International Journal of Analysis and Applications
ISSN 2291-8639
Volume 3, Number 1 (2013), 35-46
http://www.etamaths.com

TWO-POINT FUZZY OSTROWSKI TYPE INEQUALITIES

MUHAMMAD AMER LATIF[1] AND SABIR HUSSAIN[2,*]

Abstract. Two-point fuzzy Ostrowski type inequalities are proved for fuzzy Hölder and fuzzy differentiable functions. The two-point fuzzy Ostrowski type inequality for M-lipshitzian mappings is also obtained. It is proved that only the two-point fuzzy Ostrowski type inequality for M-lipshitzian mappings is sharp and as a consequence generalize the two-point fuzzy Ostrowski type inequalities obtained for fuzzy differentiable functions.

1. Introduction

In 1938, A. M. Ostrowski proved an interesting integral inequality, estimating the absolute value of deviation of a differentiable function by its integral mean as:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b). If f' is bounded on (a,b), that is*

$$\|f'\| := \sup_{t\in(a,b)} |f(t)| < \infty,$$

then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

It is easy to observe that (1.1) can be rewritten in equivalent from as follow:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty . \tag{1.2}$$

Since that time when A. Ostrowski proved this famous inequality, many mathematician have been working on it and have been applying it in numerical analysis and probability, etc.

N. S. Barnett and S. S. Dragomir [5], proved, as a generalization of (1.1), the following result:

If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and if $[c,d]\subset[a,b]$, then

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(s)ds\right|\\ &\le\left\{\frac{b-a}{4}+\frac{d-c}{2}+\frac{1}{b-a}\left[\left|\frac{c+d}{2}-\frac{a+b}{2}\right|-\frac{d-c}{2}\right]^2\right\}\|f'\|_\infty .\end{aligned}\tag{1.3}$$

2000 *Mathematics Subject Classification.* 26D07, 26D15, 26E50.

Key words and phrases. Ostrowski inequality, Fuzzy inequalities, Fuzzy real analysis.

It is to be noted that for $c = d = x$, one can assume $\frac{1}{d-c}\int_c^d f(s)ds = f(x)$, as a limit case, and hence (1.3) takes the from of (1.1).

In [9], M. Matić and J. Pečarić gave a two-point Ostrowski type inequality, as a generalization of (1.3), by replacing the condition of differentiability of f and boundedness of f' on (a, b) by a weaker condition that f is M-Lipschitzian on $[a, b]$, that is

$$|f(t) - f(s)| \leq M\,|t - s|\,,\ \forall t, s \in [a, b]\,,\ M > 0.$$

It was also proved that the two-point Ostrowski inequality established in [9] is sharp and gives tighter estimate than those of (1.3). The main result from [9] is the following one:

Theorem 2. *Let $a, b, c, d \in \mathbb{R}$ be such that*

$$a \leq c < d \leq b,\ c - a + b - d > 0.$$

(i) *If $f : [a, b] \to \mathbb{R}$ is M-Lipschitzian on $[a, b]$, with some constant $M > 0$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(s)ds\right| \leq \frac{(c-a)^2 + (b-d)^2}{2\,(c-a+b-d)}M. \tag{1.4}$$

(ii) *If $f_0 : [a, b] \to \mathbb{R}$ defined as*

$$f_0\,(t) = |t - s_0|\,,$$

where

$$s_0 = \frac{bc - ad}{c - a + b - d},$$

then f_0 is 1-Lipshitzian on $[a, b]$ and we havd

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(s)ds\right| = \frac{(c-a)^2 + (b-d)^2}{2\,(c-a+b-d)}.$$

Since fuzziness is a natural reality different than randomness and determinism, therefore an attempt has been made by George A. Anastassiou [2] to extend (1.1) to fuzzy setting context in 2003. In fact, George A. Anastassiou [2] proved the following important results for fuzzy Hölder and fuzzy differentiable functions respectively:

Theorem 3. *Let $f \in C\,([a, b], \mathbb{R}_{\mathcal{F}})$, the space of fuzzy continous functions, $x \in [a, b]$ be fixed. If f fulfills the Hölder condition*

$$D\,(f(y), f(z)) \leq L_f\,|y - z|^{\alpha}\,,\ 0 < \alpha \leq 1,\ \textit{for all}\ y,\ z \in [a, b]\,,$$

for some $L_f > 0$. Then

$$\begin{aligned} &D\left(\frac{1}{b-a} \odot (FR)\int_a^b f(y)dy, f(x)\right) \\ &\leq L_f\left(\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{(\alpha+1)\,(b-a)}\right). \end{aligned} \tag{1.5}$$

