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Positive Implicative Hyper BCK-Ideals of Hyper BCK-Algebras Under an Interval-Valued Intuitionistic Fuzzy Environment

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Abstract. This paper contains some new theorems related to hyper BCK-ideals positive implicative hyper BCK-ideals of types-1, 2, 3, 4 of hyper BCK-algebras (HBCKA) under an interval-valued intuitionistic fuzzy environment. Henceforth, the connection between these ideas and their relevant characteristics is discussed.

1. Introduction

In 1966, Imai and Iséki [11] coined the notion of BCK-algebras (BCKAs) by extending the concepts of set-theoretic difference and propositional calculus. Research works on BCKAs have been progressing rapidly since their inception. Marty [13] invented hyperstructures theory, also re-knocked as multi-algebras. Jun et al. [12] introduced the notion of hyper BCKAs (HBCKAs) as an extension of BCKAs and discussed their characteristics. The notions of fuzzy positive implicative hyper BCK-ideals (FPIHBCKIs) of types-1, 2, 3, 4 were proposed by Bakhshi et al. [5]. Meanwhile, the fuzzy set [20] was extended to the intuitionistic fuzzy set (IFS) by Atanassov [1,2]

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by employing both membership and non-membership degrees for each object. Atanassov [3] defined new operations over IFSs. Later on, Atanassov and Gargov [4] presented the notion of interval-valued IFSs (IVIFSs) using interval-valued membership and non-membership degrees. Satyanarayana et al. [18] introduced the notions of IVIFS-hyper (weak, s-weak, strong) BCKIs of BCKAs, and investigated the connections between those concepts. Borzooei has studied hyper BCK-algebras extensively, which can be seen in [6–10]. After that, Ramesh et al. [14–17] applied different algebras and IVIFS concepts. In [19], Satyanarayana et al. introduced the notions of IVIFPIHCKIs of types-$1, 2, \ldots, 8$ of HBCKAs. Additionally, we discover the connections between IVIFS-hyper (weak, s-weak, strong) BCKIs of BCKAs, IVIF-(weak, strong, reflexive) HBCKIs and IVIFPIHBCKIs of types-$1, 2, \ldots, 8$ of HBCKAs and the associated properties are explored.

This paper establishes characterizations of IVIFHBCKIs and IVIFPIHBCKIs of types-$1, 2, 3, 4$, and a few of its related characteristics are demonstrated.

Let H be a set with a hyper operation that is non-empty and, $\circ$ is a mapping from $H \times H$ into $P^*(H) = P(H) \setminus \{\emptyset\}$. Using any two subsets T and R of H, denoted by $T \circ R$ the set, $\bigcup_{a\in T, b\in R} a \circ b$. We shall use $j_1 \circ j_2$ instead of $j_1 \circ \{j_2\}, \{j_1\} \circ j_2$, or $\{j_1\} \circ \{j_2\}$.

2. Preliminaries

Definition 2.1. *[12] In an HBCKA, a non-null set H is considered along with a hyper operation $\circ$ and a constant 0 obeying the axioms mentioned below:*
(HK-1) $(j_1 \circ j_3) \circ (j_2 \circ j_3) \ll j_1 \circ j_2$,
(HK-2) $(j_1 \circ j_2) \circ j_3 = (j_1 \circ j_3) \circ j_2$,
(HK-3) $j_1 \circ H \ll \{j_1\}$,
(HK-4) $j_1 \ll j_2$ *and* $j_2 \ll j_1 \Rightarrow j_1 = j_2$, *for all* $j_1, j_2, j_3 \in H$.
We can define a relation "$\ll$" on H by letting $j_1 \ll j_2$ if and only if $0 \in j_1 \circ j_2$ and for every $T, R \subseteq H, T \ll R$ is defined by $\forall a \in T, \exists b \in R$ such that $a \ll b$. In such case, we call "$\ll$" the hyper order in H.

Note that the condition (HK-3) is equivalent to the condition:
(P_1) $j_1 \circ j_2 \ll \{j_1\}$, for all $j_1, j_2 \in H$.
In any HBCKA, the following hold:
(P_2) $j_1 \circ 0 \ll \{j_1\}, 0 \circ j_1 \ll \{j_1\}$ and $0 \circ 0 \ll \{0\}$,
(P_3) $(T \circ R) \circ Q = (T \circ Q) \circ R, T \circ R \ll T$ and $0 \circ T \ll \{0\}$,
(P_4) $0 \circ 0 = \{0\}$,
(P_5) $0 \ll j_1$,
(P_6) $j_1 \ll j_1$,
(P_7) $T \ll T$,
(P_8) $T \subseteq R \Rightarrow T \ll R$,
(P_9) $0 \circ j_1 = \{0\}$,
(P_{10}) $j_1 \circ 0 = \{j_1\}$,
(P_{11}) $0 \circ T = \{0\}$,

(P_{12}) $T \ll \{0\} \Rightarrow T = \{0\}$,
(P_{13}) $T \circ R \ll T$,
(P_{14}) $j_1 \in j_1 \circ 0$,
(P_{15}) $j_1 \circ 0 \ll \{j_2\} \Rightarrow j_1 \ll j_2$,
(P_{16}) $j_2 \ll j_3 \Rightarrow j_1 \circ j_3 \ll j_1 \circ j_2$,
(P_{17}) $j_1 \circ j_2 = \{0\} \Rightarrow (j_1 \circ j_3) \circ (j_2 \circ j_3) = \{0\}$ and $j_1 \circ j_3 \ll j_2 \circ j_3$,
(P_{18}) $T \circ \{0\} = \{0\} \Rightarrow T = 0$, for all $j_1, j_2, j_3 \in H$ and for all non-empty subsets T, R and Q of H.

