Int. J. Anal. Appl. (2024), 22:226

International Journal of Analysis and Applications

Topological Indices of Inverse Graph for Generalized Quaternion Group

Makkiah Suleiman Makki*

Department of Mathematics, Faculty of Science, Umm Al-Qura University, Makkah, Saudi Arabia

**Corresponding author:* msmakki@uqu.edu.sa

Abstract. In this paper, some of the famous and known topological indices for the inverse graph of Generalized quaternion group, including: Hosoya polynomial, Wiener index, hyper-Wiener index, first Zagreb index, second Zagreb index, ABC index, eccentric index, eccentric-connectivity index, total eccentric index, first Zagreb eccentric index, second Zagreb Eccentric index, graph energy index, and Estrada index.

1. Introduction

The graph theory has very important applications not only in mathematics but in many branches of science such as physics, biology, and chemistry. The graph denoted by $G = (V, E)$ is a mathematical tool used to model the objects as vertices V and the relations between these objects as edges E. One of the important concept related to the graph is the topological index, also called a molecular descriptor, which is a mathematical formula that can be applied to any graph. From this index, it is possible to analyze mathematical values and investigate some properties of the graph.To achieve the objective of our research which involves studying some of this indices for a type of graph-related to finite groups called the Inverse graph, this section has two subsections:

1.1. Literature Review of Topological Indices:

The concept of topological indices first occurred in 1947 by Harold Wiener [19]. Over than 3000 topological graph indices are registered in Chemical databases. A natural way for their classification is by the origin of parameters used in their definitions. According to that, we have the following classification of the toplogical indices as four major categories. Each category of them includes many types of topological indices and in this paper we focus on the following indices as

Received: Oct. 10, 2024.

2020 *Mathematics Subject Classification.* 05C09, 05C25, 05C50.

Key words and phrases. Hosoya polynomial; Wiener index; hyper wiener index; first and second Zagreb index; ABC-index; eccentric-connectivity index; total eccentricity index; first and second Zagreb eccentricity index; graph energy; Estrada index.

https://doi.org/10.28924/2291-8639-22-2024-226
ISSN: 2291-8639

demonstrated below by this diagram:

I. Distance-based topological indices.

- Wiener index.
- Hosoya polynomial.
- Hyper-Wiener index.

II. Degree-based topological indices.

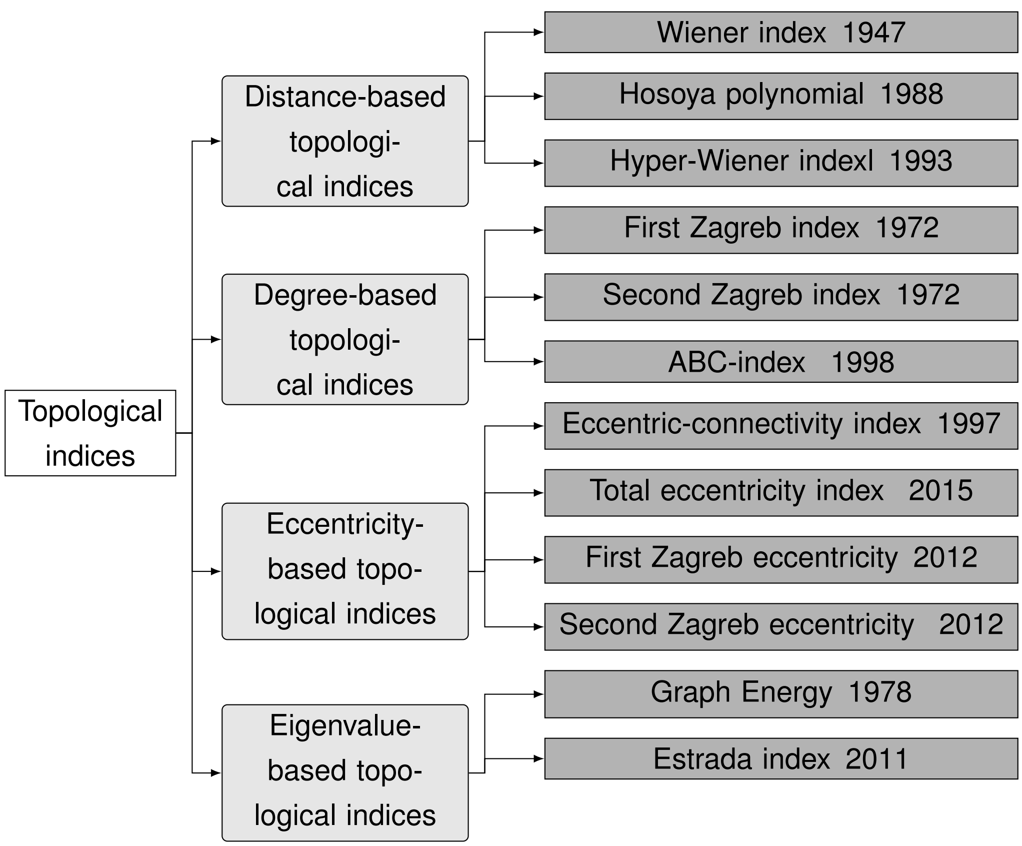
- First Zagreb index.
- Second Zagreb index.
- ABC-index.

