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ALMOST PERIODIC SOLUTIONS FOR IMPULSIVE FRACTIONAL STOCHASTIC EVOLUTION EQUATIONS

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Abstract. In this paper, we consider the existence of square-mean piecewise almost periodic solutions for impulsive fractional stochastic evolution equations involving Caputo fractional derivative. The main results are obtained by means of the theory of operators semi-group, fractional calculus, fixed point technique and stochastic analysis theory and methods adopted directly from deterministic fractional equations. Some known results are improved and generalized.

1. Introduction

The study of fractional differential equations has been gaining importance in recent years due to the fact that fractional order derivatives provide a tool for the description of memory and hereditary properties of various phenomena. Due to this fact, the fractional order models are capable to describe more realistic situation than the integer order models. Fractional differential equations have been used in many field like fractals, chaos, electrical engineering, medical science, etc. In recent years, we have seen considerable development on the topics of fractional differential equations, for instance, we refer to the monographs of Abbas et al. [2], Kilbas et al. [14], Miller and Ross [18], Podlubny [20], and the papers [3, 4, 31].

In particular, differential equations with impulsive conditions constitute an important field of research due to their numerous applications in ecology, medicine biology, electrical engineering and other areas of science. Many physical phenomena in evolution processes are modeled as impulsive fractional differential equations and existence results for such equations have been studied by several authors [9, 23, 30]. One of the important problems in the qualitative theory of impulsive differential equations is the existence of almost periodic solutions. At the present time, many results on the existence, uniqueness and stability of these solutions have been obtained (see [1, 15, 24, 26] and the references therein). However, only few papers deal with the existence of almost periodic solutions for impulsive fractional differential equations. Recently, Debbouche et al. [11] studied the existence of almost periodic and optimal mild solutions of fractional evolution equations with analytic semigroup in a Banach space. El-Borai et al. [12] established the existence and uniqueness of almost periodic solutions of a class of nonlinear fractional differential equations with analytic semigroup in Banach space, and very recently, Stamov et

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al. [27] studied the existence of almost periodic solutions for fractional differential equations with impulsive effects.

In many cases, deterministic models often fluctuate due to environmental noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. Taking the disturbances into account, the theory of differential equations has been generalized to stochastic case. The existence, uniqueness, stability, controllability and other quantitative and qualitative properties of solutions of stochastic evolution equations have recently received a lot of attention (see [13, 17, 28] and the references therein). The existence of almost periodic solutions for stochastic differential equations has been discussed in [5, 7, 21]. The existence of almost periodic solutions for impulsive stochastic evolution equations has been reported in [8, 16]. However, up to now the problem for the existence of almost periodic solutions for impulsive fractional stochastic evolution equations have not been considered in the literature. In order to fill this gap, this paper studies the existence of square-mean piecewise almost periodic solutions of the following impulsive fractional stochastic differential equations in the form
(1)

$$^{c}D_t^{\alpha}x(t) + Ax(t) = F(t, x(t)) + \Sigma(t, x(t))\frac{dw(t)}{dt} + \sum_{k=-\infty}^{\infty} G_k(x(t))\delta(t-\tau_k), \quad t \in \mathbb{R},$$

where the state $x(\cdot)$ takes values in the space $L^2(\mathbb{P}, \mathcal{H})$, $\mathcal{H}$ is a separable real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$; the fractional derivative $^{c}D^{\alpha}$, $\alpha \in (0, 1)$, is understood in the Caputo sense; $-A : \mathcal{D}(A) \subset L^2(\mathbb{P}, \mathcal{H}) \to L^2(\mathbb{P}, \mathcal{H})$ is the infinitesimal generator of an analytic semigroup of a bounded linear operator $S(t)$, $t \geq 0$, on $L^2(\mathbb{P}, \mathcal{H})$ satisfying the exponential stability; $\{w(t) : t \geq 0\}$ is a given $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mathcal{K}$ is another separable Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{K}}$ and norm $\|\cdot\|_{\mathcal{K}}$; $G_k : \mathcal{D}(G_k) \subset L^2(\mathbb{P}, \mathcal{H}) \to L^2(\mathbb{P}, \mathcal{H})$ are continuous impulsive operators, $\delta(\cdot)$ is Dirac's delta-function, $F(t, x) : \mathbb{R} \times L^2(\mathbb{P}, \mathcal{H}) \to L^2(\mathbb{P}, \mathcal{H})$ and $\Sigma(t, x) : \mathbb{R} \times L^2(\mathbb{P}, \mathcal{H}) \to L^2(\mathbb{P}, L_2^0(\mathcal{K}, \mathcal{H}))$ are jointly continuous functions (here, $L_2^0(\mathcal{K}, \mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{H}$, which is going to be defined below).

The structure of this paper is as follows. In Sect. 2, we will recall briefly some preliminaries fact which will be used in paper. Section 3, we establish criteria of the existence of an almost periodic solution and its exponential stability.

2. Preliminaries

In this section, we introduce some basic definitions, notation and lemmas which are used throughout this paper. Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ be two real separable Hilbert spaces, and we denote by $L(\mathcal{K}, \mathcal{H})$ the set of all linear bounded operators from $\mathcal{K}$ into $\mathcal{H}$, equipped with the usual operator norm $\|\cdot\|$. We will use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of $\mathcal{K}$. Suppose that $w = (w_t)_{t\geq 0}$ is a cylindrical $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{i=1}^{\infty} \tilde{\lambda}_i = \tilde{\lambda} < \infty$, which satisfies $Qe_i = \tilde{\lambda}_i e_i$. So, actually, $w(t) = \sum_{i=1}^{\infty} \sqrt{\tilde{\lambda}_i} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w. Let $L_2^0 = L_2(Q^{\frac{1}{2}}\mathcal{K}, \mathcal{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathcal{K}$ to $\mathcal{H}$ with the inner product $(\varphi, \phi)_{L_2^0} = Tr[\varphi Q \phi^*]$. For more details, we refer the reader to Da Prato and Zabczyk [10].

The collection of all measurable, square integrable, $\mathcal{H}$-valued random variables, denoted by $L^2(\mathbf{P}, \mathcal{H})$ is a Banach space equipped with norm $\|x(\cdot)\|_{L^2} = (\mathbf{E}\|x(\cdot)\|^2)^{\frac{1}{2}}$, where $\mathbf{E}(\cdot)$ denotes the expectation with respect to the measure $\mathbf{P}$. Let $\mathcal{C}(\mathbb{R}, L^2(\mathbf{P}, \mathcal{H}))$ be the Banach space of all continuous maps from $\mathbb{R}$ into $L^2(\mathbf{P}, \mathcal{H})$ satisfying the condition $\sup_{t\in\mathbb{R}} \mathbf{E}\|x(t)\|^2 < \infty$. Let $L^2_{\mathcal{F}_0}(\mathbf{P}, \mathcal{H})$ denote the family of all $\mathcal{F}_0$-measurable, $\mathcal{H}$-valued random variables $x(0)$.

Let us now recall some basic definitions and results of fractional calculus. For more details see [14, 18, 20].