Let M be a non-empty subset of an HBCKA H and $0 \in M$. Then M is called an HBCK-subalgebra of H if $j_1 \circ j_2 \subseteq M$, for all $j_1, j_2 \in M$, weak HBCKI of H if $j_1 \circ j_2 \subseteq M$ and $j_2 \in M$ imply $j_1 \in M$, for all $j_1, j_2 \in H$, an HBCKI of H if $j_1 \circ j_2 \ll M$ and $j_2 \in M$ imply $j_1 \in M$, for all $j_1, j_2 \in H$, a SHBCKI of H if $(j_1 \circ j_2) \cap M \neq \emptyset$ and $j_2 \in M$ imply $j_1 \in M$, for all $j_1, j_2 \in H$, M is said to be reflexive if $j_1 \circ j_1 \subseteq M$, for all $j_1 \in H$, S-reflexive if it satisfies $(j_1 \circ j_2) \cap M \neq \emptyset$ implies $j_1 \circ j_2 \ll M$, for all $j_1, j_2 \in H$, closed if $j_1 \ll j_2$ and $j_2 \in M$ imply $j_1 \in M$, for all $j_1 \in H$. It is easy to see that every S-reflexive subset of H is reflexive.

Let M be a non-empty subset of H and $0 \in M$. Then M is said to be a positive implicative hyper BCK-ideal (PIHBCKI) of

(i) type-1 if $(j_1 \circ j_2) \circ j_3 \subseteq M$ and $j_2 \circ j_3 \subseteq M \Rightarrow j_1 \circ j_3 \subseteq M$,
(ii) type-2 if $(j_1 \circ j_2) \circ j_3 \ll I$ and $j_2 \circ j_3 \subseteq M \Rightarrow j_1 \circ j_3 \subseteq M$,
(iii) type-3 if $(j_1 \circ j_2) \circ j_3 \ll M$ and $j_2 \circ j_3 \ll M \Rightarrow j_1 \circ j_3 \subseteq M$,
(iv) type-4 if $(j_1 \circ j_2) \circ j_3 \subseteq M$ and $j_2 \circ j_3 \ll I \Rightarrow j_1 \circ j_3 \subseteq M$,
(v) type-5 if $(j_1 \circ j_2) \circ j_3 \subseteq M$ and $j_2 \circ j_3 \subseteq M \Rightarrow j_1 \circ j_3 \ll M$,
(vi) type-6 if $(j_1 \circ j_3) \circ j_3 \ll M$ and $j_2 \circ j_3 \ll M \Rightarrow j_1 \circ j_3 \ll M$,
(vii) type-7 if $(j_1 \circ j_2) \circ j_3 \subseteq M$ and $j_2 \circ j_3 \ll I \Rightarrow j_1 \circ j_3 \ll M$,
(viii) type-8 if $(j_1 \circ j_2) \circ j_3 \ll M$ and $j_2 \circ j_3 \subseteq M \Rightarrow j_1 \circ j_3 \ll M$, for all $j_1, j_2, j_3 \in H$.

A mapping $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T) : L \to D[0,1] \times D[0,1]$ is called an IVIFS in L if $0 \leq \phi_T^+(j_1) + \psi_T^+(j_1) \leq 1$ and $0 \leq \phi_T^-(j_1) + \psi_T^-(j_1) \leq 1$, for all $j_1 \in L$ (that is, $T^+ = (\phi_T^+, \psi_T^+)$ and $T^- = (\phi_T^-, \psi_T^-)$ are IFSs), where the mappings $\tilde{\phi}_T(j_1) = [\phi_T^-(j_1), \phi_T^+(j_1)] : L \to D[0,1]$ and $\tilde{\psi}_T(j_1) = [\psi_T^-(j_1), \psi_T^+(j_1)] : L \to D[0,1]$ represent the degree of membership (namely $\tilde{\psi}_T(j_1)$) each component $j_1 \in L$ to T respectively, where $D[0,1]$ is the set of all closed sub-intervals of $[0,1]$.

3. Interval-valued intuitionistic fuzzy hyper BCK-ideals of hyper BCK-algebras

Definition 3.1. *[18] An IVIFS $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ in H is called an interval-valued intuitionistic fuzzy hyper BCK-ideal (IVIFHBCKI) of H if it fulfils:*

(k1) $j_1 \ll j_2 \Rightarrow \tilde{\phi}_T(j_1) \geq \tilde{\phi}_T(j_2)$ *and* $\tilde{\psi}_T(j_1) \leq \tilde{\psi}_T(j_2)$,
(k2) $\tilde{\phi}_T(j_1) \geq \min\{\inf_{r \in j_1 \circ j_2}\{\tilde{\phi}_T(r)\}, \tilde{\phi}_T(j_2)\}$,
(k3) $\tilde{\psi}_T(j_1) \leq \max\{\sup_{t \in j_1 \circ j_2}\{\tilde{\psi}_T(t)\}, \tilde{\psi}_T(j_2)\}$, *for all* $j_1, j_2 \in H$.

Definition 3.2. *[18] An IVIFS $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ in H is called an IVIF-strong HBCKI (IVIFSHBCKI) of H if it satisfies:*

(i) $\inf_{r\in j_1\circ j_2}\{\tilde{\phi}_T(r)\} \geq \tilde{\phi}_T(j_1) \geq \min\{\sup_{t\in j_1\circ j_2}\{\tilde{\phi}_T(t)\}, \tilde{\phi}_T(j_2)\}$,

(ii) $\sup_{v\in j_1\circ j_2}\{\tilde{\psi}_T(v)\} \leq \tilde{\psi}_T(j_1) \leq \max\{\inf_{s\in j_1\circ j_2}\{\tilde{\psi}_T(s)\}, \tilde{\psi}_T(j_2)\}$, *for all* $j_1, j_2 \in H$.

