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BEST PROXIMITY POINTS FOR $\mathcal{K}-$ PROXIMAL CONTRACTION

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Abstract. In this paper, we define $\mathcal{K}-$ proximal contraction and prove best proximity point theorem for this contraction. We also provide an illustrative example.

1. Introduction and preliminaries

Banach contraction principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Due to its wide applications in various fields, a huge number of generalizations and extended versions of this principle appear in the literature. Some important and interesting generalizations of Banach's principle is given in [2, 9, 10, 12, 18].

In 1997, Alber and Guerre [1] introduced the notion of weakly contractive self mapping.

Definition 1.1. Let (X, d) be a metric space and A be a nonempty subset of X. A mapping $T : A \to A$ is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)),$$

for all $x, y \in A$, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

If A is bounded, then the infinity condition can be omitted [1].
They [1] further proved that, if A is a closed convex subset of a Hilbert space, then a weakly contractive self mapping T on A has a unique fixed point.

Rhoades [15] extended and improved result of [1] to metric space and established the following:

Theorem 1.1. *Let (X, d) be a complete metric space, and suppose that $T : X \to X$ satisfies the following inequality*

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \tag{1}$$

for all $x, y \in X$ where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

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If we take $\psi(t) = (1-k)t$, where $0 < k < 1$, then the inequality (1) reduced to Banach contraction, hence the above Theorem 1.1 extends Banach's contraction principle.

The function ψ involved in the inequality (1) is known as alternating distance (also called control function). It was initially used in 1984 by Khan and Sessa [11]. This function and its generalizations have been used in fixed point problems in metric and probabilistic metric spaces, see for example, [13, 14, 17]

Motivated by Chatterjea [7] contraction, recently Choudhary [8] introduced weakly $\mathcal{C}$-contraction mapping.

Definition 1.2. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be weakly $\mathcal{C}-$contraction if for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)),$$

where $\psi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

If we take $\psi(x, y) = k(x + y)$ where $0 < k < \frac{1}{2}$ then it reduces to Chatterjea contraction.

Choudhary [8] further proved the following theorem:

Theorem 1.2. *Let (X, d) be a complete metric space. Then a weakly $\mathcal{C}-$contraction $T : X \to X$ has a unique fixed point.*

Most of the extensions and generalizations of Banach's contraction principle focused on weakening the contractive condition of the operator or weakening the completeness of the metric space. One method of weakening the contractive condition is to consider non self mapping. Sankar Raj [16] defined the notion of non-self-weakly contractive mappings as follows:

Definition 1.3. Let A, B be nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a weakly contractive mapping if

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)),$$

for all $x, y \in A$, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

An element $x \in A$ is said to be a fixed point of a map $T : A \to B$ if $Tx = x$. Clearly, $T(A) \cap A \neq \emptyset$; is a necessary (but not sufficient) condition for the existence of a fixed point of T. If $T(A) \cap A = \emptyset$, then $d(x, Tx) > 0$ for all $x \in A$, that is, the set of fixed points of T is empty. In such a situation, one often attempts to find an element x which is in some sense closest to Tx. If such a point exists, we call it best proximity point, i.e., an element $x \in A$ is called a best proximity point of T if $d(x, Tx) = \text{dist}(A, B)$; where $\text{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. Best proximity point analysis has been developed in this direction. The goal of the best proximity point theory is to furnish sufficient conditions that assure the existence of best proximity points.

Before proceeding further let us fix the following notations:

$$A_0 = \{x \in A : d(x,y) = \operatorname{dist}(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = \operatorname{dist}(A,B) \text{ for some } x \in A\}.$$

If A and B are closed subsets of a normed linear space such that $\operatorname{dist}(A,B) > 0$, then A_0 and B_0 are contained in the boundaries of A and B, respectively [6].

Let A and B be two nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the $\mathcal{P}-$ *property* [16] if and only if

$$\left.\begin{array}{l} d(x_1,y_1) = \operatorname{dist}(A,B) \\ d(x_2,y_2) = \operatorname{dist}(A,B) \end{array}\right\} \Rightarrow d(x_1,x_2) = d(y_1,y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

For any nonempty subset A of X, the pair (A,A) has the $\mathcal{P}-$*property.*

Sankar Raj [16] proved the following result:

Theorem 1.3. *Let A and B be two nonempty closed subsets of a metric space (X,d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a weakly contractive mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A,B) has the $\mathcal{P}-$ property. Then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = \operatorname{dist}(A,B)$.*

Basha [3] introduced the notion of proximal contraction as follows:

Definition 1.4. A mapping $T : A \to B$ is said to be *proximal contraction* if there exists a non-negative number $\alpha < 1$ such that, for all $u_1, u_2, x_1, x_2 \in A$

$$\left.\begin{array}{l} d(u_1,Tx_1) = \operatorname{dist}(A,B) \\ d(u_2,Tx_2) = \operatorname{dist}(A,B) \end{array}\right\} \Rightarrow d(u_1,u_2) \leq \alpha\, d(x_1,x_2)\,.$$

Every self mapping that is a proximal contraction is essentially a contraction.