III. Eccentricity-based topological indices.

- Eccentric-connectivity index.
- Total eccentricity index.
- First Zagreb eccentricity.
- Second Zagreb eccentricity

IV. Eigenvalue-based topological indices.

- Graph Energy.
- Estrada index.



1.1.1. ***Distance-based topological indices:***

One of the most investigated categories of topological indices used in mathematical chemistry, which are defined in terms of the distance between the vertices of the graph. The first distance-based index is the wiener index [19]. Another distance-based indices listed below:

- Hosoya polynomial:

The Hosoya polynomial was introduced by Haruo Hosoya in 1988 [10], as a counting polynomial it counts the number of distances of paths of different lengths in a molecular graph. If G is a connected graph with n vertices and m edges then the Hosoya polynomial is defined as:

$$H(G,x) := \sum_{k\geq 0} d(G,k)x^k \tag{1.1}$$

where $d(G,k)$ is the number of pairs of its vertices at distance k. Note that $d(G,1) = m$ and $d(G,0) = 0$ if G simple graph has no loop.

- Wiener index:

The Wiener index was first introduced in 1947 by Harold Wiener [19] and defined as:

$$W(G) := \sum_{u,v\in V(G)} d(u,v) \tag{1.2}$$

Remark 1.1. *There is a relation between the Wiener index and the Hosoya polynomial given as follows:*

$$W(G) = \frac{\partial}{\partial x} H(G,x)|_{x=1} \tag{1.3}$$

- Hyper-Wiener index:

The Hyper-Wiener index of cyclic graphs was introduced by Randić in 1993 [16], as a generalization of the Wiener index, and defined as:

$$WW(G) := \frac{1}{2}W(G) + \frac{1}{2}\sum_{\{u,v\}\subset V(G)} d^2(u,v) \tag{1.4}$$

where $d^2(u,v) = (d(u,v))^2$. There is another formula for this index given by:

$$WW(G) := \frac{1}{2}\sum_{k\geq 1}^{diam(G)} k(k+1)d(G,k) \tag{1.5}$$

Remark 1.2. *There is a relation between the Hyper-Wiener index and the Hosoya polynomial given as follows:*

$$WW(G) = \frac{1}{2}(\frac{\partial^2}{\partial x^2}H(G,x))|_{x=1} \tag{1.6}$$

1.1.2. ***Degree-based topological indices:***

Degree-based topological indices are defined in terms of the degrees of the vertices of a graph. The first degree-based topological index was put forward in 1975 by Milan Randić [15]. There are many degree-based indices, such as:

- Zagreb index :

One of the oldest graph index is the well-known Zagreb index, which was first introduced by Gutman and Trinajstić in 1972 [8]. There are the first Zagreb index and the second Zagreb index defined as follows:

- First Zagreb index:

The first Zagreb index M_1 is equal to the sum of squares of the degrees of vertices. Some authors call M_1 the Gutman index:

$$M_1(G) := \sum_{u \in V(G)} d^2(u) \tag{1.7}$$

- Second Zagreb index:

The second Zagreb index is equal to the sum of the products of the degrees of pairs of adjacent vertices :

$$M_2(G) := \sum_{\{u,v\} \in E(G)} d(u).d(v) \tag{1.8}$$

- ABC index or Atom-Bond-Connectivity index:

In 1998, Estrada and authors [3], introduced a new index, which was later known as the Atom-Bond Connectivity abbreviated as ABC index and defined as:

$$ABC(G) := \sum_{\{u,v\} \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)d(v)}} \tag{1.9}$$

1.1.3. ***Eccentricity-based topological indices:***

Topological indices, which are based on eccentricity of the vertices in a graph G, are known as Eccentricity-based topological indices.