Definition 3.3. *[18] An IVIFS $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ in H is known as an IVIF s-weak HBCKI (IVIFSWHBCKI) of H if it satisfies:*

(s1) $\tilde{\phi}_T(0) \geq \tilde{\phi}_T(j_1)$ *and* $\tilde{\psi}_T(0) \leq \tilde{\psi}_T(j_1)$, *for all* $j_1 \in H$,

(s2) for every $j_1, j_2 \in H$, *there exist* $a, b \in j_1 \circ j_2$ *such that*

$$\tilde{\phi}_T(j_1) \geq \min\{\tilde{\phi}_T(r), \tilde{\phi}_T(j_2)\} \text{ and } \tilde{\psi}_T(j_1) \leq \max\{\tilde{\psi}_T(t), \tilde{\psi}_T(j_2)\}.$$

Definition 3.4. *[18] An IVIFS $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ in H is called an IVIF-weak HBCKI (IVIFWHBCKI) of H if it satisfies:*

(i) $\tilde{\phi}_T(0) \geq \tilde{\phi}_T(j_1) \geq \min\{\inf_{r\in j_1\circ j_2}\{\tilde{\phi}_T(r)\}, \tilde{\phi}_T(j_2)\}$,

(ii) $\tilde{\psi}_T(0) \leq \tilde{\psi}_T(j_1) \leq \max\{\sup_{t\in j_1\circ j_2}\{\tilde{\psi}_T(t)\}, \tilde{\psi}_T(j_2)\}$, *for all* $j_1, j_2 \in H$.

Definition 3.5. *[18] An IVIFS $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ in H is called an IVIF-HBCK-subalgebra (IVIFHBCKSA) of H if it satisfies:*

(i) $\inf_{r\in j_1\circ j_2}\{\tilde{\phi}_T(r)\} \geq \min\{\tilde{\phi}_T(j_1), \tilde{\phi}_T(j_2)\}$,

(ii) $\sup_{t\in j_1\circ j_2}\{\tilde{\psi}_T(t)\} \leq \max\{\tilde{\psi}_T(j_1), \tilde{\psi}_T(j_2)\}$, *for all* $j_1, j_2 \in H$.

Definition 3.6. *[18] An IVIFS $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ in H is said to satisfy **"sup-inf"** a property if any subset D of H, there exist $t_0, s_0 \in D$ such that*

$$\tilde{\phi}_T(t_0) = \sup_{j\in D}\{\tilde{\phi}_T(j)\} \text{ and } \tilde{\psi}_T(s_0) = \inf_{j\in D}\{\tilde{\psi}_T(j)\}.$$

Theorem 3.1. *Let $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ be an IVIFS in H, then the following statements hold:*

(i) $\tilde{T}$ is an IVIFHBCKI of H if and only if for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$, $U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are HBCKIs of H,

(ii) if $\tilde{T}$ is an IVIFSHBCKI of H, then for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$, $U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are SHBCKIs of H,

*(iii) if $\tilde{T}$ is an IVIFS of H which satisfies the **sup-inf** property and for every $\tilde{s_1}, \tilde{t_1} \in D[0,1]$, $U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are SHBCKIs of H, then $\tilde{T}$ is an IVIFSHBCKI of H.*

Proof. The proof is straightforward. □

4. Interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideals of hyper BCK-algebras

Definition 4.1. *[14] Let $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ be an IVIFS of H, and $\tilde{\phi}_T(0) \geq \tilde{\phi}_T(j)$ and $\tilde{\psi}_T(0) \leq \tilde{\psi}_T(j)$, for all $j \in H$. Then $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is said to be an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal (IVIFPIHBCKI) of*

(i) type-1 if for all $\mathfrak{q} \in j_1 \circ j_3$, $\tilde{\phi}_T(\mathfrak{q}) \geq \min\{\inf_{a_1\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(a_1)\}, \inf_{a_2\in j_2\circ j_3}\{\tilde{\phi}_T(a_2)\}\}$ *and* $\tilde{\psi}_T(\mathfrak{q}) \leq \max\{\sup_{a_3\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(a_3)\}, \sup_{a_4\in j_2\circ j_3}\{\tilde{\psi}_T(a_4)\}\}$,

(ii) type-2 if for all $\mathfrak{q} \in j_1 \circ j_3$, $\tilde{\phi}_T(\mathfrak{q}) \geq \min\{\sup_{a_1 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(a_1)\}, \inf_{a_2 \in j_2 \circ j_3}\{\tilde{\phi}_T(a_2)\}\}$ *and* $\tilde{\psi}_T(\mathfrak{q}) \leq \max\{\inf_{a_3 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(a_3)\}, \sup_{a_4 \in j_2 \circ j_3}\{\tilde{\psi}_T(a_4)\}\}$,

(iii) type-3 if for all $\mathfrak{q} \in j_1 \circ j_3$, $\tilde{\phi}_T(\mathfrak{q}) \geq \min\{\sup_{a_1 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(a_1)\}, \sup_{a_2 \in j_2 \circ j_3}\{\tilde{\phi}_T(a_2)\}\}$ *and* $\tilde{\psi}_T(\mathfrak{q}) \leq \max\{\inf_{a_3 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(a_3)\}, \inf_{a_4 \in j_2 \circ j_3}\{\tilde{\psi}_T(a_4)\}\}$,

(iv) type-4 if for all $\mathfrak{q} \in j_1 \circ j_3$, $\tilde{\phi}_T(\mathfrak{q}) \geq \min\{\inf_{a_1 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(a_1)\}, \sup_{a_2 \in j_2 \circ j_3}\{\tilde{\phi}_T(a_2)\}\}$ *and* $\tilde{\psi}_T(\mathfrak{q}) \leq \max\{\sup_{a_3 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(a_3)\}, \inf_{a_4 \in j_2 \circ j_3}\{\tilde{\psi}_T(a_4)\}\}$, *for all* $j_1, j_2, j_3 \in H$.

