2}{\left[1 - \left(\dfrac{1}{W_2}\right)^{a-1}\right]}$$

$$\times \begin{pmatrix} S - \left(\dfrac{W_1}{\lambda P}\right)^2 & -\left(\dfrac{W_1}{\lambda P}\right)^2 \left\{\left(\dfrac{1}{W_2}\right)^{\frac{1}{2}} \left(\dfrac{1}{W_2}\right) \cdots \left(\dfrac{1}{W_2}\right)^{\frac{a-1}{2}}\right\} \\ 0 & \left(\dfrac{W_1}{\lambda P}\right)\left\{\left(\dfrac{1}{W_2}\right)^{\frac{1}{2}} \left(\dfrac{1}{W_2}\right) \cdots \left(\dfrac{1}{W_2}\right)^{\frac{a-1}{2}}\right\} \end{pmatrix} \begin{bmatrix} Z_1^{(p)} \\ Z_2^{(p)} \\ \vdots \\ Z_a^{(p)} \end{bmatrix}$$

which on further simplification gives

$$\hat{\alpha} = Z_1^{(p)} - \frac{W_1}{\lambda\, p}\hat{\beta}$$

and

$$\hat{\beta} = \frac{2\ (\lambda\ p)}{(W_1)^2} \frac{W_2}{\left[1 - \left(\frac{1}{W_2}\right)^{a-1}\right]}$$

$$\times \left(\frac{W_1}{\lambda P}\right) \left\{ \left(\frac{1}{W_2}\right)^{\frac{1}{2}} Z_2^{(p)} + \left(\frac{1}{W_2}\right) Z_3^{(p)} + \cdots + \left(\frac{1}{W_2}\right)^{\frac{a-1}{2}} Z_a^{(p)} \right\}$$

Hence,

$$Var\ (\hat{\alpha})\ = \beta^2 W_2 \left(\frac{W_1}{\lambda P}\right)^2 \left[1 + \frac{2W_2}{\lambda\ p} \left\{1 - \left(\frac{1}{W_2}\right)^{a-1}\right\}^{-1}\right] \tag{4.11}$$

$$Var\ (\hat{\beta})\ = \frac{2\ \beta^2 (W_2)^2}{(\lambda\ p)} \left[1 - \left(\frac{1}{W_2}\right)^{a-1}\right]^{-1} \tag{4.12}$$

and

$$Cov\ (\hat{\alpha}, \hat{\beta})\ = -\ 2\ \beta^2 W_1 \left(\frac{W_2}{\lambda P}\right)^2 \left[1 - \left(\frac{1}{W_2}\right)^{a-1}\right]^{-1} \tag{4.13}$$

The generalized variance $\hat{\sum}$ of $\hat{\alpha}$ and $\hat{\beta}$ $(\hat{\sum} = Var(\hat{\alpha})\ Var(\hat{\beta}) - (Cov(\hat{\alpha}, \hat{\beta})^2)$ is

$$2\ \beta^4 (W_1)^2 \left(\frac{W_2}{\lambda P}\right)^3 \left[1 - \left(\frac{1}{W_2}\right)^{a-1}\right]^{-1}$$

The coefficients for BLUE of the location and scale parameter of LE distribution based on record values have been obtained but not presented here.

On considering the two $p-th$ upper record values $V_s^{(p)}$ and $V_r^{(p)}$ $(s > r)$ it follows from (2.1) and (4.6) that the best linear unbiased estimates of α and β based on these two $p-th$ record values are given by

$$\alpha^* = Z_r^{(p)} - [(W_1)^r - 1]\beta^*$$

$$\beta^* = \frac{(W_2)^{\frac{s-r}{2}}}{[(W_1)^r - 1]} Z_s^{(p)}$$

The variance and covariance of α^* and β^* are

$$Var(\alpha^*) = \beta^2 \; [(W_2)^r - (W_1)^{2r}]\,[1 + (W_2)^{s-r}] \tag{4.14}$$

$$Var(\beta^*) = \frac{\beta^2 \; [(W_2)^r - (W_1)^{2r}](W_2)^{s-r}}{[(W_1)^r - 1]^2} \tag{4.15}$$

and

$$Cov\;(\alpha^*, \beta^*) = -\frac{\beta^2\,(W_2)^{s-r}[(W_2)^r - (W_1)^{2r}]}{[(W_1)^r - 1]} \tag{4.16}$$