Theorem 4. *Let $f \in C^1([a,b],\mathbb{R}_{\mathcal{F}})$, the space of one time continuously differentiable functions in the fuzzy sense. Then for $x \in [a,b]$,*

$$D\left(\frac{1}{b-a}\odot (FR)\int_a^b f(y)dy, f(x)\right) \le \left(\sup_{t\in[a,b]} D\left(f'(t),\tilde{o}\right)\right)\left(\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right). \tag{1.6}$$

The inequalities in (1.5) and (1.6) are sharp as equalities are attained by the choice of simple fuzzy real number valued functions. For further details on these inequalities we refer the interested readers to [2].

The main purpose of the present paper is to establish two-point fuzzy Ostrowski type inequalties for fuzzy Hölder, fuzzy differentiable and fuzzy M-Lipshitzian functions in Section 2, which actually generalize the inequalities (1.5) and (1.6).

2. Preliminaries

In this section we point out some basic definitions and results which would help us in the sequel of the paper, we begin with:

Definition 1. [11] *Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of real axis $\mathbb{R}$ (i.e. $u:\mathbb{R}\longrightarrow[0,1]$), satisfying the following properties:*

(1) *$\forall u \in \mathbb{R}_{\mathcal{F}}$, u is normal i.e.with $u(x)=1$.*
(2) *$\forall u \in \mathbb{R}_{\mathcal{F}}$, u is convex fuzzy set i.e.*

$$u(tx+(1-t)y) \ge \min\{u(x),u(y)\}, \forall t\in[0,1].$$

(3) *$\forall\ u \in \mathbb{R}_{\mathcal{F}}$, u is upper semi-continuous on $\mathbb{R}$.*
(4) *$\overline{\{x\in\mathbb{R}:u(x)>0\}}$ is compact.*
The set $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy real numbers.

Remark 1. *It is clear that $\mathbb{R}\subset\mathbb{R}_{\mathcal{F}}$, because any real number $x_0\in\mathbb{R}$, can be described as the fuzzy number whose value is 1 for $x=x_0$ and zero otherwise.*

We will collect some further definitions and notations needed in the sequel. For $0<r\le 1$ and $u\in\mathbb{R}_F$, we define

$$[u]^r = \{x\in\mathbb{R}:u(x)\ge r\}$$
$$[u]^0 = \overline{\{x\in\mathbb{R}:u(x)>0\}}$$

Now it is well known that for each $r\in[0,1]$, $[u]^r$, is bounded closed interval. For $u,v\in\mathbb{R}_{\mathcal{F}}$ and $\lambda\in\mathbb{R}$, we have the sum $u\oplus v$ and the product $\lambda\odot u$ are defined by $[u\oplus v]^r=[u]^r+[v]^r$, $[\lambda\odot u]^r=\lambda[u]^r$, $\forall r\in[0,1]$, where $[u]^r+[v]^r$ means the usual addition of two intervals as subsets of $\mathbb{R}$ and $\lambda[u]^r$means the usual product between a scalar and a subset of $\mathbb{R}$.

Now we define $D:\mathbb{R}_{\mathcal{F}}\times\mathbb{R}_{\mathcal{F}}\longrightarrow\mathbb{R}\cup\{0\}$ by

$$D(u,v)=\sup_{r\in[0,1]}\left(\max\left\{\left|u_-^r-v_-^r\right|,\left|u_+^r+u_+^r\right|\right\}\right),$$

where $[u]^r=\left[u_-^r,u_+^r\right]$, $[v]^r=\left[v_-^r,v_+^r\right]$, then $(D,\mathbb{R}_{\mathcal{F}})$ is a metric space and it possesses the following properties:

(i) $D(u\oplus w,v\oplus w)=D(u,v)$, $\forall u,v,w\in\mathbb{R}_{\mathcal{F}}$.
(ii) $D(\lambda\odot\ u,\ \lambda\odot v)=\lambda D(u,v),\forall u,v\in\mathbb{R}_{\mathcal{F}},\forall\lambda\in\mathbb{R}$.