Definition 4.2. *[14] Suppose* $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ *denotes an IVIFS of H. Then* $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ *is said to be an IVIFPIHBCKI of*

(i) type-5 if $\exists \mathfrak{q} \in j_1 \circ j_3$ *such that* $\tilde{\phi}_T(\mathfrak{q}) \geq \min\{\inf_{a_1 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(a_1)\}, \inf_{a_2 \in j_2 \circ j_3}\{\tilde{\phi}_T(a_2)\}\}$ *and* $\tilde{\psi}_T(\mathfrak{q}) \leq \max\{\sup_{a_3 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(a_3)\}, \sup_{a_4 \in j_2 \circ j_3}\{\tilde{\psi}_T(a_4)\}\}$,

(ii) type-6 if $\exists \mathfrak{q} \in j_1 \circ j_3$ *such that* $\tilde{\phi}_T(\mathfrak{q}) \geq \min\{\sup_{a_2 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(a_2)\}, \sup_{a_2 \in j_2 \circ j_3}\{\tilde{\phi}_T(a_2)\}\}$ *and* $\tilde{\psi}_T(\mathfrak{q}) \leq \max\{\inf_{a_3 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(a_3)\}, \inf_{a_4 \in j_2 \circ j_3}\{\tilde{\psi}_T(a_4)\}\}$,

(iii) type-7 if $\exists \mathfrak{q} \in j_1 \circ j_3$ *such that* $\tilde{\phi}_T(\mathfrak{q}) \geq \min\{\inf_{a_1 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(a_1)\}, \sup_{a_2 \in j_2 \circ j_3}\{\tilde{\phi}_T(a_2)\}\}$ *and* $\tilde{\psi}_T(\mathfrak{q}) \leq \max\{\sup_{a_3 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(a_3)\}, \inf_{a_4 \in j_2 \circ j_3}\{\tilde{\psi}_T(a_4)\}\}$,

(iv) type-8 if $\exists \mathfrak{q} \in j_1 \circ j_3$ *such that* $\tilde{\phi}_T(\mathfrak{q}) \geq \min\{\sup_{a_1 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(a_1)\}, \inf_{a_2 \in j_2 \circ j_3}\{\tilde{\phi}_T(a_2)\}\}$ *and* $\tilde{\psi}_T(\mathfrak{q}) \leq \max\{\inf_{a_3 \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(a_3)\}, \sup_{a_4 \in j_2 \circ j_3}\{\tilde{\psi}_T(a_4)\}\}$, *for all* $j_1, j_2, j_3 \in H$.

Theorem 4.1. *Suppose* $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ *denotes an IVIFS in H, then*

(i) $\tilde{T}$ *is an IVIFPIHBCKI of type-1 if and only if for all* $\tilde{s_1}, \tilde{t_1} \in D[0,1], L(\tilde{\psi}_T, \tilde{t_1}) \neq \emptyset \neq U(\tilde{\phi}_T, \tilde{s_1})$ *are PIHBCKIs of type-1,*

(ii) $\tilde{T}$ *is an IVIFPIHBCKI of type-2 (type-3) if and only if for all* $\tilde{s_1}, \tilde{t_1} \in D[0,1], U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ *are PIHBCKIs of type-2 (type-3),*

(iii) if for all $\tilde{s_1}, \tilde{t_1} \in D[0,1], U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ *are PIHBCKIs of type-2 (type-3) and* $\tilde{T}$ *fulfils the **sup-inf** property, then* $\tilde{T}$ *is an IVIFPIBCKI of type-2 (type-3),*

(iv) if $\tilde{T}$ *is an IVIF-closed and IVIFPIHBCKI of type-4, then for all* $\tilde{s_1}, \tilde{t_1} \in D[0,1], U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ *are PIHBCKIs of type-4,*

(v) if $\tilde{T}$ *is an IIVIF-closed, fulfils the **sup-inf** property and for all* $\tilde{s_1}, \tilde{t_1} \in D[0,1]$, $U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ *are reflexive PIHBCKIs of type-4, then* $\tilde{T}$ *is an IVIFPIHBCKI of type-4.*

Proof. (i) Assume $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFPIHBCKI of type-1. Let $j_1, j_2, j_3 \in H$ and $\tilde{s_1} \in D[0,1]$ be such that $(j_1 \circ j_2) \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$ and $j_2 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Then $r, t \in U(\tilde{\phi}_T, \tilde{s_1})$, for all $r \in (j_1 \circ j_2) \circ j_3$ and $t \in j_2 \circ j_3$. Thus $\tilde{\phi}_T(r) \geq \tilde{s_1}$ and $\tilde{\phi}_T(t) \geq \tilde{s_1}$, for all $r \in (j_1 \circ j_2) \circ j_3$ and $t \in j_2 \circ j_3$ imply that $\inf_{r \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(r)\} \geq \tilde{s_1}$ and $\inf_{t \in j_2 \circ j_3}\{\tilde{\phi}_T(t)\} \geq \tilde{s_1}$. Thus by hypothesis, for all $u \in j_1 \circ j_3$, $\tilde{\phi}_T(u) \geq \min\{\inf_{r \in (j_1 \circ j_2) \circ j_3}\{\tilde{\phi}_T(r)\}, \inf_{t \in j_2 \circ j_3}\{\tilde{\phi}_T(t)\}\} \geq \min\{\tilde{s_1}, \tilde{s_1}\} = \tilde{s_1}$ *imply* $j_1 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Let $j_1, j_2, j_3 \in H$ be such that $(j_1 \circ j_2) \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$ and $j_2 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. Then $l, m \in L(\tilde{\psi}_T, \tilde{t_1})$, for all $l \in (j_1 \circ j_2) \circ j_3$ and $l \in j_2 \circ j_3$ imply $\tilde{\psi}_T(l) \leq \tilde{t_1}$ and $\tilde{\psi}_T(m) \leq \tilde{t_1}$, for all $l \in (j_1 \circ j_2) \circ j_3$ and $m \in j_2 \circ j_3$, imply that $\sup_{l \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(l)\} \leq \tilde{t_1}$ and $\sup_{d \in j_2 \circ j_3}\{\tilde{\psi}_T(d)\} \leq \tilde{t_1}$. Thus by hypothesis, for all $v \in j_1 \circ j_3$, $\tilde{\psi}_T(v) \leq \max\{\sup_{l \in (j_1 \circ j_2) \circ j_3}\{\tilde{\psi}_T(l)\}, \sup_{d \in j_2 \circ j_3}\{\tilde{\psi}_T(d)\}\} \leq \max\{\tilde{t_1}, \tilde{t_1}\} = \tilde{t_1}$ implies $j_1 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. Thus $U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are PIHBCKIs of type-1, for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$.