Definition 2.1. *The fractional integral of order α with the lower limit zero for a function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbf{N}$, is defined as*

$${}^{(R-L)}D^\alpha_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3. *The Caputo derivative of order $\alpha > 0$ for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$D^\alpha f(t) = D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n - 1 < \alpha < n.$$

Remark 2.4. (i) If $f(t) \in \mathcal{C}^n[0, \infty)$, then

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, n - 1 < \alpha < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in $\mathcal{H}$, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochners sense.

Let $\mathcal{B} = \{\{\tau_k\} : \tau_k \in \mathbb{R},\ \tau_k < \tau_{k+1},\ k \in \mathbf{Z}\}$ be the set of all sequences unbounded and strictly increasing. We consider the impulsive fractional differential equation (1), and denote by $x(t) = x(t; t_0, x_0)$, $t_0 \in \mathbb{R}$, $x_0 \in \mathcal{H}$, the solution of (1) with the initial condition

$$x(t_0) = x_0. \tag{2}$$

Definition 2.5 ([16]). *A stochastic process $x : \mathbf{R} \to L^2(\mathbf{P}, \mathcal{H})$, is said to be stochastically bounded if there exists $N > 0$ such that $\mathbf{E}\|x(t)\|^2 \le N$ for all $t \in \mathbf{R}$.*

Definition 2.6 ([16]). *A stochastic process $x : \mathbf{R} \to L^2(\mathbf{P}, \mathcal{H})$, is said to be stochastically continuous in $s \in \mathbf{R}$, if $\lim_{t\to s} \mathbf{E}\|x(t) - x(s)\|^2 = 0$.*

For $\{\tau_k\} \in \mathcal{B}$ and $k \in \mathbf{Z}$, let $\mathcal{PC}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$ be the space consisting of all stochastically bounded functions $\phi : \mathbf{R} \to L^2(\mathbf{P}, \mathcal{H})$ such that $\phi(\cdot)$ is stochastically continuous at t for any $t \notin \{\tau_k\}$, $\tau_k \in \mathbf{R}$, $k \in \mathbf{Z}$ and $\phi(\tau_k^-) = \phi(\tau_k)$. In particular, we introduce the space $\mathcal{PC}(\mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}), L^2(\mathbf{P}, \mathcal{H}))$ formed by all piecewise stochastically continuous stochastic processes $\phi : \mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}) \to L^2(\mathbf{P}, \mathcal{H})$ such that for any $x \in L^2(\mathbf{P}, \mathcal{H})$, $\phi(\cdot, x)$ is stochastically continuous at t for any $t \notin \{\tau_k\}$ and $\phi(\tau_k^-, x) = \phi(\tau_k, x)$ for all $k \in \mathbf{Z}$, and for any $t \in \mathbf{R}$, $\phi(t, \cdot)$ is stochastically continuous at $x \in L^2(\mathbf{P}, \mathcal{H})$.

Remark 2.7 ([16, 29]). The solution $x(t) = x(t; t_0, x_0)$ of the problem (1)-(2) is a piecewise stochastically continuous, $\mathcal{F}_t$-adapted measurable process with points of discontinuity at the moments τ_k, $k \in \mathbf{Z}$, at which it is continuous from the left.

Definition 2.8 ([26]). *The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbf{Z}$, $j \in \mathbf{Z}$, $\{\tau_k\} \in \mathcal{B}$ is said to be equipotentially almost periodic, if for arbitrary $\epsilon > 0$ there exists a relatively dense set B_ϵ of $\mathbf{R}$ such that for each $\kappa \in B_\epsilon$ there is an integer $q \in \mathbf{Z}$ such that $|\tau_{k+q} - \tau_k - \kappa| < \epsilon$ for all $k \in \mathbf{Z}$.*

Definition 2.9 ([7]). *A stochastic process $x \in \mathcal{PC}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$ is said to be square-mean piecewise almost periodic, if:*

(i) *The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbf{Z}$, $j \in \mathbf{Z}$, $\{\tau_k\} \in \mathcal{B}$ is equipotentially almost periodic.*
(ii) *For any $\epsilon > 0$, there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $x(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $\mathbf{E}\|x(t') - x(t'')\|^2_{\mathcal{H}} < \epsilon$.*
(iii) *For any $\epsilon > 0$, there exists a relatively dense set T such that if $\tau \in T$, then $\mathbf{E}\|x(t + \tau) - x(t)\|^2_{\mathcal{H}} < \epsilon$, satisfying the condition $|t - \tau_k| > \epsilon$, $k \in \mathbf{Z}$. The elements of T are called ϵ-translation number of x.*

We denote by $\mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$ the collection of all the square-mean piecewise almost periodic processes, it thus is a Banach space with the norm $\|x\|_\infty = \sup_{t\in\mathrm{R}} \|x(t)\|_{L^2} = \sup_{t\in\mathrm{R}} (\mathbf{E}\|x(t)\|^2)^{\frac{1}{2}}$ for $x \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$.

Lemma 2.10 ([16]). *Let $F \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$. Then, $R(F)$, the range of F is a relatively compact set of $L^2(\mathbf{P}, \mathcal{H})$.*

Definition 2.11 ([8]). *For $\{\tau_k\} \in \mathcal{B}$, $k \in \mathbf{Z}$, the function $F(t, x) \in \mathcal{PC}(\mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}), L^2(\mathbf{P}, \mathcal{H}))$ is said to be square-mean piecewise almost periodic in $t \in \mathbf{R}$ and uniform on compact subset of $L^2(\mathbf{P}, \mathcal{H}))$ if for every $\epsilon > 0$ and every compact subset $K \subseteq L^2(\mathbf{P}, \mathcal{H}))$, there exists a relatively dense subset T of $\mathbf{R}$ such that*

$$\mathbf{E}\|F(t + \tau, x) - F(t, x)\|^2 < \epsilon,$$

for all $x \in K$, $\tau \in T$, $t \in \mathbf{R}$ satisfying $|t - \tau_k| > \epsilon$, $k \in Z$. The collection of all such processes is denoted $\mathcal{AP}(\mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}), L^2(\mathbf{P}, \mathcal{H}))$.

Lemma 2.12 ([16]). *Suppose that $F(t, x) \in \mathcal{AP}(\mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}), L^2(\mathbf{P}, \mathcal{H}))$ and $F(t, \cdot)$ is uniformly continuous on each compact subset $K \subseteq L^2(\mathbf{P}, \mathcal{H})$ uniformly* *for $t \in \mathbf{R}$. That is, for all $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in \mathcal{K}$ and $\mathbf{E}\|x-y\|^2 < \delta$ implies that $\mathbf{E}\|F(t,x)-F(t,y)\|^2 < \epsilon$ for all $t \in \mathbf{R}$. Then $F(\cdot, x(\cdot)) \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$ for any $x \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$.*

We obtain the following corollary as an immediate consequence of Lemma 2.12.

Corollary 2.13. *Let $F(t,x) \in \mathcal{AP}(\mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}), L^2(\mathbf{P}, \mathcal{H}))$ and F is Lipschitz, i.e., there is a number $c > 0$ such that*

$$\mathbf{E}\|F(t,x) - F(t,y)\|^2 \leq c\mathbf{E}\|x - y\|^2,$$

for all $t \in \mathbf{R}$ and $x, y \in L^2(\mathbf{P}, \mathcal{H})$, if for any $x \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$, then $F(\cdot, x(\cdot)) \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$.