(iii) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$, $\forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$

Moreover it is well known that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Also we have the following theorem:

Theorem 5. [11]

- i *If we denote* $\widetilde{o} = \mathcal{X}_{\{0\}}$ *then* $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ *is neutral element with respect to* $\oplus$, *i.e.* $u \oplus \widetilde{o} = \widetilde{o} \oplus u$, *for all* $u \in \mathbb{R}_{\mathcal{F}}$.
- ii *With respect to* $\widetilde{0}$ *none of* $u \in \mathbb{R}_{\mathcal{F}} \backslash \mathbb{R}$ *has opposite in* $\mathbb{R}_{\mathcal{F}}$ *with respect to* $\oplus$.
- iii *For any* $a, b \in \mathbb{R}$ *with* $a, b \geq 0$ *or* $a, b \leq 0$, *any* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $(a+b) \odot u = a \odot u \oplus b \odot u . \forall a, b \in \mathbb{R}$ *the above property does not hold.*
- iv *For any* $\lambda \in \mathbb{R}$ *and any* $u, v \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.
- v *For any* $\lambda, \mu \in \mathbb{R}$ *and any* $u \in \mathbb{R}$, *we have* $\lambda \odot (\mu \odot v) = (\lambda . \mu) \odot v$.
- vi *If we denote* $\|u\|_{\mathcal{F}} = D(u, \widetilde{o})$, $\forall u \in \mathbb{R}_{\mathcal{F}}$ *then* $\|.\|_{\mathcal{F}}$ *has the properties of a usual norm on* $\mathbb{R}_{\mathcal{F}}$, *i.e.* $\|u\|_{\mathcal{F}} = 0$ *if and only if* $u = \widetilde{o}$, $\|\lambda \odot u\|_{\mathcal{F}} = |\lambda| . \|u\|_{\mathcal{F}}$ *and* $\|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, |\|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}| \leq D(u, v)$.

Remark 2. *The propositions (ii) and (iii) in theorem show us that* $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ *is not a linear space over* $\mathbb{R}$ *and consequently* $(\mathbb{R}_{\mathcal{F}}, \|.\|_{\mathcal{F}})$ *cannot be a normed space. However, the properties of* D *and those in theorem (iv)-(vi), have as an effect that most of the metric properties of a functions defined on* $\mathbb{R}$ *with values in a Banach space, can be extended to functions* $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$, *called fuzzy number valued functions.*

Definition 2. *A function* $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ *is said to be continuous at* $x_0 \in \mathbb{R}$ *if for every* $\varepsilon > 0$ *we can find* $\delta > 0$ *such that* $D(f(x), f(x_0)) < \varepsilon$, *whenever* $|x - x_0| < \delta$. f *is said to be continuous on* $\mathbb{R}$ *if it is continuous at every* $x \in \mathbb{R}$.

Lemma 1. *For any* $a,\ b \in \mathbb{R}$, $a,\ b \geq 0$ *and* $u \in \mathbb{R}_{\mathcal{F}}$, *we have*

$$D\left(a \odot u, b \odot u\right) \leq |a - b|\, D(u, \widetilde{o}),$$

where $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ *is defined by* $\widetilde{o} := \mathcal{X}_{\{0\}}$.

Definition 3. *Let* $x,\ y \in \mathbb{R}_{\mathcal{F}}$. *If there exists a* $z \in \mathbb{R}_{\mathcal{F}}$ *such that* $x = y \oplus z$, *then we call* z *the* H*-difference of* x *and* y, *denoted by* $z = x \ominus y$.

Definition 4. *Let* $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, *with* $\beta > 0$. *A function* $f : T \longrightarrow \mathbb{R}_{\mathcal{F}}$ *is* H*-differentiable at* $x \in T$ *if there exists a* $f'(x) \in \mathbb{R}_{\mathcal{F}}$ *such that the limits (with respect to the metric* D*)*

$$\lim_{h \to 0^+} \frac{f(x+h) \ominus f(x)}{h}, \ \lim_{h \to 0^+} \frac{f(x) \ominus f(x-h)}{h}$$

exist and are equal to $f'(x)$. *We call* f' *the derivative or* h*-derivative of* f *at* x. *If* f *is* H*-differentiable at any* $x \in T$, *we call* f *differentiable or* H*-differentiable and it has* H*-derivative over* T *the function* f'.