Conversely, let for all $\tilde{s_1}, \tilde{t_1} \in D[0,1], U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are PIHBCKIs of type-1 and put $\tilde{s_1} = \min\{\inf_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\}, \inf_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\}\}$. Then $\inf_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\} \geq \tilde{s_1}$ and $\inf_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\} \geq \tilde{s_1}$. So, $\tilde{\phi}_T(r) \geq \tilde{s_1}$ and $\tilde{\phi}_T(t) \geq \tilde{s_1}$, for all $r \in (j_1 \circ j_2) \circ j_3$ and $t \in j_2 \circ j_3$. Hence, $r \in U(\tilde{\phi}_T, \tilde{s_1})$ and $t \in U(\tilde{\phi}_T, \tilde{s_1})$, for all $r \in (j_1 \circ j_2) \circ j_3$ and $t \in j_2 \circ j_3$. That is $(j_1 \circ j_2) \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$ and $j_2 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$ and so by hypothesis, $j_1 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Thus for all $u \in j_1 \circ j_3$, $\tilde{\phi}_T(u) \geq \tilde{s_1} = \min\{\inf_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\}, \inf_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\}\}$. Put, $\tilde{t_1} = \max\{\sup_{l\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(l)\}, \sup_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\}\}$ implies $\tilde{t_1} \geq \sup_{l\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(l)\}$ and $\tilde{t_1} \geq \sup_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\}$. So, $\tilde{\psi}(m) \leq \tilde{t_1}$, for all $l \in (j_1 \circ j_2) \circ j_3$ and $m \in (j_2 \circ j_3)$ imply $l \in L(\tilde{\psi}_T, \tilde{t_1})$ and $m \in L(\tilde{\psi}_T, \tilde{t_1})$. Hence, $(j_1 \circ j_2) \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$ and $j_2 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. By hypothesis, $j_1 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. Thus for all $v \in j_1 \circ j_3$, $\tilde{\psi}_T(v) \leq \tilde{t_1} = \max\{\sup_{l\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(l)\}, \sup_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\}\}$. Thus $\tilde{T}$ is an IVIFPIHBCKI of type-1.

(ii) Suppose $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFPIHBCKI of type-2. Let $j_1, j_2, j_3 \in H$ and $\tilde{s_1} \in D[0,1]$ be such that $(j_1 \circ j_2) \circ j_3 \ll U(\tilde{\phi}_T, \tilde{s_1})$ and $j_2 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Then for all $r \in (j_1 \circ j_2) \circ j_3$, there exists $p \in U(\tilde{\phi}_T, \tilde{s_1})$ such that $r \ll p$. By Corollary 3.10 [14], we have $\tilde{\phi}_T(r) \geq \tilde{\phi}_T(p) \geq \tilde{s_1}$. Thus $\tilde{\phi}_T(r) \geq \tilde{s_1}$, for all $r \in (j_1 \circ j_2) \circ j_3$, so $\sup_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\} \geq \tilde{s_1}$. Moreover, since $j_2 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$ implies $\tilde{\phi}_T(t) \geq \tilde{s_1}$, for all $t \in j_2 \circ j_3$. Thus $\inf_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\} \geq \tilde{s_1}$ and for all $u \in j_1 \circ j_3, \tilde{\phi}_T(u) \geq \min\{\sup_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\}, \inf_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\}\} \geq \min\{\tilde{s_1}, \tilde{s_1}\} = \tilde{s_1}$. Therefore, $\tilde{\phi}_T(u) \geq \tilde{s_1}$, so $u \in U(\tilde{\phi}_T, \tilde{s_1})$, for all $u \in j_1 \circ j_3$. Thus $j_1 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Suppose $(j_1 \circ j_2) \circ j_3 \ll L(\tilde{\psi}_T, \tilde{t_1})$ and $j_2 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. Then for all $l \in (j_1 \circ j_2) \circ j_3$, there exists $q \in L(\tilde{\psi}_T, \tilde{t_1})$ such that $l \ll q$. By Corollary 3.10 [14], we have $\tilde{\psi}_T(l) \leq \tilde{\psi}_T(q) \leq \tilde{t_1}$ implies $\tilde{\psi}_T(l) \leq \tilde{t_1}$, for all $l \in (j_1 \circ j_2) \circ j_3$. Thus $\inf_{l\in(j_1\circ j_2)\circ j_3} \tilde{\psi}_T(l) \leq \tilde{t_1}$. Since $j_2 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$, we have $m \in L(\tilde{\psi}_T, \tilde{t_1})$, for all $m \in j_2 \circ j_3$. So, $\tilde{\psi}_T(m) \leq \tilde{t_1}$, for all $m \in j_2 \circ j_3$. This implies that $\sup_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\} \leq \tilde{t_1}$. Thus for all $v \in j_1 \circ j_3$, $\tilde{\psi}_T(v) \leq \max\{\inf_{l\in(j_1\circ j_2)\circ j_3} \tilde{\psi}_T(l), \sup_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\}\} \leq \tilde{t_1}$. Therefore, $j_1 \circ j_2 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. Thus $U(\tilde{\phi}_T, \tilde{s_1})$ and $L(\tilde{\psi}_T, \tilde{t_1})$ are PIHBCKIs of type-2, for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$. Similarly, we can prove for type-3.