Definition 2.14. *A sequence $x : \mathbf{Z} \to L^2(\mathbf{P}, \mathcal{H})$ is called a square-mean almost periodic sequence if ϵ-translation set of x*

$$\mathcal{I}(x; \epsilon) = \{\tau \in \mathbf{Z} : \mathbf{E}\|x(n + \tau) - x(t)\|^2 < \epsilon, \quad \text{for all } n \in \mathbf{Z}\}$$

is a relatively dense set in $\mathbf{Z}$ for all $\epsilon > 0$.

The collection of all square-mean almost periodic sequences $x : \mathbf{Z} \to L^2(\mathbf{P}, \mathcal{H})$ will be denoted by $\mathcal{AP}(\mathbf{Z}, L^2(\mathbf{P}, \mathcal{H}))$.

Remark 2.15. If $x(n) \in \mathcal{AP}(\mathbf{Z}, L^2(\mathbf{P}, \mathcal{H}))$, then $\{x(n) : n \in \mathbf{Z}\}$ is stochastically bounded.

Lemma 2.16 ([16])**.** *Assume that $F \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$, the sequence $\{x_k : k \in \mathbf{Z}\}$ is almost periodic in $L^2(\mathbf{P}, \mathcal{H})$ and $\{\tau_k^j\}$, $j \in \mathbf{Z}$, is equipotentially almost periodic. Then for each $\epsilon > 0$ there are relatively dense sets T_{ϵ,F,x_k} of $\mathbf{R}$ and $\widehat{T}_{\epsilon,F,x_k}$ of $\mathbf{Z}$ such that the following conditions hold:*

(i) *$\mathbf{E}\|F(t+\tau) - F(t)\|^2 < \epsilon$ for all $t \in \mathbf{R}$, $|t - \tau_k| > \epsilon$, $\tau \in T_{\epsilon,F,x_k}$ and $k \in \mathbf{Z}$.*
(ii) *$\mathbf{E}\|x_{k+q} - x_k\|^2 < \epsilon$ for all $q \in \widehat{T}_{\epsilon,F,x_k}$ and $k \in \mathbf{Z}$.*
(iii) *For every $\tau \in T_{\epsilon,F,x_k}$, there exists at least one number $q \in \widehat{T}_{\epsilon,F,x_k}$ such that $|\tau_k^q - \tau| < \epsilon$, $k \in \mathbf{Z}$.*

Consider the linear fractional impulsive stochastic differential equation corresponding to (1)

$$(3) \qquad {}^cD_t^\alpha x(t) + Ax(t) = f(t) + \sigma(t)\frac{dw(t)}{dt} + \sum_{k=-\infty}^{\infty} g_k \delta(t - \tau_k),$$

where $f \in \mathcal{PC}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$, $\sigma \in \mathcal{PC}(\mathbf{R}, L^2(\mathbf{P}, L_2^0))$ and $g_k : \mathcal{D}(g_k) \subset L^2(\mathbf{P}, \mathcal{H}) \to L^2(\mathbf{P}, \mathcal{H})$.

Let us introduce the following conditions.

(**C1**) The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbf{Z}$, $j \in \mathbf{Z}$, $\{\tau_k\} \in \mathcal{B}$ is equipotentially almost periodic and there exists $\theta > 0$ such that $\inf_k \tau_k^1 = \theta$.

(**C2**) The function f is in $\mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$ and locally Hölder continuous with points of discontinuity at the moments τ_k, $k \in \mathbf{Z}$ at which it is continuous from the left.

(**C3**) The function σ is in $\mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, L_2^0))$ and locally Hölder continuous with points of discontinuity at the moments τ_k, $k \in \mathbf{Z}$ at which it is continuous from the left.

(**C4**) $\{g_k\}$, $k \in \mathbf{Z}$, of impulsive operators is a square-mean almost periodic sequence.

Lemma 2.17 ([15],[16]). *Let the condition* (**C1**) *holds. Then*

(i) *There exists a constant $p > 0$ such that, for every $t \in \mathbf{R}$*

$$\lim_{T\to\infty} \frac{\iota(t, t+T)}{T} = p.$$

(ii) *For each $p > 0$ there exists a positive integer N such that each interval of length p has no more than N elements of the sequence $\{\tau_k\}$, that is,*

$$\iota(s,t) \le N(t-s) + N,$$

where $\iota(s,t)$ is the number of points τ_k in the interval (s,t).

The following Lemma is an immediate consequence of Lemma 2.16.

Lemma 2.18. *Let the conditions* (**C1**)-(**C4**) *hold. Then, for each $\epsilon > 0$ there are relatively dense sets $T_{\epsilon,f,\sigma,g_k}$ of $\mathbf{R}$ and $\widehat{T}_{\epsilon,f,\sigma,g_k}$ of $\mathbf{Z}$ such that the following relations hold:*

(i) $\mathbf{E}\|f(t+\tau) - f(t)\|^2 < \epsilon$, $t \in \mathbf{R}$, $\tau \in T_{\epsilon,f,\sigma,g_k}$, $|t - \tau_k| > \epsilon$, $k \in \mathbf{Z}$.

(ii) $\mathbf{E}\|\sigma(t+\tau) - \sigma(t)\|^2 < \epsilon$, $t \in \mathbf{R}$, $\tau \in T_{\epsilon,f,\sigma,g_k}$, $|t - \tau_k| > \epsilon$, $k \in \mathbf{Z}$.

(iii) $\mathbf{E}\|g_{k+q} - g_k\|^2 < \epsilon$, $k \in \mathbf{Z}$, $q \in \widehat{T}_{\epsilon,f,\sigma,g_k}$.

(iv) *For each $\tau \in T_{\epsilon,f,\sigma,g_k}$, $\exists q \in \widehat{T}_{\epsilon,f,\sigma,g_k}$, such that $|\tau_{k+q} - \tau_k - \tau| < \epsilon$, $k \in \mathbf{Z}$.*

Now, we present the definition of mild solutions for the problem (2)-(3) based on the paper [25].