Definition 5. *Let* $f : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}}$. *We say that* f *is Fuzzy- Riemann integrable to* $I \in \mathbb{R}_{\mathcal{F}}$, *if for every* $\varepsilon > 0$, *there exsit* $\delta > 0$ *such that for any division* $P = \{[u, v]\,; \xi\}$ *of* $[a, b]$ *with the norms* $\Delta(P) < \delta$, *we have*

$$D\left(\sum\nolimits^{*} (v - u) \odot f(\xi), I\right) < \varepsilon,$$

where $\sum^*$ *denotes the fuzzy summation. We choose to write*

$$I := (FR) \int_a^b f(x)dx.$$

We also call an f *as above* (FR)*-integrable.*

Corollary 1. *If* $f \in C([a,b], \mathbb{R}_{\mathcal{F}})$ *then* f *is* (FR)*-integrable.*

Lemma 2. *If* $f, g : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ *are fuzzy continuous (with respect to the metric* D*), then the function* $F : [a,b] \longrightarrow \mathbb{R}_+ \cup \{0\}$ *defined by* $F(x) := D(f(x), g(x))$ *is continuous on* $[a,b]$*, and*

$$D\left((FR)\int_a^b f(u)du, (FR)\int_a^b g(u)du\right) \leq \int_a^b D(f(x), g(x))\, dx.$$

Lemma 3. *Let* $f : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ *be fuzzy continuous. Then*

$$(FR)\int_a^x f(t)dt$$

is fuzzy continuous function in $x \in [a,b]$.

Proposition 1. *Let* $F(t) := t^n \odot u$, $t \geq 0$, $n \in \mathbb{N}$ *and* $u \in \mathbb{R}_{\mathcal{F}}$ *be fixed. The (the* H*-derivative)*

$$F'(t) = nt^{n-1} \odot u.$$

In particular when $n = 1$ *then* $F'(t) = u$.

Proposition 2. *Let* I *be an open interval of* $\mathbb{R}$ *and let* $f : I \longrightarrow \mathbb{R}_{\mathcal{F}}$ *be* H*-fuzzy differentiable,* $c \in \mathbb{R}$*. Then* $(c \odot f)'$ *exist and* $(c \odot f(x))' = c \odot f'(x)$.

Theorem 6. *Let* $f : [a,b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ *be fuzzy differentiable function on* $[a,b]$ *with* H*-derivative* f' *which is assumed to be fuzzy continuous. Then*

$$D(f(d), f(c)) \leq (d-c) \sup_{t\in[c,d]} D(f'(t), \tilde{o}),$$

for any c, $d \in [a,b]$ *with* $d \geq c$.

Theorem 7. *Let* I *be closed interval in* $\mathbb{R}$*. Let* $g : I \to \zeta := g(I) \subseteq \mathbb{R}$ *be differentiable, and* $f : g(I) \to \mathbb{R}_{\mathcal{F}}$ *be* H*-differentiable. Assume that* g *is strictly increasing. Then* $(f \circ g)'(x)$ *exists and*

$$(f \circ g)'(x) = f(g(x)) \odot g'(x),\ \forall x \in I.$$

3. Main Results

In this section we prove a two-point Ostrowski type inequalities for fuzzy Hölder and fuzzy differentiable functions in a similar fashion as in [2].

We first prove a two point Ostrowski inequality like (1.3) but for fuzzy differeitaible functions in:

Theorem 8. *Let $f \in C^1([a,b],\mathbb{R}_{\mathcal{F}})$, the space of one time continuously differentiable functions in the fuzzy sense. if $x \in [c,d] \subset [a,b]$, then*

$$D\left(\frac{1}{b-a}\odot(FR)\int_a^b f(t)\,dt, \frac{1}{d-c}\odot(FR)\int_a^b f(s)\,ds\right)$$

(3.1)

$$\leq \left(\sup_{t\in[a,b]} D(f'(t),\tilde{o})\right)\left\{\frac{b-a}{4}+\frac{d-c}{2}+\frac{1}{b-a}\left[\left|\frac{c+d}{2}-\frac{a+b}{2}\right|-\frac{d-c}{2}\right]^2\right\}.$$

Proof. By using the properties of the metric D and (1.5), we have

$$D\left(\frac{1}{b-a}\odot(FR)\int_a^b f(t)\,dt, \frac{1}{d-c}\odot(FR)\int_a^b f(s)\,ds\right)$$

$$= D\left(\left(\frac{1}{b-a}\odot(FR)\int_a^b f(t)\,dt\right)\oplus f(x), \left(\frac{1}{d-c}\odot(FR)\int_a^b f(s)\,ds\right)\oplus f(x)\right)$$

$$= D\left(\left(\frac{1}{b-a}\odot(FR)\int_a^b f(t)\,dt\right)\oplus f(x), f(x)\oplus\left(\frac{1}{d-c}\odot(FR)\int_a^b f(s)\,ds\right)\right)$$

$$\leq D\left(\frac{1}{b-a}\odot(FR)\int_a^b f(t)\,dt, f(x)\right)+D\left(\frac{1}{d-c}\odot(FR)\int_a^b f(s)\,ds, f(x)\right)$$

$$\leq \left(\sup_{t\in[a,b]} D(f'(t),\tilde{o})\right)\left[\left\{\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\}(b-a)+\left\{\frac{1}{4}+\left(\frac{x-\frac{c+d}{2}}{d-c}\right)^2\right\}(d-c)\right].$$