(iii) Let $j_1, j_2, j_3 \in H$. Put $\tilde{s_1} = \min\{\sup_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\}, \inf_{t\in j_2\circ j_3} \tilde{\phi}_T(t)\}$. Since $\tilde{\phi}_T$ satisfies the **sup** property, then there exists $a_0 \in (j_1 \circ j_2) \circ j_3$ such that $\tilde{\phi}_T(a_0) = \sup_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\} \geq \tilde{s_1}$ and so $a_0 \in U(\tilde{\phi}_T, \tilde{s_1})$. Hence, $((j_1 \circ j_2) \circ j_3) \cap U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset$, since by Theorem 2.8 (ii) [5], $U(\tilde{\phi}_T, \tilde{s_1})$ is an HBCKI of H. By hypothesis and Theorem 3.5 (ii) [5], we have $(j_1 \circ j_2) \circ j_3 \ll U(\tilde{\phi}_T, \tilde{s_1})$. Moreover, for all $u \in j_2 \circ j_3, \tilde{\phi}_T(u) \geq \inf_{t\in j_2\circ j_3} \tilde{\phi}_T(t) \geq \tilde{s_1}$. Then $j_2 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Since $U(\tilde{\phi}_T, \tilde{s_1})$ is a PIHBCKI of type-2, we have $j_1 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. This implies that for all $v \in j_1 \circ j_3$, $\tilde{\phi}_T(v) \geq \tilde{s_1} = \min\{\sup_{a\in(j_1\circ j_2)\circ z}\{\tilde{\phi}_T(a)\}, \inf_{b\in j_2\circ j_3}\{\tilde{\phi}_T(b)\}\}$. Since $L(\tilde{\psi}_T, \tilde{t_1})$ is a PIHBCKI of type-2 and for $j_1, j_2, j_3 \in H$, put $\tilde{t_1} = \max\{\inf_{c\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(c)\}, \sup_{d\in j_2\circ j_3}\{\tilde{\psi}_T(d)\}\}$. Since $\tilde{\psi}_T$ satisfies the **inf** property, there exists $c_0 \in (j_1 \circ j_2) \circ j_3$ such that $\tilde{\psi}_T(c_0) = \inf_{a\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(a)\} \leq \tilde{t_1}$ and so $c_0 \in L(\tilde{\psi}_T, \tilde{t_1})$. Hence, $((j_1 \circ j_2) \circ j_3 \cap L(\tilde{\psi}_T, \tilde{s_1}) \neq \emptyset$. By Theorem 2.8 (ii) [5], $L(\tilde{\psi}_T, \tilde{t_1})$ is an HBCKI of H. By hypothesis and Theorem 3.5 (ii) [5], we have $(j_1 \circ j_2) \circ j_3 \ll L(\tilde{\psi}_T, \tilde{t_1})$. Moreover, for all $u' \in j_2 \circ j_3$, $\tilde{\psi}_T(u') \leq \sup_{d\in j_2\circ j_3}\{\tilde{\psi}_T(d)\} \leq \tilde{t_1}$ implies $j_2 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. Since $L(\tilde{\psi}_T, \tilde{t_1})$ is a PIBCKI of type-2, we have $j_1 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. This implies for all $v' \in j_1 \circ j_3$,

$\tilde{\psi}_T(v') \leq \tilde{t_1} = \max\{\inf_{c\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(c)\}, \sup_{d\in j_2\circ j_3}\{\tilde{\psi}_T(d)\}\}$. Thus $\tilde{T}$ is an IVIFPIHBCKI of type-2. Similarly, we can prove for type-3.

(iv) Suppose $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFPIHBCKI of type-4 and IVIF-closed. Let $\tilde{s_1}, \tilde{t_1} \in D[0,1]$. Let $j_1, j_2, j_3 \in H$ be such that $(j_1 \circ j_2) \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$ and $j_2 \circ j_3 \ll U(\tilde{\phi}_T, \tilde{s_1})$. Then $r \in U(\tilde{\phi}_T, \tilde{s_1})$, for all $r \in (j_1 \circ j_2) \circ j_3$ and for all $t \in j_2 \circ j_3$, there exists $p \in U(\tilde{\phi}_T, \tilde{s_1})$ such that $t \ll p$. Since $\tilde{\phi}_T$ is fuzzy closed, we have $\tilde{\phi}_T(t) \geq \tilde{\phi}_T(p) \geq \tilde{s_1}$, for all $t \in j_2 \circ j_3$, so $\sup_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\} \geq \tilde{s_1}$. Since $a \in U(\tilde{\phi}_T, \tilde{s_1})$, for all $r \in (j_1 \circ j_2) \circ j_3$, we have $\tilde{\phi}_T(r) \geq \tilde{s_1}$, for all $r \in (j_1 \circ j_2) \circ j_3$. Therefore, $\inf_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\} \geq \tilde{s_1}$. Thus for all $u \in j_1 \circ j_3, \tilde{\phi}_T(u) \geq \min\{\inf_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\}, \sup_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\}\} \geq \tilde{s_1}$. Therefore, $j_1 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Let $j_1, j_2, j_3 \in H$ be such that $(j_1 \circ j_2) \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$ and $j_2 \circ j_3 \ll L(\tilde{\psi}_T, \tilde{t_1})$. Then $l \in L(\tilde{\psi}_T, \tilde{t_1})$, for all $l \in (j_1 \circ j_2) \circ j_3$ and for all $m \in j_2 \circ j_3$, there exists $q \in L(\tilde{\psi}_T, \tilde{t_1})$ such that $m \ll q$. Since $\tilde{\psi}_T$ is anti-fuzzy closed, we have $\tilde{\psi}_T(m) \leq \tilde{\phi}_T(q) \leq \tilde{t_1}$, for all $m \in j_2 \circ j_3$, so $\inf_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\} \leq \tilde{t_1}$. Since $l \in L(\tilde{\psi}_T, \tilde{t_1})$, for all $l \in (j_1 \circ j_2) \circ j_3$, we have $\tilde{\phi}_T(l) \leq \tilde{t_1}$, for all $l \in (j_1 \circ j_2) \circ j_3$. Therefore, $\sup_{l\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(l)\} \leq \tilde{t_1}$. Thus for all $v \in j_1 \circ j_3$, $\tilde{\psi}_T(v) \leq \max\{\sup_{l\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(l)\}, \inf_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\}\} \leq \tilde{t_1}$. Therefore, $j_1 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{s_1})$. Thus $U(\tilde{\phi}_T, \tilde{s_1})$ and $L(\tilde{\psi}_T, \tilde{s_1})$ are PIHBCKIs of type-4, for all $\tilde{s_1} \in D[0,1]$.