- Eccentric-connectivity index:

Sharma, Goswami, and Madan would introduce the Eccentric-connectivity index in 1997 [17], which is defined as:

$$\xi^c(G) := \sum_{u \in V(G)} d(u).Ecc(u) \tag{1.10}$$

- Total eccentricity index:

The Total Eccentricity index was introduced by Farooq in 2015 [4] and defined as:

$$\xi(G) := \sum_{u \in V(G)} Ecc(u) \tag{1.11}$$

Some new modified versions of Zagreb indices are expressed in terms of Eccentricity [5] as follows:

- The First Zagreb eccentricity index:

$$E_1(G) := \sum_{u \in V(G)} (Ecc(u))^2 \tag{1.12}$$

- The Second Zagreb eccentricity index:

$$E_2(G) := \sum_{\{u,v\} \in E(G)} Ecc(u).Ecc(v) \tag{1.13}$$

1.1.4. ***Eigenvalue-based topological indices:***

- Graph energy:
The first Eigenvalue-based topological index was introduced in 1978 [7] by Ivan Gutman called the Graph energy index. This index is defined using the eigenvalues of an adjacency matrix $A(G)$ in the following way:

$$E(G) := \sum_{i=1}^{n} |\lambda_i| \tag{1.14}$$

where λ_i is the i-th eigenvalue of $A(G)$ for all $i = 1, 2, 3, ..., n$.

- Estrada index:
Next to Graph energy, the second most investigated topological index based on eigenvalues of an adjacency matrix known as the Estrada index and defined as follows:

$$EE(G) := \sum_{i=1}^{n} e^{\lambda_i} \tag{1.15}$$

Graph measurements	
symbol	**term and definition**
$d(v)$	degree of a vertex v is the number of edges in G that are incident with v
$d(u,v)$	distance between any two vertices u and v is the length of the shortest path between u and v.
$D(v)$	$= \sum_{u \in G} d(u,v)$; the sum of distances between the vertex v and vertex u of G
$d(G,k)$	the number of vertex pairs at distance k.
Ecc(u)	the eccentricity of a vertex u is the maximum $d(u,v)$ for any v in G ; $Ecc(u) := max\{d(u,v) : v \in V(G)\}$.
$diam(G)$	the diameter of G is the greatest distance between any two vertices in G

Table 1. Graph measurements

Some of the graph measurements we need and used in our paper given in Table 1 above.

1.2. **Literature Review of the Inverse graph of generalized quaternion group:**
The inverse graph $G_S(\Gamma)$ associated with a finite group $(\Gamma, \star)$ was first introduced by alfuraidan and Zakariya in 2017 [1]. It is a simple undirect graph with a vertex set $V(G_S)$ coinciding with the elements of the group Γ and two distinct vertices u and v are adjacent in $G_S(\Gamma)$ if and only if either $u \star v \in S$ or $v \star u \in S$ where $S = \{a \in \Gamma : a \neq a^{-1}\}$. For any finite group Γ, the inverse graph $G_S(\Gamma)$ is empty if and only if $|S| = 0$. Hence the inverse graph of a finite group with all of its elements being self-invertible is empty. For more properties and information about the inverse graph [1].