Since the rest of the proof is similar to that of (1.3), we therefore omit the detals. □

Theorem 9. *Let $a,b,c,d \in \mathbb{R}$ be such that*

$$a \leq c < d \leq b,\ c-a+b-d > 0.$$

$f \in C([a,b],\mathbb{R}_{\mathcal{F}})$, the space of fuzzy continuous functions. Suppose f fulfills the Hölder condition, that is

$$D(f(y),f(z)) \leq L_f\,|y-z|^\alpha,\ 0<\alpha\leq 1, \text{ for all } y,\ z\in[a,b],$$

for some $L_f > 0$. Then

$$D\left(\frac{1}{b-a}\odot(FR)\int_a^b f(t)\,dt, \frac{1}{d-c}\odot(FR)\int_c^d f(s)\,ds\right)$$

$$\leq L_f\left(\frac{(c-a)^{\alpha+1}+(b-d)^{\alpha+1}}{(\alpha+1)(c-a+b-d)}\right). \tag{3.2}$$

Proof. By using the substitution

$$t=\frac{b-a}{d-c}s-\frac{bc-ad}{d-c},\ s\in[c,d],$$

by (v) of Theorem 5 and Lemma 2, we have that

$$\frac{1}{b-a}\odot(FR)\int_a^b f(t)\,dt = \frac{1}{d-c}\odot(FR)\int_c^d f\left(\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}\right)ds.$$

Thus

$$
\begin{aligned}
&D\left(\frac{1}{b-a}\odot(FR)\int_a^b f(t)\,dt, \frac{1}{d-c}\odot(FR)\int_c^d f(s)\,ds\right) \\
&= D\left(\frac{1}{d-c}\odot(FR)\int_c^d f\left(\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}\right)ds, \frac{1}{d-c}\odot(FR)\int_c^d f(s)\,ds\right) \\
&= \frac{1}{d-c}D\left((FR)\int_c^d f\left(\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}\right)ds, (FR)\int_c^d f(s)\,ds\right) \\
&\leq \frac{1}{d-c}\int_c^d D\left(f\left(\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}\right), f(s)\right)ds \\
&\leq \frac{L_f}{d-c}\int_c^d \left|\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}-s\right|^\alpha ds \\
&= \frac{L_f}{d-c}\int_c^d \left|\frac{c-a+b-d}{d-c}s-\frac{bc-ad}{d-c}\right|^\alpha ds \\
&= \frac{L_f(c-a+b-d)^\alpha}{(d-c)^{\alpha+1}}\int_c^d \left|s-\frac{bc-ad}{c-a+b-d}\right|^\alpha ds
\end{aligned}
$$

(3.3)

$$
= \frac{L_f(c-a+b-d)^\alpha}{(d-c)^{\alpha+1}}\int_c^d |s-s_0|^\alpha\,ds,
$$

where $s_0 = \frac{bc-ad}{c-a+b-d}$.
We now observe that

$$
s_0 - c = \frac{(d-c)(c-a)}{c-a+b-d} \geq 0
$$

and

$$
d - s_0 = \frac{(d-c)(b-d)}{c-a+b-d} \geq 0,
$$

and hence $s_0 \in [c,d]$.
Therefore,

$$
\begin{aligned}
\int_c^d |s-s_0|^\alpha\,ds &= \int_c^{s_0}(s_0-s)^\alpha\,ds + \int_{s_0}^d (s-s_0)^\alpha\,ds \\
&= \frac{1}{\alpha+1}\left[(s_0-c)^{\alpha+1}+(s_0-d)^{\alpha+1}\right] \\
&= \frac{(d-c)^{\alpha+1}}{(\alpha+1)(c-a+b-d)^{\alpha+1}}\left[(c-a)^{\alpha+1}+(b-d)^{\alpha+1}\right] \qquad (3.4)
\end{aligned}
$$

Substitution of (3.4) in (3.3) gives (3.2).
This completes the proof. ☐

Remark 3. *The inequalities (3.1) and (3.2) generalize the inequalitites (1.3) and (1.6) respectively but are not sharp as the equality cannot be attained by a particular choice of the fuzzy real number valued functions, since if we choose* $f^*(t) = |t-s_0|^\alpha \odot u$, $0 < \alpha \leq 1$, *with* $u \in \mathbb{R}_{\mathcal{F}}$ *fixed,* $t \in [a,b]$ *and* $s_0 = \frac{bc-ad}{c-a+b-d}$. *Then*