(v) Let for all $\tilde{s_1} \in D[0,1], U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{s_1})$ are reflexive PIHBCKIs of type-4. Let $j_1, j_2, j_3 \in H$ and put $\tilde{s} = \min\{\inf_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\}, \sup_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\}\}$. Then $\inf_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\} \geq \tilde{s_1}$ and

$\sup_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\} \geq \tilde{s_1}$. Hence, $\tilde{\phi}_T(r) \geq \tilde{s_1}$, for all $r \in (j_1 \circ j_2) \circ j_3$ and so $(j_1 \circ j_2) \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Moreover, since $\tilde{\phi}_T$ satisfies the **sup** property, there exists $b_0 \in U(\tilde{\phi}_T, \tilde{s_1})$ such that $\tilde{\phi}_T(b_0) = \sup_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\} \geq \tilde{s_1}$ and so $\tilde{\phi}_T(b_0) \geq \tilde{s_1}$. This is, $b_0 \in U(\tilde{\phi}_T, \tilde{s_1})$. Hence, $(j_2 \circ j_3) \cap U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset$. Since $U(\tilde{\phi}_T, \tilde{s_1})$ is a PIBCKI of type-4 and hence type-1 by Theorem 2.7 [5], then by Theorem 2.8 (ii) [5], we have $U(\tilde{\phi}_T, \tilde{s_1})$ is a weak HBCKI of H. Also, since $\tilde{\phi}_T$ is fuzzy closed, we have $U(\tilde{\phi}_T, \tilde{s_1})$ is closed and so by Lemma 2.3 (iv) [5], $U(\tilde{\phi}_T, \tilde{s_1})$ is an HBCKI of H. Now, $U(\tilde{\phi}_T, \tilde{s_1})$ is a reflexive HBCKI of H and $(j_2 \circ j_3) \cap U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset$, so $j_2 \circ j_3 \ll U(\tilde{\phi}_T, \tilde{s_1})$ by Theorem 3.5 (ii) [5]. Since $U(\tilde{\phi}_T, \tilde{s_1})$ is a PIHBCKI of type-4, we have $j_1 \circ j_3 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$. Hence, for all $u \in j_1 \circ j_3, \tilde{\phi}_T(u) \geq \tilde{s} = \min\{\inf_{r\in(j_1\circ j_2)\circ j_3}\{\tilde{\phi}_T(r)\}, \sup_{t\in j_2\circ j_3}\{\tilde{\phi}_T(t)\}\}$. Put $\tilde{t_1} = \max\{\sup_{l\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(l)\}, \inf_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\}\}$. Then $\sup_{l\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(l)\} \leq \tilde{t_1}$ and $\inf_{m\in j_2\circ j_3}\{\tilde{\psi}_T(m)\} \leq \tilde{t_1}$. Hence, $\tilde{\psi}_T(l) \leq \tilde{t_1}$, for all $l \in (j_1 \circ j_2) \circ j_3$ and so $((j_1 \circ j_2) \circ j_3) \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. Moreover, since $\tilde{\psi}_T$ satisfies **inf** property, there exists $d_0 \in L(\tilde{\psi}_T, \tilde{t_1})$ such that $\tilde{\psi}_T(d_0) = \inf_{d\in j_2\circ j_3}\{\tilde{\psi}_T(d)\} \leq \tilde{t_1}$ and so $\tilde{\psi}_T(d_0) \leq \tilde{t_1}$. That is, $d_0 \in L(\tilde{\psi}_T, \tilde{t_1})$. Hence, $(j_2 \circ j_3) \cap L(\tilde{\psi}_T, \tilde{t_1}) \neq \emptyset$. Since $L(\tilde{\psi}_T, \tilde{t_1})$ is a PIHBCKI of type-4 and hence of type-1 by Theorem 2.7 [5], then by Theorem 2.8 [5], we have $L(\tilde{\psi}_T, \tilde{t_1})$ is a WHBCKI of H. Also, since $\tilde{\psi}_T$ is anti-fuzzy closed, we have $L(\tilde{\psi}_T, \tilde{t_1})$ is closed and so by Lemma 2.3 (iv) [5], $L(\tilde{\psi}_T, \tilde{t_1})$ is an HBCKI of H. Now, $L(\tilde{\psi}_T, \tilde{t_1})$ is a reflexive HBCKI of H and $(j_2 \circ j_3) \cap L(\tilde{\psi}_T, \tilde{t_1}) \neq \emptyset$ imply that $j_2 \circ j_3 \ll L(\tilde{\psi}_T, \tilde{t_1})$, so $j_1 \circ j_3 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$. Hence, for all $t \in j_1 \circ j_3, \tilde{\psi}_T(t) \leq \tilde{t_1} = \max\{\sup_{c\in(j_1\circ j_2)\circ j_3}\{\tilde{\psi}_T(c)\}, \inf_{d\in j_2\circ j_3}\{\tilde{\psi}_T(d)\}\}$. Thus $\tilde{T}$ is an IVIFPIHBCKI of type-4. □