Theorem 1.1. *For any finite abelian group Γ with at least three elements and a non-empty subset S of non-self-invertible elements, then the inverse graph $G_S(\Gamma)$ is connected.*

Theorem 1.2. *Let Γ be a finite group. The diameter of a connected inverse graph $diam(G_S(\Gamma)) = 2$.*

In group theory, a generalized quaternion group of order 2^n has the following presentation:

$$Q_{2^n} =< a, b / a^{2^{n-1}} = e, a^{2^{n-2}} = b^2, b^{-1}ab = a^{-1} > \tag{1.16}$$

Example 1.1. (i) *At $n = 2$ the quaternion group:*

$$Q_{2^2} = \{e, a, a^2, a^3\} \cong Z_4$$

where the sets of nonself invertible and self invertible elements are $S = \{a, a^3\}$ and $S' = \{e, a^2\}$ respectively. which is the commutative group and $G_S(Q_{2^2}) \cong G_S(Z_4)$ is a 4-cycle graph given in Figure 1.

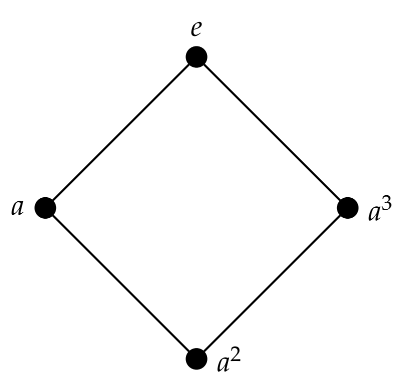


FIGURE 1. $G_S(Q_{2^2})$

(ii) *If $n = 3$ we have the standard quaternion group of order 8.*

$$Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

The inverse graph $G_S(Q_8)$ is given in Figure 2 where the sets of nonself invertible and self invertible elements are $S = \{a, a^3, b, ab, a^2b, a^3b\}$ and $S' = \{e, a^2\}$ respectively.

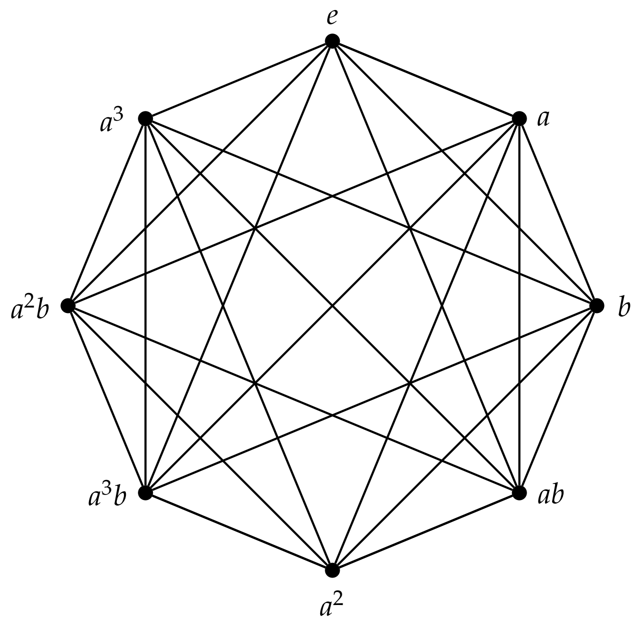


Figure 2. $G_S(Q_{2^3})$

(iii) *If $n = 4$ we have the generalized quaternion group of order 16.*

$$Q_{16} = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^5b, a^6b, a^7b\}$$

The inverse graph $G_S(Q_{16})$ where the sets of nonself invertible and self invertible elements are $S = \{a, a^2, a^3, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^5b, a^6b, a^7b\}$ and $S' = \{e, a^4\}$ respectively.

In general, the inverse graph $G_S(Q_{2^n})$ has the set of self invertible elements $S' = \{e, a^{2^{n-2}}\}$.

Theorem 1.3. *Let Γ be a finite group and the set $S \neq \phi$ is a non-empty. The associated inverse graph $G_S(\Gamma)$ is:*

(i) *2-regular if Γ is a group of four elements.*
(ii) *$(2^n - 2)$-regular graphs if Γ is a generalized quaternion group of order 2^n where $n > 2$.*

Definition 1.1. *Let G be a simple undirect graph of order n. The adjacency matrix $A(G) = (a_{ij})_{n\times n}$ is a zero-one symmetric square matrix of dimension $n \times n$ where:*

$$a_{ij} = \begin{cases} 1 & \textit{if } \{v_i, v_j\} \in E \\ 0 & \textit{if } \{v_i, v_j\} \notin E \end{cases}$$