$f^* \in C([a,b],\mathbb{R}_{\mathcal{F}})$, as for letting $t_n \to t$, $t_n \in [a,b]$ and using Lemma 1, we have

$$\begin{aligned} D\left(f^*\left(t_n\right), f^*\left(t\right)\right) &= D\left(\left|t_n - s_0\right|^{\alpha} \odot u, \left|t - s_0\right|^{\alpha} \odot u\right) \\ &\leq \left|\left|t_n - s_0\right|^{\alpha} - \left|t - s_0\right|^{\alpha}\right| D\left(u, \tilde{o}\right) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Furthermore

$$\begin{aligned} D\left(f^*\left(t\right), f^*\left(s\right)\right) &= D\left(\left|t - s_0\right|^{\alpha} \odot u, \left|s - s_0\right|^{\alpha} \odot u\right) \\ &\leq \left|\left|t - s_0\right|^{\alpha} - \left|s - s_0\right|^{\alpha}\right| D\left(u, \tilde{o}\right) \\ &\leq \left|t - s\right|^{\alpha} D\left(u, \tilde{o}\right). \end{aligned}$$

This shows that for $L_{f^*} = D(u,\tilde{o})$, we have

$$D\left(f^*\left(t\right), f^*\left(s\right)\right) \leq L_{f^*}\left|t - s\right|^{\alpha}, \ 0 < \alpha \leq 1, \text{ for any } t, s \in [a,b].$$

and therefore f^* satisfies the Hölder condition.
Finally by the properties of (FR)-integrable functions and (iii) of Theorem 5, we have

$$\begin{aligned} &\frac{1}{d-c} \odot (FR) \int_c^d f^*(s)\, ds \\ &= \frac{(d-c)^{\alpha}}{(\alpha+1)(c-a+b-d)^{\alpha+1}} \left[(c-a)^{\alpha+1} + (b-d)^{\alpha+1}\right] \odot u. \end{aligned} \tag{3.5}$$

Since

$$s_0 - a = \frac{(b-a)(c-a)}{c-a+b-d} \geq 0$$

and

$$b - s_0 = \frac{(b-a)(b-d)}{c-a+b-d} \geq 0$$

implies that $s_0 \in [a,b]$.
By similar arguments as in obtaining (3.4), we get that

$$\begin{aligned} &\frac{1}{b-a} \odot (FR) \int_a^b f^*(t)\, dt \\ &= \frac{(b-a)^{\alpha}}{(\alpha+1)(c-a+b-d)^{\alpha+1}} \left[(c-a)^{\alpha+1} + (b-d)^{\alpha+1}\right] \odot u \end{aligned} \tag{3.6}$$

Now it is evident from (3.5), (3.6) and Lemma 1 that

$$\begin{aligned} &D\left(\frac{1}{b-a} \odot (FR) \int_a^b f^*(t), \frac{1}{d-c} \odot (FR) \int_c^d f^*(s)\, ds\right) \\ &= \left(\frac{(c-a)^{\alpha+1} + (b-d)^{\alpha+1}}{(\alpha+1)(c-a+b-d)}\right)\left(\frac{(b-a)^{\alpha} - (d-c)^{\alpha}}{(c-a+b-d)^{\alpha}}\right) D\left(u, \tilde{o}\right). \end{aligned}$$

This shows that (1.6) is not sharp.

Our next result is about fuzzy differentiable functions and is stated as follow:

Theorem 10. *Let $a,b,c,d \in \mathbb{R}$ be such that*

$$a \leq c < d \leq b, \ c-a+b-d > 0.$$

$f \in C^1([a,b],\mathbb{R}_{\mathcal{F}})$, the space of fuzzy one time continuously differentiable functions. Then

$$D\left(\frac{1}{b-a}\odot (FR)\int_a^b f(t)\,dt, \frac{1}{d-c}\odot (FR)\int_c^d f(s)\,ds\right)$$

$$\leq \left(\sup_{t\in[a,b]} D\left(f'(t),\tilde{o}\right)\right)\left(\frac{(c-a)^2+(b-d)^2}{2(c-a+b-d)}\right). \tag{3.7}$$

Proof. Again by using the substitution

$$t=\frac{b-a}{d-c}s-\frac{bc-ad}{d-c},\ s\in[c,d],$$

and by (v) of Theorem 5, we have that

$$\frac{1}{b-a}\odot (FR)\int_a^b f(t)\,dt=\frac{1}{d-c}\odot (FR)\int_c^d f\left(\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}\right)ds.$$