Definition 2.19. *A stochastic process $x \in \mathcal{PC}(J, L^2(\mathbf{P},\mathcal{H}))$, $J \subset \mathbf{R}$ is called a mild solution of the problem* (2)-(3) *if*

(i) $x_0 \in L^2_{\mathcal{F}_0}(\mathbf{P},\mathcal{H})$;

(ii) *$x(t) \in L^2(\mathbf{P},\mathcal{H})$ has càdlàg paths on $t \in J$ a.s., and it satisfies the following integral equation*

(4)

$$x(t) = \begin{cases} \mathcal{T}(t-t_0)x_0 + \displaystyle\int_{t_0}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds \\ \qquad + \displaystyle\int_{t_0}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dw(s), & t \in [t_0, \tau_1], \\ \mathcal{T}(t-t_0)x_0 + \mathcal{T}(t-\tau_1)g_1 + \displaystyle\int_{t_0}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds \\ \qquad + \displaystyle\int_{t_0}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dw(s), & t \in (\tau_1, \tau_2], \\ \vdots \\ \mathcal{T}(t-t_0)x_0 + \displaystyle\sum_{t_0<\tau_k<t} \mathcal{T}(t-\tau_k)g_k + \displaystyle\int_{t_0}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds \\ \qquad + \displaystyle\int_{t_0}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dw(s), & t \in (\tau_k, \tau_{k+1}], \end{cases}$$

where

$$\mathcal{T}(t) = \int_0^\infty \xi_\alpha(\theta)S(t^\alpha\theta)d\theta, \qquad \mathcal{S}(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)S(t^\alpha\theta)d\theta,$$

and for $\theta \in (0,\infty)$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha\left(\theta^{-\frac{1}{\alpha}}\right) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),$$

ξ_α *is a probability density function defined on* $(0,\infty)$*, that is*

$$\xi_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \text{and} \quad \int_0^\infty \xi_\alpha(\theta)d\theta = 1.$$

Remark 2.20.

$$\int_0^\infty \theta^\nu \xi_\alpha(\theta)d\theta = \int_0^\infty \theta^{-\alpha\nu}\varpi_\alpha(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}.$$

In this paper, we will also assume that $\xi_\alpha^2 \in L^1((0,\infty))$.

Let the operator $-A$ in (1) and (3) be an infinitesimal generator of an analytic semigroup $S(t)$ in $L^2(\mathbb{P},\mathcal{H})$ and $0 \in \rho(A)$, the resolvent set of A. For any $\beta > 0$, we define the fractional power $A^{-\beta}$ of the operator a by

$$A^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}S(t)dt,$$

where $A^{-\beta}$ is bounded, bijective and $A^\beta = (A^{-\beta})^{-1}$, $\beta > 0$ a closed linear operator on its domain $\mathcal{D}(A^\beta)$ and such that $\mathcal{D}(A^\beta) = \mathcal{R}(A^{-\beta})$ where $\mathcal{R}(A^{-\beta})$ is the range of $A^{-\beta}$. Furthermore, the subspace $\mathcal{D}(A^\beta)$ is dense in $L^2(\mathbb{P},\mathcal{H})$ and the expression

$$\|x\|_\beta = \|A^\beta x\|, \qquad x \in \mathcal{D}(A^\beta),$$

defines a norm on $L^2(\mathbb{P},\mathcal{H}_\beta) := \mathcal{D}(A^\beta)$. The following properties are well known.

Lemma 2.21 ([19])**.** *Suppose that the preceding conditions are satisfied. Then*

(i) $S(t) : L^2(\mathbb{P},\mathcal{H}) \to \mathcal{D}(A^\beta)$ *for every* $t > 0$ *and* $\beta \geq 0$.

(ii) *For every* $x \in \mathcal{D}(A^\beta)$*, the following equality* $S(t)A^\beta x = A^\beta S(t)x$ *holds.*

(iii) *For every* $t > 0$*, the operator* $A^\beta S(t)$ *is bounded and*

$$\|A^\beta S(t)\| \leq K_\beta t^{-\beta}e^{-\lambda t}, \qquad K_\beta > 0, \lambda > 0.$$

(iv) *For* $0 < \beta \leq 1$ *and* $x \in \mathcal{D}(A^\beta)$*, we have*

$$\|S(t)x - x\| \leq C_\beta t^\beta \|A^\beta x\|, \quad C_\beta > 0.$$

When $-A$ generates a semi-group with negative exponent, we deduce that if $x(t)$ is a bounded solution of (3) on $\mathbb{R}$, then we take the limit as $t_0 \to -\infty$ and using (4), we obtain (see [6])

(5)

$$x(t) = \int_{-\infty}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds + \int_{-\infty}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dw(s) + \sum_{\tau_k<t}\mathcal{T}(t-\tau_k)g_k.$$

3. Main results

In this section, we present and prove our main theorems.

Theorem 3.1. *Assume the conditions* (**C1**)-(**C4**) *are satisfied and* $-A$ *is the infinitesimal generator of an analytic semi-group* $S(t)$*, then system* (2)-(3) *has a square-mean piecewise almost periodic mild solution.*

Proof. First, we shall show that the right-hand side of (5) is well defined. From conditions (**C2**)-(**C4**), it follows that $f(t)$, $\sigma(t)$ and $\{g_k\}$ are stochastically bounded, and let

$$\max\Big\{\mathbf{E}\|f(t)\|^2_{\mathcal{PC}}, \mathbf{E}\|\sigma(t)\|^2_{\mathcal{PC}}, \mathbf{E}\|g_k\|^2_{L^2(\mathrm{P},\mathcal{H})}\Big\} \leq N_0, \qquad N_0 > 0.$$

In view of Lemma 2.21 and the definition of the norm in $\mathcal{H}_\beta$, we obtain

(6)

$$\begin{aligned}
&\mathbf{E}\|x(t)\|^2_\beta = \mathbf{E}\|A^\beta x(t)\|^2\\
\leq\ & 3\mathbf{E}\Big\|\int_{-\infty}^t (t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)f(s)ds\Big\|^2\\
&+3\mathbf{E}\Big\|\int_{-\infty}^t (t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)\sigma(s)dw(s)\Big\|^2 + 3\mathbf{E}\Big\|\sum_{\tau_k<t}A^\beta\mathcal{T}(t-\tau_k)g_k\Big\|^2\\
\leq\ & 3\alpha^2\mathbf{E}\Big\|\int_{-\infty}^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^\beta S((t-s)^\alpha\theta)f(s)d\theta ds\Big\|^2\\
&+3\alpha^2\mathbf{E}\Big\|\int_{-\infty}^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^\beta S((t-s)^\alpha\theta)\sigma(s)d\theta dw(s)\Big\|^2\\
&+3\sum_{\tau_k<t}\mathbf{E}\Big\|\int_0^\infty \xi_\alpha(\theta)A^\beta S((t-\tau_k)^\alpha\theta)g_k d\theta\Big\|^2\\
\leq\ & 3\alpha^2\mathbf{E}\bigg[\int_{-\infty}^t\int_0^\infty \|\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^\beta S((t-s)^\alpha\theta)f(s)\|d\theta ds\bigg]^2\\
&+3\alpha^2Tr(Q)\mathbf{E}\bigg[\int_{-\infty}^t\int_0^\infty \|\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)A^\beta S((t-s)^\alpha\theta)\sigma(s)\|^2_{L_2^0}d\theta ds\bigg]\\
&+3\sum_{\tau_k<t}\mathbf{E}\bigg[\int_0^\infty \|\xi_\alpha(\theta)A^\beta S((t-\tau_k)^\alpha\theta)g_k\|d\theta\bigg]^2\\
\leq\ & 3\alpha^2K_\beta^2\int_0^\infty \xi_\alpha(\theta)\int_{-\infty}^t \theta^{1-\beta}(t-s)^{-\alpha\beta+\alpha-1}e^{-\lambda\theta(t-s)^\alpha}dsd\theta\\
&\quad\times\int_0^\infty \xi_\alpha(\theta)\int_{-\infty}^t \theta^{1-\beta}(t-s)^{-\alpha\beta+\alpha-1}e^{-\lambda\theta(t-s)^\alpha}\mathbf{E}\|f(s)\|^2dsd\theta\\
&+3\alpha^2K_\beta^2Tr(Q)\int_{-\infty}^t\int_0^\infty \theta^{2(1-\beta)}\xi_\alpha^2(\theta)(t-s)^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta(t-s)^\alpha}\mathbf{E}\|\sigma(s)\|^2_{L_2^0}d\theta ds\\
&+3K_\beta^2\sum_{\tau_k<t}\mathbf{E}\bigg[\int_0^\infty \theta^{-\beta}\xi_\alpha(\theta)(t-\tau_k)^{-\alpha\beta}e^{-\lambda\theta(t-\tau_k)^\alpha}\|g_k\|d\theta\bigg]^2.
\end{aligned}$$