Corollary 4.1. *Let $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ be an IVIFS of H which satisfies the **sup-inf** property and for all $\tilde{s_1}, \tilde{t_1} \in D[0,1], U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are reflexive. Then $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFPIHBCKI of type-2 if and only if it is an IVIFPIHBCKI of type-3.*

Proof. Assume that $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFPIHBCKI of type-2. Then by Theorem 4.1 (ii), $U(\tilde{\phi}_T, \tilde{s_1})$ and $L(\tilde{\psi}_T, \tilde{t_1})$ are PIHBCKIs of type-2 and hence type-3. Hence, by Theorem 4.1 (iii), $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFPIBCKI of type-3.

The converse follows from Theorem 3.3 (i) [14]. □

Theorem 4.2. *Let H be a PIHBCKA. Then the following statements are equivalent:*

(i) $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFWHBCKI of H,
(ii) $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFPIHBCKI of type-1.

Proof. (i) ⇒ (ii) Assume $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFWHBCKI of H. For all $\tilde{s_1}, \tilde{t_1} \in D[0,1], U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$. Let $j_1, j_2, j_3 \in H$ be such that $j_1 \circ j_2 \subseteq U(\tilde{\phi}_T, \tilde{s_1})$ and $j_2 \in U(\tilde{\phi}_T, \tilde{s_1})$. Then $r \in U(\tilde{\phi}_T, \tilde{s_1})$, for all $r \in j_1 \circ j_2$ and $j_2 \in U(\tilde{\phi}_T, \tilde{s_1})$, so $\tilde{\phi}_T(r) \geq \tilde{s_1}$, for all $r \in j_1 \circ j_2$ and $\tilde{\phi}_T(j_2) \geq \tilde{s_1}$ imply that $\inf_{r \in j_1 \circ j_2}\{\tilde{\phi}_T(r)\} \geq \tilde{s_1}$ and $\tilde{\phi}_T(j_2) \geq \tilde{s_1}$. Thus $\tilde{\phi}_T(j_1) \geq \min\{\inf_{r \in j_1 \circ j_2}\{\tilde{\phi}_T(r)\}, \tilde{\phi}_T(j_2)\} \geq \tilde{s_1}$, imply $j_1 \in U(\tilde{\phi}_T, \tilde{s_1})$. Let $j_1, j_2, j_3 \in H$ be such that $j_1 \circ j_2 \subseteq L(\tilde{\psi}_T, \tilde{t_1})$ and $j_2 \in L(\tilde{\psi}_T, \tilde{t_1})$. Then $t \in L(\tilde{\psi}_T, \tilde{t_1})$, for all $t \in j_1 \circ j_2$ and $j_2 \in L(\tilde{\psi}_T, \tilde{t_1})$, so $\tilde{\psi}_T(t) \leq \tilde{t_1}$, for all $t \in j_1 \circ j_2$ and $\tilde{\psi}_T(j_2) \leq \tilde{t_1}$. Thus $\sup_{t \in j_1 \circ j_2}\{\tilde{\psi}_T(t)\} \leq \tilde{t_1}$ and $\tilde{\psi}_T(j_2) \leq \tilde{t_1}$. Thus $\tilde{\psi}_T(t) \leq \max\{\sup_{t \in j_1 \circ j_2}\{\tilde{\psi}_T(t)\}, \tilde{\psi}_T(j_2)\} \leq \tilde{t_1}$, imply $j_1 \in L(\tilde{\psi}_T, \tilde{t_1})$. Thus $L(\tilde{\psi}_T, \tilde{t_1})$ and $U(\tilde{\phi}_T, \tilde{s_1})$ are WHBCKIs of H, for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$. By hypothesis and Theorem 2.8 (iii) [5], $U(\tilde{\phi}_T, \tilde{s_1})$ and $L(\tilde{\psi}_T, \tilde{t_1})$ are PIHBCKIs of type-1, for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$. By Theorem 4.1 (i), we have $\tilde{T}$ is an IVIFPIHBCKI of type-1.

(ii) ⇒ (i) The proof follows from Theorem 3.6 (i) [14]. □

Theorem 4.3. *Let H be a PIHBCKA and $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ be an IVIFS of H which satisfies the **sup-inf** property and for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$, $U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are reflexive. If $\tilde{T}$ is an IVIFHBCKI of H, then $\tilde{T}$ an IVIFPIHBCKI of type-2 (type-3).*

Proof. Assume $\tilde{T} = (\tilde{\phi}_T, \tilde{\psi}_T)$ is an IVIFHBCKI of H. By Theorem 3.1 (i), for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$, $U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are HBCKIs of H. By Theorem 2.8 (iii) [5], for all $\tilde{s_1}, \tilde{t_1} \in D[0,1], U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are PIHBCKIs of type-2. By Theorem 2.7 [5], for all $\tilde{s_1}, \tilde{t_1} \in D[0,1]$, $U(\tilde{\phi}_T, \tilde{s_1}) \neq \emptyset \neq L(\tilde{\psi}_T, \tilde{t_1})$ are PIHBCKIs of type-3. By Theorem 4.1 (iii), we have $\tilde{T}$ is an IIVIFPIHBCKI of type-2 (type-3). □

Theorem 4.4. *Every IVIFPIHBCKI of type-2 is an IVIFHBCKI of H.*

Proof. The proof is straightforward. □

5. Conclusion

In this paper, we presented hyper BCK-ideals positive implicative hyper BCK-ideals of types-1, 2, 3, 4 of hyper BCK-algebras under an interval-valued intuitionistic fuzzy environment. The connection between these ideas and their relevant characteristics is discussed.

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