In [12], we study and find the adjacency matrix for the inverse graph of the generalized quaternion group Q_{2^n}, the characteristic polynomial of it, and its eigenvalues, given as:

Theorem 1.4. *For the inverse graph $G_S(Q_{2^n})$ we have:*

$$A((G_S(Q_{2^n})) = \begin{array}{c} \\ e \\ a^{2^{n-2}} \\ a \\ a^{-1} \\ \vdots \\ b \\ b^{-1} \\ \cdots \end{array} \begin{array}{c} \begin{array}{cccccccc} e & a^{2^{n-2}} & a & a^{-1} & \cdots & b & b^{-1} & \cdots \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 & \cdots \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \end{bmatrix}_{2^n \times 2^n} \end{array}$$

Theorem 1.5. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n \geq 3$, the characteristic polynomial of the adjacency matrix $A(G_S(Q_{2^n}))$ is:*

$$P_{A((G_S(Q_{2^n}))}(\lambda) = \lambda^{\frac{n}{2}}(\lambda+2)^{\frac{n}{2}-1}[\lambda - (2^n - 2)] \tag{1.17}$$

Corollary 1.1. *Let $G_S(Q_{2^n})$ be an inverse graph of generalized quaternion group Q_{2^n} where $n \geq 3$, the adjacency spectrum of it is given as:*

$$Spec(A(G_S(Q_{2^n})) = \begin{pmatrix} 2^n - 2 & 0 & -2 \\ 1 & \frac{2^n}{2} & \frac{2^n}{2} - 1 \end{pmatrix}$$

From theorem 1.5, we get the important result:

Corollary 1.2. *For any $n \geq 2$ the inverse graph $G_S(Q_n)$ is connected.*

2. Methodology

A simple connected graph $G = (V, E)$ is a finite nonempty set of vertices $V(G)$ with a finite nonempty set $E(G)$ of unordered pairs of distinct vertices of G called edges. To compute the topological indices for graph G, we need to compute some of the graph measurements as; length, distance, diameter, eccentricity, radius, and so on as given in Table 1. For this aim, we use mathematical induction on the order and the size of the graph and apply fundamental calculus. Moreover, we use some software such as Matlab and Maple for mathematical calculations and verifications.

3. Main Results

In this section, we will compute and prove some topological indices for the inverse graph $G_S(Q_n)$. Accordingly, this section is devided into four subsections:

3.1. Distance-based topological indices of $G_S(Q_n)$:

Consider the inverse graph $G_S(Q_n)$ of order 2^n of the generalized quaternion group Q_{2^n} given in (1.16). From the regularity of $G_S(Q_n)$ we have the following:

Lemma 3.1. (1) *$deg(v) = 2^n - 2$ for all $v \in V(G_S(Q_n))$.*

(2) *The size of $G_S(Q_n)$ is $|E(G_S(Q_n)| = \frac{2^n(2^n-2)}{2}$.*

Lemma 3.2. *The group Q_{2^n} has only two self-invertible elements; $\{e, a^{2^{n-2}}\}$, so $S = Q_{2^n} \setminus \{e, a^{2^{n-2}}\}$.*

Theorem 3.1. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$. The Hosoya polynomial of it given by:*

$$H(G_S(Q_{2^n}), x) := (2^{2n-1} - 2^n - 2^{n-1} - 2^{n-2} + 2)x + (2^{n-1} + 2^{n-2} - 2)x^2$$

Proof. Since $diam(G_S(\Gamma)) = 2$, then $d(u, v) = 1$ or 2 for all $u, v \in V(G_S)$, $u \neq v$. From the definition of the inverse graph $G_S(Q_{2^n})$:

- The vertex $v \neq e$ with $v^2 \in S'$ is not adjacent to its inverse v^{-1}.
- The vertex $v \neq e$ with $v^2 \notin S'$ is not adjacent to its inverse v^{-1}. Also it is not adjacent to the vertex u such that $u.v \equiv a^{2^{n-2}}(mod 2^{n-1})$ which their number is $2^{n-2} - 1$. Hence the number of vertices having distance 2 is:

$$\frac{2^n - 2}{2} + 2^{n-2} - 1 = 2^{n-1} + 2^{n-2} - 2$$

- The number of vertices with distances 1 is $\frac{2^n(2^n-2)}{2} - (2^{n-1} + 2^{n-2} - 2) = 2^{2n-1} - 2^n - 2^{n-1} - 2^{n-2} + 2$.