We have, for $\eta = t-s$,

(7)

$$\begin{aligned}
\mathbf{E}\|x(t)\|^2_\beta \leq\ & 3\alpha^2K_\beta^2N_0\int_0^\infty \xi_\alpha(\theta)\int_0^\infty \theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}d\eta d\theta\\
&\times\int_0^\infty \xi_\alpha(\theta)\int_0^\infty \theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}d\eta d\theta\\
&+3\alpha^2K_\beta^2N_0Tr(Q)\int_0^\infty \xi_\alpha^2(\theta)\int_0^\infty \theta^{2(1-\beta)}\eta^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta\eta^\alpha}d\eta d\theta\\
&+3K_\beta^2N_0R(\theta),
\end{aligned}$$

where

$$R(\theta)=\Bigg(\int_0^\infty \xi_\alpha(\theta)\Big[\sum_{0<t-\tau_k\le 1}(\theta(t-\tau_k)^\alpha)^{-\beta}e^{-\lambda\theta(t-\tau_k)^\alpha}$$

$$+\sum_{j=1}^{\infty}\sum_{j<t-\tau_k\le j+1}(\theta(t-\tau_k)^\alpha)^{-\beta}e^{-\lambda\theta(t-\tau_k)^\alpha}\Big]d\theta\Bigg)^2.$$

By a standard calculation, we can deduce that

$$\begin{aligned}&\left(\alpha\int_0^\infty \xi_\alpha(\theta)\int_0^\infty \theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}d\eta d\theta\right)^2\\ &=\left(\frac{1}{\lambda^{1-\beta}}\int_0^\infty \xi_\alpha(\theta)\int_0^\infty (\lambda\theta\eta^\alpha)^{-\beta}e^{-\lambda\theta\eta^\alpha}d\lambda\theta\eta^\alpha d\theta\right)^2\\ &=\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}.\end{aligned}\tag{8}$$

Since $\xi_\alpha^2\in L^1((0,\infty))$, we further derive that

$$\alpha^2\int_0^\infty \xi^2_\alpha(\theta)\int_0^\infty \theta^{2(1-\beta)}\eta^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta\eta^\alpha}d\eta d\theta\le N_1\frac{\Gamma(1-2\beta)}{\lambda^{2-2\beta}},\tag{9}$$

where $N_1=\sup_{\theta\ge 0}\xi^2_\alpha(\theta)$.
By the help of (**C1**) and Lemma 2.17, we have

$$\begin{aligned}R(\theta)&\le\left(\int_0^\infty \xi_\alpha(\theta)\left(\frac{2N}{N_2^\beta}+\frac{2N}{e^\lambda-1}\right)d\theta\right)^2\\ &=4N^2\left(\frac{1}{N_2^\beta}+\frac{1}{e^\lambda-1}\right)^2,\end{aligned}\tag{10}$$

$N_2=\min\{\theta(t-\tau_k)^\alpha, 0<t-\tau_k\le 1\}$.
Recalling (7), from (8)-(10), we obtain
(11)

$$\mathbf{E}\|x(t)\|^2_\alpha\le 3K^2_\beta N_0\left[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\left(\frac{1}{N_2^\beta}+\frac{1}{e^\beta-1}\right)^2\right],$$

and $x(t)\in\mathcal{PC}(\mathbf{R},L^2(\mathbf{P},\mathcal{H}))$.

Let $\epsilon>0$, $\tau\in T_{\epsilon,f,\sigma,g_k}$ and $q\in\widehat{T}_{\epsilon,f,\sigma,g_k}$, $k\in\mathbf{Z}$, where the sets $T_{\epsilon,f,\sigma,g_k}$ and $\widehat{T}_{\epsilon,f,\sigma,g_k}$ are defined as in Lemma 2.18. We have

$$
\begin{aligned}
x(t+\tau)-x(t) = &\Big(\int_{-\infty}^{t+\tau}(t+\tau-s)^{\alpha-1}\mathcal{S}(t+\tau-s)f(s)ds \\
&+\int_{-\infty}^{t+\tau}(t+\tau-s)^{\alpha-1}\mathcal{S}(t+\tau-s)\sigma(s)dw(s)+\sum_{\tau_k<t}\mathcal{T}(t+\tau-\tau_k)g_k\Big) \\
&-\Big(\int_{-\infty}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)f(s)ds \\
&+\int_{-\infty}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)\sigma(s)dw(s)+\sum_{\tau_k<t}\mathcal{T}(t-\tau_k)g_k\Big) \\
= &\int_{-\infty}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)[f(s+\tau)-f(s)]ds \\
&+\int_{-\infty}^{t}(t-s)^{\alpha-1}\mathcal{S}(t-s)[\sigma(s+\tau)-\sigma(s)]d\tilde{w}(s) \\
&+\sum_{\tau_k<t}\mathcal{T}(t-\tau_k)[g_{k+q}-g_k],
\end{aligned}
$$

where $\tilde{w}(s)=w(s+\tau)-w(\tau)$ is also a Brownian motion and has the same distribution as w.

Then

(12)

$$
\begin{aligned}
\mathbf{E}\|x(t+\tau)-x(t)\|_\beta^2 &= \mathbf{E}\|A^\beta(x(t+\tau)-x(t))\|^2 \\
&\leq 3\mathbf{E}\Big\|\int_{-\infty}^{t}(t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)[f(s+\tau)-f(s)]ds\Big\|^2 \\
&\quad +3\mathbf{E}\Big\|\int_{-\infty}^{t}(t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)[\sigma(s+\tau)-\sigma(s)]d\tilde{w}(s)\Big\|^2 \\
&\quad +3\mathbf{E}\Big\|\sum_{\tau_k<t}\mathcal{T}(t-\tau_k)[g_{k+q}-g_k]\Big\|^2 \\
&\leq M_\beta\epsilon,
\end{aligned}
$$

where $|t-\tau_k|>\epsilon$ and

$$
M_\beta = K_\beta\left[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\left(\frac{1}{N_2^\beta}+\frac{1}{e^\beta-1}\right)^2\right].
$$

The last inequality implies that $x(t)$ is a square-mean piecewise almost periodic process, so system (2)-(3) has a square-mean piecewise almost periodic solution. The proof is complete.