Now by Lemma 2 and Theorem 6, we get that

$$\begin{aligned}
&D\left(\frac{1}{b-a}\odot (FR)\int_a^b f(t)\,dt, \frac{1}{d-c}\odot (FR)\int_c^d f(s)\,ds\right)\\
&=D\left(\frac{1}{d-c}\odot (FR)\int_c^d f\left(\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}\right)ds, \frac{1}{d-c}\odot (FR)\int_c^d f(s)\,ds\right)\\
&=\frac{1}{d-c}D\left((FR)\int_c^d f\left(\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}\right)ds, (FR)\int_c^d f(s)\,ds\right)\\
&\leq \frac{1}{d-c}\int_c^d D\left(f\left(\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}\right), f(s)\right)ds\\
&\leq \frac{\sup_{s\in[c,d]} D\left(f'(s),\tilde{o}\right)}{d-c}\int_c^d \left|\frac{b-a}{d-c}s-\frac{bc-ad}{d-c}-s\right|ds\\
&=\frac{\sup_{s\in[c,d]} D\left(f'(s),\tilde{o}\right)}{d-c}\int_c^d \left|\frac{c-a+b-d}{d-c}s-\frac{bc-ad}{d-c}\right|ds\\
&\leq \frac{\sup_{t\in[a,b]} D\left(f'(t),\tilde{o}\right)(c-a+b-d)}{(d-c)^2}\int_c^d \left|s-\frac{bc-ad}{c-a+b-d}\right|ds
\end{aligned}$$

(3.8)

$$=\frac{\sup_{t\in[a,b]} D\left(f'(t),\tilde{o}\right)(c-a+b-d)}{(d-c)^2}\int_c^d |s-s_0|\,ds,$$

where $s_0=\frac{bc-ad}{c-a+b-d}$.
We now observe that

$$s_0-c=\frac{(d-c)(c-a)}{c-a+b-d}\geq 0$$

and

$$d-s_0=\frac{(d-c)(b-d)}{c-a+b-d}\geq 0,$$

and hence $s_0 \in [c, d]$.
Therefore,

$$\int_c^d |s - s_0|\, ds = \int_c^{s_0} (s_0 - s)\, ds + \int_{s_0}^d (s - s_0)\, ds$$
$$= \frac{1}{2}\left[(s_0 - c)^2 + (s_0 - d)^2\right]$$
$$= \frac{(d-c)^2}{2\,(c-a+b-d)^2}\left[(c-a)^2 + (b-d)^2\right] \tag{3.9}$$

Substituting (3.9) in (3.8), we get (3.7).

□

Remark 4. *Inequality (3.7) in Theorem 9 is not sharp. Moreover, we note that if $c = d = x$ we can assume $\frac{1}{d-c} \odot (FR) \int_c^d f(s)\, ds = f(x)$, as a limit case, so (3.2) and (3.7) reduce to (1.5) and (1.6) respectively. This fact also reveals that although our results are not sharp but generalize the inequalities (1.5) and (1.6).*

Our last result is about fuzzy M-Lipshitzian mappings and is stated as follow:

Theorem 11. *Let $a, b, c, d \in \mathbb{R}$ be such that*

$$a \le c < d \le b,\ c - a + b - d > 0.$$

$f \in C([a, b], \mathbb{R}_{\mathcal{F}})$, the space of fuzzy continuous functions. Suppose f is M-Lipshitzian, that is

$$D(f(y), f(z)) \le M\,|y - z|\,,\ \textit{for all } y,\ z \in [a, b]\,,$$

for some $M > 0$. Then

$$D\left(\frac{1}{b-a} \odot (FR) \int_a^b f(t)\, dt, \frac{1}{d-c} \odot (FR) \int_c^d f(s)\, ds\right)$$
$$\le M\left(\frac{(c-a)^2 + (b-d)^2}{2\,(c-a+b-d)}\right). \tag{3.10}$$

Inequality (3.10) is sharp, in fact attained by $f^*(t) = |t - s_0| \odot u$, $u \in \mathbb{R}_{\mathcal{F}}$ being fixed and $s_0 = \frac{bc-ad}{c-a+b-d}$.