It can be shown that the generalized variance $\sum^*$ of α^* and β^*

$$(\textstyle\sum^* = Var(\alpha^*)\; Var(\beta^*) - (Cov(\alpha^*, \beta^*)^2)$$

$$\textstyle\sum^* = \dfrac{\beta^4\,(W_2)^{s-r}[(W_2)^r - (W_1)^{2r}]}{[(W_1)^r - 1]^2}$$

is minimum when $s = a$ and $r = 1$. Hence the best linear unbiased estimates of α and β based on two $p-th$ record values are

$$\tilde{\alpha} = Z_1^{(p)} - \frac{W_1}{\lambda\, p}\tilde{\beta}$$

$$\tilde{\beta} = \frac{\lambda p (W_2)^{\frac{a-1}{2}}}{W_1} Z_a^{(p)}$$

Also

$$Var(\tilde{\alpha}) = \beta^2 \;\; [W_2 - (W_1)^2][1 + (W_2)^{a-1}] \tag{4.17}$$

$$Var(\tilde{\beta}) = \beta^2 \left(\frac{\lambda P}{W_1}\right)^2 [W_2 - (W_1)^2](W_2)^{a-1} \tag{4.18}$$

$$Cov\;(\tilde{\alpha}, \tilde{\beta}) = -\;\beta^2 \left(\frac{\lambda P}{W_1}\right)^2 (W_2)^{\,a-1}[W_2 - (W_1)^2] \tag{4.19}$$

$$e_1 = \frac{Var(\hat{\alpha})}{Var(\tilde{\alpha})}, \quad e_2 = \frac{Var(\hat{\beta})}{Var(\tilde{\beta})}, \quad e_{12} = \frac{Var(\hat{\alpha}, \hat{\beta})}{Var(\tilde{\alpha}, \tilde{\beta})}$$

Thus the generalized variance $\tilde{\sum}$ of $\tilde{\alpha}$ and $\tilde{\beta}$ is

$$\tilde{\textstyle\sum} = \beta^4 \left(\frac{\lambda P}{W_1}\right)^2 (W_2)^{\,a-1}[W_2 - (W_1)^2]$$

Further, it can be seen that $e_{12} = e_2$.

5. Unbiased Estimates of λ When α and β are Known

In view of (1.3), (1.6) and (1.7), the likelihood function of LED is given as

$$L = \left(\frac{p}{\beta}\right)^a \lambda^a \prod_{i=1}^{a-1}\left(1+\frac{v_i-\alpha}{\beta}\right)^{-1}\left(1+\frac{v_a-\alpha}{\beta}\right)^{-(\lambda p+1)},$$

Hence

$$\log L = a\log\lambda + \log C - (\lambda p+1)\log\left(1+\frac{v_a-\alpha}{\beta}\right)$$

where

$$C = \left(\frac{p}{\beta}\right)^a \prod_{i=1}^{a-1}\left(1+\frac{v_i-\alpha}{\beta}\right)^{-1}$$

The maximum likelihood estimates of λ is

$$\lambda^{\approx} = \frac{a}{p\log\left(1+\frac{v_a-\alpha}{\beta}\right)}$$

Further,

$$E(\lambda^{\approx}) = \frac{p^{a-1}\lambda^a}{(a-1)!}\int_\alpha^\infty \frac{a}{\log\left(1+\frac{v-\alpha}{\beta}\right)}\left[\lambda\log\left(1+\frac{v-\alpha}{\beta}\right)\right]^{a-1}\left(1+\frac{v-\alpha}{\beta}\right)^{-\lambda p+\lambda}$$

$$\times\frac{\lambda}{\beta}\left(1+\frac{v-\alpha}{\beta}\right)^{-\lambda-1} dv$$

$$= \frac{(\lambda a)\ (\lambda p)^{a-1}}{(a-1)!}\int_0^\infty t^{a-2}e^{-(\lambda p)t}dt$$

$$E(\lambda^{\approx}) = \frac{a}{a-1}\lambda$$

Note that $\lambda^{\approx}$ is a biased estimator for λ. The unbiased estimator λ^{**} for λ is given by

$$\lambda^{**} = \frac{a-1}{a}\lambda^{\approx} = \frac{a-1}{p\log\left(1+\frac{v_a-\alpha}{\beta}\right)}$$

and it can easily be verified that

$$Var(\lambda^{**}) = \frac{\lambda^2}{a-2}, \qquad a > 2$$

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