□

In order to obtain the existence of square-mean piecewise almost periodic solution to system (1)-(2), we introduce the following conditions:

(**C5**) $-A:\mathcal{D}(A)\subseteq L^2(\mathbf{P},\mathcal{H})\to L^2(\mathbf{P},\mathcal{H})$ is the infinitesimal generator of an exponentially stable analytic semi-group $S(t)$, $t\in\mathbf{R}$, on $L^2(\mathbf{P},\mathcal{H})$.

(**C6**) $F(t,x)\in\mathcal{AP}(\mathbf{R}\times L^2(\mathbf{P},\mathcal{H}_\beta),L^2(\mathbf{P},\mathcal{H}))$ with respect to $t\in\mathbf{R}$ uniformly in $x\in K$, for each compact set $K\subseteq L^2(\mathbf{P},\mathcal{H})$, and there exist constants $\tilde{c}>0$, $0<\kappa<1$, $0<\beta<1$, such that

$$
\mathbf{E}\|F(t_1,x_1)-F(t_2,x_2)\|^2\leq\tilde{c}\Big(|t_1-t_2|^\kappa+\mathbf{E}\|x_1-x_2\|_\beta^2\Big),
$$

where $(t_i, x_i) \in \mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}_\beta)$, $i = 1, 2$.

(**C7**) $\Sigma(t, x) \in \mathcal{AP}(\mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}_\beta), L^2(\mathbf{P}, L_2^0))$ with respect to $t \in \mathbf{R}$ uniformly in $x \in K$, for each compact set $K \subseteq L^2(\mathbf{P}, \mathcal{H})$, and there exist constants $\hat{c} > 0$, $0 < \kappa < 1$, $0 < \beta < 1$, such that

$$\mathbf{E}\|\Sigma(t_1, x_1) - \Sigma(t_2, x_2)\|_{L_2^0}^2 \leq \hat{c}\Big(|t_1 - t_2|^\kappa + \mathbf{E}\|x_1 - x_2\|_\beta^2\Big),$$

$(t_i, x_i) \in \mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}_\beta)$, $i = 1, 2$.

(**C8**) The sequence $\{G_k(x)\}$ is almost periodic in $k \in \mathbf{Z}$ uniformly in $x \in K \subseteq L^2(\mathbf{P}, \mathcal{H})$, and there exist constants $\bar{c} > 0$, $0 < \beta < 1$, such that

$$\mathbf{E}\|G_k(x_1) - G_k(x_2)\|^2 \leq \bar{c}\mathbf{E}\|x_1 - x_2\|_\beta^2,$$

where $x_1, x_2 \in L^2(\mathbf{P}, \mathcal{H}_\beta)$.

Theorem 3.2. *Assume that the conditions* (**C1**), (**C5**)-(**C8**) *are satisfied, then the impulsive fractional stochastic system* (1)-(2) *admits a unique square-mean piecewise almost periodic mild solution.*

Proof. Let B the set of all $x \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$ with discontinuities of the first type at the points τ_k, $k \in \mathbf{Z}$, $\{\tau_k\} \in \mathcal{B}$, satisfying the inequality $\mathbf{E}\|x\|^2 \leq r$, $r > 0$. Obviously, B is a closed set of $\mathcal{AP}(\mathbf{R}, L^2(P, \mathcal{H}))$.

Define the operator Θ in B by

$$\begin{aligned} \Theta x(t) &= \int_{-\infty}^t (t-s)^{\alpha-1} A^\beta \mathcal{S}(t-s) F(s, A^{-\beta}x(s))ds \\ &\quad + \int_{-\infty}^t (t-s)^{\alpha-1} A^\beta \mathcal{S}(t-s)\Sigma(s, A^{-\beta}x(s))dw(s) \\ &\quad + \sum_{\tau_k < t} A^\beta \mathcal{T}(t-\tau_k) G_k(A^{-\beta}x(\tau_k)). \end{aligned} \tag{13}$$

Proceeding in the same way as in the proof of Theorem 3.1, from conditions (**C6**)-(**C8**) and Lemma 2.2 in [8], we can show that Θ is well defined and $\Theta x(t) \in \mathcal{PC}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$.

First, we shall show that $\Theta x(t) \in B$. We define

$$\begin{aligned} \Theta_1 x(t) &= \int_{-\infty}^t (t-s)^{\alpha-1} A^\beta \mathcal{S}(t-s) F(s, A^{-\beta}x(s))ds \\ \Theta_2 x(t) &= \int_{-\infty}^t (t-s)^{\alpha-1} A^\beta \mathcal{S}(t-s)\Sigma(s, A^{-\beta}x(s))dw(s). \end{aligned}$$

Let us show that $\Theta_1 x \in B$, let $x \in B$. Using condition (**C6**), since A^β is closed and $F(t, x) \in \mathcal{AP}(\mathbf{R} \times L^2(\mathbf{P}, \mathcal{H}_\beta), L^2(\mathbf{P}, \mathcal{H}))$, we have from Corollary 2.13 that $A^{-\beta}x \in B$ and $F(\cdot, A^{-\beta}x(\cdot)) \in \mathcal{AP}(\mathbf{R}, L^2(\mathbf{P}, \mathcal{H}))$. Therefore, it follows from Definition 2.9 and Lemma 2.16 that for any $\epsilon > 0$, there exists a relatively dense set T such that for $\tau \in T$ the following property

$$\mathbf{E}\|F(t+\tau, A^{-\beta}x(t+\tau)) - F(t, A^{-\beta}x(t))\|^2 < \frac{\epsilon\lambda^{2(1-\beta)}}{K_\beta^2 \Gamma^2(1-\beta)}$$

hold, satisfying the condition $|t-\tau_k|>\epsilon$, for each $t\in\mathbf{R}$ and $k\in\mathbf{Z}$.

By virtue of Lemma 2.21, we have

$$\begin{aligned}
&\mathbf{E}\|\Theta_1 x(t+\tau)-\Theta_1 x(t)\|^2\\
=\;&\mathbf{E}\Big\|\int_{-\infty}^{t}(t-s)^{\alpha-1}A^{\beta}\mathcal{S}(t-s)[F(s+\tau,A^{-\beta}x(s+\tau))-F(s,A^{-\beta}x(s))]ds\Big\|^2\\
\le\;&\alpha^2K_\beta^2\int_0^\infty\xi_\alpha(\theta)\int_0^\infty\theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}d\eta d\theta\int_0^\infty\xi_\alpha(\theta)\int_0^\infty\theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}\\
&\times\mathbf{E}\|F(t+\tau-\eta,A^{-\beta}x(t+\tau-\eta))-F(t-\eta,A^{-\beta}x(t-\eta))\|^2d\eta d\theta\\
\le\;&\alpha^2K_\beta^2\left(\int_0^\infty\xi_\alpha(\theta)\int_0^\infty\theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^\alpha}d\eta d\theta\right)^2\\
&\times\sup_{t\in R}\mathbf{E}\|F(t+\tau,A^{-\beta}x(t+\tau))-F(t,A^{-\beta}x(t))\|^2\\
=\;&K_\beta^2\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}\sup_{t\in R}\mathbf{E}\|F(t+\tau,A^{-\beta}x(t+\tau))-F(t,A^{-\beta}x(t))\|^2\\
<\;&K_\beta^2\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}\times\frac{\epsilon\lambda^{2(1-\beta)}}{K_\beta^2\Gamma^2(1-\beta)}=\epsilon.
\end{aligned}$$

Hence, $\Theta_1 x(\cdot)\in B$.