Proof. The proof of (3.10) is similar to that of (3.2) we, therefore omit the detals for the intrested reader.
It remains only to prove that (3.10) is sharp. It is clear that $f^* \in C([a, b], \mathbb{R}_{\mathcal{F}})$, since for letting $t_n \to t$, $t_n \in [a, b]$, then Lemma 1 we have

$$D(f^*(t_n), f^*(t)) = D(|t_n - s_0| \odot u, |t - s_0| \odot u)$$
$$\le ||t_n - s_0| - |t - s_0||\, D(u, \tilde{o})$$
$$\le |t_n - t|\, D(u, \tilde{o}) \to 0, \text{ as } n \to \infty.$$

Moreover, again by Lemma 1, we get that

$$D(f^*(t), f^*(s)) = D(|t - s_0| \odot u, |s - s_0| \odot u)$$
$$\le ||t - s_0| - |s - s_0||\, D(u, \tilde{o})$$
$$\le |t - s|\, D(u, \tilde{o})\,.$$

That is for $M = D(u, \tilde{o})$, we get

$$D(f^*(t), f^*(s)) \leq M |t - s|, \forall t, s \in [a, b].$$

So f^* is M-Lipshitzian function.
Now by the similar reasoning as in Remark 1, we have

$$\begin{aligned}&\frac{1}{d-c} \odot (FR) \int_c^d f^*(s)\, ds \\ &= \frac{d-c}{2(c-a+b-d)^2} \left[(c-a)^2 + (b-d)^2\right] \odot u. \end{aligned} \tag{3.11}$$

Since

$$s_0 - a = \frac{(b-a)(c-a)}{c-a+b-d} \geq 0$$

and

$$b - s_0 = \frac{(b-a)(b-d)}{c-a+b-d} \geq 0$$

implies that $s_0 \in [a, b]$.
Similarly we also have

$$\begin{aligned}&\frac{1}{b-a} \odot (FR) \int_a^b f^*(t)\, dt \\ &= \frac{b-a}{(c-a+b-d)^2} \left[(c-a)^2 + (b-d)^2\right] \odot u \end{aligned} \tag{3.12}$$

Now it is aparent from (3.11), (3.12) and Lemma 1 that

$$\begin{aligned}&D\left(\frac{1}{b-a} \odot (FR) \int_a^b f^*(t), \frac{1}{d-c} \odot (FR) \int_c^d f^*(s)\, ds\right) \\ &= \left(\frac{(c-a)^2 + (b-d)^2}{2(c-a+b-d)}\right) D(u, \tilde{o}). \end{aligned}$$

Hence it is proved that (3.10) is sharp. □

References

[1] George A. Anastassiou, *Ostrowski type inequalities*, Proc. AMS, **123** (1995), 3775–3791.

[2] George A. Anastassiou, Fuzzy *Ostrowski type inequalities, Computational and Applied Mathematics, Vol. 22, No. 2 (2003), pp. 279-292.*

[3] George A. Anastassiou, *Univariate fuzzy-random neural network approximation operators*, Computers & Mathematics with Applications Volume **48**, Issue 9, November 2004, Pages 1263-1283.

[4] George A. Anastassiou and Sorin Gal, *On a fuzzy trigonometric approximation theorem of Weierstrass-type*, Journal of Fuzzy Mathematics, **9**, No. 3 (2001), 701–708.

[5] N. S. Barnett and S. S. Dragomir, *Issues of estimating in the monitoring of constant flow coninuous streams*, RGMIA, Research report collection, Vol. 2, No. 3 (1999), pp.275-282.

[6] A.M. Fink, *Bounds on the deviation of a function from its averages*, Czechoslavak Math. J., **42** (117) (1992), 289–310.

[7] S. Gal, *Approximation theory in fuzzy setting*, Chapter 13 in Handbook of Analytic Computational Methods in Applied Mathematics (edited by G. Anastassiou), Chapman & Hall, CRC Press, Boca Raton, NewYork, 2000, pp. 617–666.

[8] A. Ostrowski, *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226–227. Comp. Appl. Math., Vol. **22**, N. 2, 2003.

[9] M. Matić and J. Pečarić, *Two-pont Ostrowski inequality*, Mathematical Inequalities & Applications, **4** 2(2001), 215-221.
[10] M.L. Puri and D.A. Ralescu, *Differentials of fuzzy functions*, J. of Math. Analysis and Appl., **91** (1983), 552–558.
[11] CongxinWu and Zengtai Gong, *On Henstock integral of fuzzy number valued functions (I)*, Fuzzy Sets and Systems, **120**, No. 3 (2001), 523–532.
[12] L.A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353.

[1]College of Science, Department of Mathematics, University of Hail, Hail 2440, Saudi Arabia

[2]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan

*Corresponding author