Definition 1.5. [5] The set B is said to be *approximatively compact* with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x,y_n) \to d(x,B)$ for some $x \in A$ has a convergent subsequence.

Any compact set is approximatively compact, and that any set is approximatively compact with respect to itself. Further, if A is compact and B is approximatively compact with respect to A, then A_0 and B_0 are non-empty.

Definition 1.6. [4] A mapping $T : A \to B$ is said to be a *generalized proximal contraction* if, for all $u_1, u_2, x_1, x_2 \in A$,

$$\left.\begin{array}{l} d(u_1,Tx_1) = \operatorname{dist}(A,B) \\ d(u_2,Tx_2) = \operatorname{dist}(A,B) \end{array}\right\} \Rightarrow d(u_1,u_2) \leq d(x_1,x_2) - \psi(d(x_1,x_2)),$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous and non-decreasing such that ψ is positive on $(0,\infty), \psi(0) = 0$, and $\lim_{t\to\infty} \psi(t) = \infty$. Motivated by above studies and Kannan contraction, we now introduce the notion of $\mathcal{K}-$*proximal contraction*:

Definition 1.7. Let A and B be two non-empty subsets of a metric space (X,d). Then $T : A \to B$ is said to be a $\mathcal{K}-$*proximal contraction* if for all $u_1, u_2, x_1, x_2 \in A$,

$$d(u_1,Tx_1) = \operatorname{dist}(A,B) \quad \text{and} \quad d(u_2,Tx_2) = \operatorname{dist}(A,B)$$

implies that

$$d(u_1, u_2) \leq \frac{1}{2}(d(x_1, u_1) + d(x_2, u_2)) - \psi(d(x_1, u_1), d(x_2, u_2))$$

where $\psi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(x_1, x_2) = 0$ if and only if $x_1 = x_2 = 0$.

We now establish necessary and sufficient condition for the existence and uniqueness of best proximity point of the $\mathcal{K}-$ proximal contraction.

Theorem 1.4. *Let (X, d) be a complete metric space, A and B be two nonempty, closed subsets of X such that B is approximatively compact with respect to A. Suppose that A_0 and B_0 are non-empty and $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *T is a $\mathcal{K}-$proximal contraction;*
(ii) *$T(A_0) \subseteq B_0$.*

Then, there exists a unique element $x \in A$ such that $d(x, Tx) = \mathrm{dist}(A, B)$. Further, the sequence $\{x_n\}$ converges to the best proximity point x, where for a fixed $x_0 \in A_0$ the sequence $\{x_n\}$ is given by $d(x_{n+1}, Tx_n) = \mathrm{dist}(A, B)$ for all $n \geq 0$.

Proof. Let x_0 be a fixed element in A_0. Since $T(A_0) \subseteq B_0$, Tx_0 is an element of B_0. So by the definition of B_0, there exists an element $x_1 \in A_0$ such that $d(x_1, Tx_0) = \mathrm{dist}(A, B)$. Again, since $T(A_0) \subseteq B_0$ we have $Tx_1 \in B_0$, it follows that there is $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = \mathrm{dist}(A, B).$$

Continuing this process, we can derive a sequence $\{x_n\}$ in A_0, such that

$$d(x_{n+1}, Tx_n) = \mathrm{dist}(A, B), \tag{2}$$

for every $n \geq 0$. Since T is a $\mathcal{K}-$proximal contraction, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ &\quad - \psi(d(x_{n-1}, x_n), d(x_n, x_{n+1})) \\ &\leq \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})). \end{aligned}$$

Consequently, we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

So the sequence $\{d(x_{n+1}, x_n)\}$ is a monotone-decreasing sequence of nonnegative real numbers. Then there exists a number $\mu \geq 0$ such that

$$d(x_n, x_{n+1}) \to \mu \text{ as } n \to \infty. \tag{3}$$

Now, we will show that $\mu = 0$. Since T is a $\mathcal{K}-$proximal contraction, we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ &\quad - \psi(d(x_{n-1}, x_n), d(x_n, x_{n+1})). \end{aligned} \tag{4}$$