Similarly, by using condition (**C7**), since A^β is closed and $\Sigma(t,x)\in\mathcal{AP}(\mathbf{R}\times L^2(\mathbf{P},\mathcal{H}_\beta),L^2(\mathbf{P},L_2^0))$, we have from Corollary 2.13 that $A^{-\beta}x\in B$ and $\Sigma(\cdot,A^{-\beta}x(\cdot))\in\mathcal{AP}(\mathbf{R},L^2(\mathbf{P},L_2^0))$. Therefore, it follows from Definition 2.9 and Lemma 2.16 that for any $\epsilon>0$, there exists a relatively dense set T such that for $\tau\in T$ the following property

$$\mathbf{E}\|\Sigma(t+\tau,A^{-\beta}x(t+\tau))-\Sigma(t,A^{-\beta}x(t))\|_{L_2^0}^2<\frac{\epsilon\lambda^{2-2\beta}}{N_1K_\beta^2Tr(Q)\Gamma(1-2\beta)}$$

hold, satisfying the condition $|t-\tau_k|>\epsilon$, for each $t\in\mathbf{R}$ and $k\in\mathbf{Z}$.

By virtue of Lemma 2.21, for $\tilde{w}(t):=w(t+\tau)-w(\tau)$, we have

$$\begin{aligned}
&\mathbf{E}\|\Theta_2 x(t+\tau)-\Theta_2 x(t)\|^2\\
=\;&\mathbf{E}\Big\|\int_{-\infty}^{t}(t-s)^{\alpha-1}A^{\beta}\mathcal{S}(t-s)[\Sigma(s+\tau,A^{-\beta}x(s+\tau))-\Sigma(s,A^{-\beta}x(s))]d\tilde{w}(s)\Big\|^2\\
\le\;&\alpha^2K_\beta^2Tr(Q)\int_0^\infty\xi_\alpha^2(\theta)\int_0^\infty\theta^{2(1-\beta)}\eta^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta\eta^\alpha}\\
&\times\mathbf{E}\|\Sigma(t+\tau-\eta,A^{-\beta}x(t+\tau-\eta))-\Sigma(t-\eta,A^{-\beta}x(t-\eta))\|_{L_2^0}^2d\eta d\theta\\
\le\;&\alpha^2K_\beta^2Tr(Q)\int_0^\infty\xi_\alpha^2(\theta)\int_0^\infty\theta^{2(1-\beta)}\eta^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta\eta^\alpha}d\eta d\theta\\
&\times\sup_{t\in R}\mathbf{E}\|\Sigma(t+\tau,A^{-\beta}x(t+\tau))-\Sigma(t,A^{-\beta}x(t))\|_{L_2^0}^2\\
\le\;&K_\beta^2Tr(Q)N_1\frac{\Gamma(1-2\beta)}{\lambda^{2-2\beta}}\sup_{t\in R}\mathbf{E}\|\Sigma(t+\tau,A^{-\beta}x(t+\tau))-\Sigma(t,A^{-\beta}x(t))\|_{L_2^0}^2\\
<\;&K_\beta^2Tr(Q)N_1\frac{\Gamma(1-2\beta)}{\lambda^{2-2\beta}}\times\frac{\epsilon\lambda^{2-2\beta}}{N_1K_\beta^2Tr(Q)\Gamma(1-2\beta)}=\epsilon.
\end{aligned}$$

Thus, $\Theta_2 x(\cdot)\in B$. And in view of the above, it is clear that Θ maps B into itself.

Next, we show that Θ is a contracting operator on B. Let $x_1,x_2\in B$. Then , we have

$$\begin{aligned}
&\mathbb{E}\|\Theta x_1(t)-\Theta x_2(t)\|^2\\
\leq\;& 3\mathbb{E}\Big\|\int_{-\infty}^{t}(t-s)^{\alpha-1}A^{\beta}\mathcal{S}(t-s)[F(s,A^{-\beta}x_1(s))-F(s,A^{-\beta}x_2(s))]ds\Big\|^2\\
&+3\mathbb{E}\Big\|\int_{-\infty}^{t}(t-s)^{\alpha-1}A^{\beta}\mathcal{S}(t-s)[\Sigma(s,A^{-\beta}x_1(s))-\Sigma(s,A^{-\beta}x_2(s))]dw(s)\Big\|^2\\
&+3\mathbb{E}\Big\|\sum_{\tau_k<t}A^{\beta}\mathcal{T}(t-\tau_k)[G_k(A^{-\beta}x_1(\tau_k))-G_k(A^{-\beta}x_2(\tau_k))]\Big\|^2\\
\leq\;& 3K_{\beta}^2c_*\Bigg[\alpha^2\bigg(\int_0^{\infty}\xi_{\alpha}(\theta)\int_0^{\infty}\theta^{1-\beta}\eta^{-\alpha\beta+\alpha-1}e^{-\lambda\theta\eta^{\alpha}}d\eta d\theta\bigg)^2+\alpha^2Tr(Q)\int_0^{\infty}\xi_{\alpha}^2(\theta)\\
&\times\int_0^{\infty}\theta^{2(1-\beta)}\eta^{2(\alpha-\alpha\beta-1)}e^{-2\lambda\theta\eta^{\alpha}}d\eta d\theta+R(\theta)\Bigg]\sup_{t\in\mathbb{R}}\mathbb{E}\|x_1(t)-x_2(t)\|^2,
\end{aligned}$$

where $c_*=\max\{\hat{c},\tilde{c},\bar{c}\}>0$ is sufficiently small and $R(\theta)$ is defined as in above.

By following similar arguments like those used in (7), we have

$$\begin{aligned}
&\mathbb{E}\|\Theta x_1(t)-\Theta x_2(t)\|^2\\
\leq\;& 3c_*K_{\beta}^2\Bigg[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}\\
&+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\bigg(\frac{1}{N_2^{\beta}}+\frac{1}{e^{\beta}-1}\bigg)^2\Bigg]\sup_{t\in\mathbb{R}}\mathbb{E}\|x_1(t)-x_2(t)\|^2.
\end{aligned}$$

Therefore, if c_* is chosen in the form

$$c_*\leq\left(3K_{\beta}^2\Bigg[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\bigg(\frac{1}{N_2^{\beta}}+\frac{1}{e^{\beta}-1}\bigg)^2\Bigg]\right)^{-1},$$

we have

$$\begin{aligned}
&\mathbb{E}\|\Theta x_1(t)-\Theta x_2(t)\|^2\\
\leq\;& 3c_*K_{\beta}^2\Bigg[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\bigg(\frac{1}{N_2^{\beta}}+\frac{1}{e^{\beta}-1}\bigg)^2\Bigg]\|x_1-x_2\|_{\infty}^2,
\end{aligned}$$

implies that,

$$\|\Theta x_1-\Theta x_2\|_{\infty}\leq\sqrt{\Lambda}\|x_1-x_2\|_{\infty},$$

$$\Lambda=3c_*K_{\beta}^2\Bigg[\frac{\Gamma^2(1-\beta)}{\lambda^{2(1-\beta)}}+Tr(Q)N_1\frac{\Gamma^2(1-2\beta)}{\lambda^{2(1-\beta)}}+4N^2\bigg(\frac{1}{N_2^{\beta}}+\frac{1}{e^{\beta}-1}\bigg)^2\Bigg].$$

Thus, Θ is a contracting operator on B. So by the contraction principle, we conclude that there exists a unique fixed point x for Θ in B, such that $x=\Theta x$, that is

$$\begin{aligned}
x(t)\;=\;&\int_{-\infty}^{t}(t-s)^{\alpha-1}A^{\beta}\mathcal{S}(t-s)F(s,A^{-\beta}x(s))ds\\
&+\int_{-\infty}^{t}(t-s)^{\alpha-1}A^{\beta}\mathcal{S}(t-s)\Sigma(s,A^{-\beta}x(s))dw(s)\\
&+\sum_{\tau_k<t}A^{\beta}\mathcal{T}(t-\tau_k)G_k(A^{-\beta}x(\tau_k)),
\end{aligned}\tag{14}$$

for all $t\in\mathbb{R}$.

Now, From conditions (**C6**)-(**C8**) and since A^{β} is closed, $G_k\in\mathcal{AP}(\mathbf{Z},L^2(\mathbb{P},\mathcal{H})))$,

$F(t,x) \in \mathcal{AP}(\mathbb{R}\times L^2(\mathbb{P},\mathcal{H}_\beta), L^2(\mathbb{P},\mathcal{H}))$ and $\Sigma(t,x) \in \mathcal{AP}(\mathbb{R}\times L^2(\mathbb{P},\mathcal{H}_\beta), L^2(\mathbb{P}, L_2^0))$, we have from Corollary 2.13 that $F(\cdot, A^{-\beta}x(\cdot)) \in \mathcal{AP}(\mathbb{R}, L^2(\mathbb{P},\mathcal{H}))$, $\Sigma(\cdot, A^{-\beta}x(\cdot)) \in \mathcal{AP}(\mathbb{R}, L^2(\mathbb{P}, L_2^0))$ and $G_k(A^{-\beta}x(\cdot))$ is square-mean almost periodic sequence. Therefore, by Lemma 2.2 in [8] and Remark 2.15, it follows that $F(\cdot, A^{-\beta}x(\cdot))$, $\Sigma(\cdot, A^{-\beta}x(\cdot))$ and $G_k(A^{-\beta}x(\cdot))$ are stochastically bounded, and $\mathbb{E}\|F(t, A^{-\beta}x(t))\|^2$, $\mathbb{E}\|\Sigma(t, A^{-\beta}x(t))\|^2_{L_2^0}$ are uniformly continuous in t. We also get

$$\begin{aligned} A^{-\beta}x(t) &= \int_{-\infty}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}x(s))ds \\ &\quad + \int_{-\infty}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}x(s))dw(s) + \sum_{\tau_k<t}\mathcal{T}(t-\tau_k)G_k(A^{-\beta}x(\tau_k)) \end{aligned}$$

with $A^{-\beta}x$ is stochastically bounded in the sense that for $r>0$, for each $t\in\mathbb{R}$, $\mathbb{E}\|A^{-\beta}x(t)\|^2 \le r$. Hence, $A^{-\beta}x \in B$ is mild solution of the problem (1)-(2).

□

Theorem 3.3. *Assume that the conditions* (**C1**), (**C5**)-(**C8**) *are satisfied, then the impulsive fractional stochastic system* (1)-(2) *has an exponentially stable almost periodic solution.*

Proof. Let $u(t)$ be the solution of the following integral equation

(15)

$$\begin{aligned} u(t) &= \int_{-\infty}^t (t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)F(s, A^{-\beta}u(s))ds \\ &\quad + \int_{-\infty}^t (t-s)^{\alpha-1}A^\beta\mathcal{S}(t-s)\Sigma(s, A^{-\beta}u(s))dw(s) + \sum_{\tau_k<t}A^\beta\mathcal{T}(t-\tau_k)G_k(A^{-\beta}u(\tau_k)). \end{aligned}$$

Consider the equation

(16)

$$^cD_t^\alpha x(t) + Ax = F(t, A^{-\beta}u(t)) + \Sigma(t, A^{-\beta}u(t))\frac{dw(t)}{dt} + \sum_{t_0<\tau_k} G_k(A^{-\beta}u(\tau_k))\delta(t-\tau_k), \quad t\in\mathbb{R}.$$

In view of Theorem 3.2, it follows that there exists a unique square-mean piecewise almost periodic mild solution in the form

(17)

$$\begin{aligned} \psi(t) &= \int_{-\infty}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}u(s))ds \\ &\quad + \int_{-\infty}^t (t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}u(s))dw(s) + \sum_{\tau_k<t}\mathcal{T}(t-\tau_k)G_k(A^{-\beta}u(\tau_k)). \end{aligned}$$

Then

$$\begin{aligned} A^\beta\psi(t) &= \int_{-\infty}^t A^\beta(t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}u(s))ds \\ &\quad + \int_{-\infty}^t A^\beta(t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}u(s))dw(s) \\ &\quad + \sum_{\tau_k<t}A^\beta\mathcal{T}(t-\tau_k)G_k(A^{-\beta}u(\tau_k)) \\ &= u(t). \end{aligned} \tag{18}$$

The last equality shows that $\psi(t) = A^{-\beta}u(t)$ is a solution of (1)-(2), and the uniqueness follows from the uniqueness of the solution of (15) from (14).

Let $u(t) = u(t; t_0, u_0)$ and $v(t) = v(t; t_0, v_0)$ be two solutions of equation (1), then

(19)

$$\begin{aligned} u(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}u(s))ds \\ &\quad + \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}u(s))dw(s) + \sum_{\tau_k<t} \mathcal{T}(t-\tau_k)G_k(A^{-\beta}u(\tau_k)), \end{aligned}$$

(20)

$$\begin{aligned} v(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)F(s, A^{-\beta}v(s))ds \\ &\quad + \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathcal{S}(t-s)\Sigma(s, A^{-\beta}v(s))dw(s) + \sum_{\tau_k<t} \mathcal{T}(t-\tau_k)G_k(A^{-\beta}v(\tau_k)), \end{aligned}$$

and $z(t) = u(t) - v(t)$ is in $\mathcal{AP}(\mathbb{R}, L^2(\mathbb{P}, \mathcal{H}))$,

$$z = \mathcal{T}(t - t_0)z(t_0). \tag{21}$$

The proof follows from (21), the estimates from Lemma 2.21 and the fact that $\iota(t_0 - t) - p(t - t_0) = o(t)$ for $t \to \infty$.

□

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