Letting $n \to \infty$ in (4), by using (3) and continuity of ψ, we get

$$\begin{aligned} \mu &\leq \frac{1}{2}(\mu + \mu) - \psi(\mu, \mu) \\ \mu &\leq \mu - \psi(\mu, \mu). \end{aligned}$$

or $\psi(\mu,\mu) \le 0$, which is a contradiction unless $\mu = 0$, that is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{5}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, for every $\varepsilon > 0$ there exists $n_k > m_k \ge k$ with

$$d(x_{m_k}, x_{n_k}) \ge \varepsilon \text{ and } d(x_{m_k}, x_{n_k-1}) < \varepsilon,$$

for each $k \in \mathbb{N}$. Then, we have,

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\le d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + d(x_{n_k-1}, x_{n_k}). \end{aligned} \tag{6}$$

Taking $k \to \infty$ in (6) and using (5), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon.$$

By (2), we have

$$\lim_{k\to\infty} d(x_{n_k+1}, Tx_{n_k}) = \operatorname{dist}(A,B)\,, \tag{7}$$

and

$$\lim_{k\to\infty} d(x_{m_k+1}, Tx_{m_k}) = \operatorname{dist}(A,B)\,. \tag{8}$$

Since T is a $\mathcal{K}-$proximal contraction, by (7), we have

$$\begin{aligned} d(x_{n_k+1}, x_{m_k+1}) \le{}& \frac{1}{2}\left(d(x_{n_k}, x_{n_k+1}) + d(x_{m_k}, x_{m_k+1})\right) \\ &- \psi\left(d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1})\right). \end{aligned} \tag{9}$$

Since,

$$d(x_{n_k}, x_{m_k}) \le d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k})\,. \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} d(x_{n_k}, x_{m_k}) \le{}& d(x_{n_k}, x_{n_k+1}) + d(x_{m_k+1}, x_{m_k}) + \frac{1}{2}\left(d(x_{n_k}, x_{n_k+1}) + d(x_{m_k}, x_{m_k+1})\right) \\ &- \psi\left(d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1})\right). \end{aligned}$$

letting $k \to \infty$, we have

$$\varepsilon \le 0 + 0 + 0 - \psi(0,0) = 0\,,$$

a contradiction.
Hence, $\{x_n\}$ is a Cauchy sequence. Since A is a closed subset of a complete metrics space, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$. Taking $n \to \infty$ in (2) and continuity of T, we have

$$d(x, Tx) = \operatorname{dist}(A,B).$$

This completes the proof. □

Now, we give an example to illustrate Theorem 1.4.

Example 1.8. Let $X = \mathbb{R}^2$ with the Euclidean metric. Suppose that

$$A = \{(1,x) : x \in \mathbb{R}, 0 \le x \le 1\},$$

and

$$B = \{(0,x) : x \in \mathbb{R}, x \ge 0\}.$$

Now we define a mapping $T : A \to B$ as below:

$$T((1,x)) = \Big(0, \frac{x}{1+x}\Big).$$

It is easy to see that $d(A,B) = 1$, $A_0 = A$ and $B_0 = B$. T is continuous, $T(A_0) \subset B_0$ and

$$d(u_1, Tx_1) = d(A,B) = 1, d(u_2, Tx_2) = d(A,B) = 1.$$

for some $u_1, u_2, x_1, x_2 \in A$. We will show that T is satisfy the condition (i) of Theorem 1.4 with $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ defined by

$$\psi(t_1, t_2) = \max\left(\frac{t_1}{1+t_1}, \frac{t_2}{1+t_2}\right) \text{ for all } (t_1,t_2) \in [0,\infty) \times [0,\infty).$$

Suppose $u_1 = u_2 = (1,0)$ and $x_1 = x_2 = (0,0)$, then

$$d(x_1,u_1) = 1\,,\;\; d(x_2,u_2) = 1\,,\;\; d(u_1,u_2) = 0$$

and

$$\psi\left(d(x_1,u_1), d(x_2,u_2)\right) = \psi(1,1) = \max\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\,.$$

We can see that

$$d(u_1,u_2) \leq \frac{1}{2}(d(x_1,u_1) + d(x_2,u_2)) - \psi(d(x_1,u_1), d(x_2,u_2))\,,$$

hence, T is a $\mathcal{K}-$proximal contraction

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