□

Theorem 3.2. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$, the Wiener index given by:*

$$W(G_S(Q_{2^n})) := 2^{2n-1} - 2^{n-2} - 2$$

Proof. From Remark (1.3) we have:

$$W(G) = \frac{\partial}{\partial x} H(G, x)|_{x=1}$$

Then from theorem 3.1 we get:

$$W(G) = 2^{2n-1} - 2^{n-2} - 2$$

□

Theorem 3.3. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$, the Hyper-Wiener index is :*

$$WW(G_S(Q_{2^n})) = 2^n + 2^{n-1} - 4$$

Proof. From (1.6) we have:

$$WW(G) = \frac{1}{2}(\frac{\partial^2}{\partial x^2}H(G,x))|_{x=1}$$

Then from theorem 3.1 we get:

$$WW(G_S(Q_{2^n})) = \frac{1}{2}(\frac{\partial^2}{\partial x^2}[(2^{2n-1} - 2^n - 2^{n-1} - 2^{n-2} + 2)x + (2^{n-1} + 2^{n-2} - 2)x^2]|_{x=1}$$

Therefore, $WW(G_S(Q_{2^n})) = 2^n + 2^{n-1} - 4$.

□

3.2. **Degree-based topological indices of Inverse graph $G_S(Q_n)$.**

Theorem 3.4. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$, The first Zagreb index:*

$$M_1(G_S(Q_{2^n})) := 2^n(2^n - 2)^2$$

Proof. From Lemma 3.1 we know that $deg(v) = 2^n - 2$ for all $v \in V(G_S(Q_n)$. Hence from (1.7) we have:

$$M_1(G_S(Q_{2^n})) = \sum_{u \in V(G)} (2^n - 2)^2 = 2^n(2^n - 2)^2$$

□

Theorem 3.5. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n}, where $n > 2$. The second Zagreb index:*

$$M_2(G_S(Q_{2^n})) = 2^{n-1}(2^n - 2)^3$$

Proof. From (1.8), the second Zagreb index M_2 is equal to the sum of the products of the degrees of pairs of adjacent vertices of the graph, and from Lemma 3.1 we know that $deg(v) = 2^n - 2$ for all $v \in V(G_S(Q_n)$. On the other hand, the number of the edges of $|E(G_S(Q_n)) = \frac{2^n(2^n-2)}{2}$. Hence from (1.8) we have:

$$M_2(G_S(Q_{2^n})) = \sum_{(u,v) \in E(G)} (2^n - 2)(2^n - 2) = \frac{2^n(2^n - 2)}{2}(2^n - 2)^2$$

$$M_2(G_S(Q_{2^n})) = 2^{n-1}(2^n - 2)^3$$

□

Theorem 3.6. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n}, where $n > 2$. The ABC-index or Atom-Bond-Connectivity index is:*

$$ABC(G) = 2^{n-1}\sqrt{2^{n+1} - 6}$$

Proof. From (1.9), we have $ABC(G) := \sum_{\{u,v\} \in E(G)} \sqrt{\frac{d(u)+d(v)-2}{d(u)d(v)}}$, then:

$$ABC(G) = \sum_{\{u,v\} \in E(G)} \sqrt{\frac{(2^n - 2) + (2^n - 2) - 2}{(2^n - 2)^2}}$$

$$ABC(G) = \frac{2^n(2^n-2)}{2}\sqrt{\frac{2(2^n-2)-2}{(2^n-2)^2}}$$

Hence, $ABC(G) = 2^{n-1}\sqrt{2^{n+1}-6}$.

□

3.3. Eccentricity-based topological indices of Inverse graph $G_S(Q_n)$.

Theorem 3.7. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$. The Eccentric-connectivity index*

$$\xi^c(G) = 2^{n+1}(2^n-2)$$

Proof. From (1.10) and $Ecc(u) = 2$ for all $u \in V(G_S)$ we have:
$\xi^c(G) = \sum_{u\in V(G)} d(u).Ecc(u) = \sum_{u\in V(G)} 2(2^n-2) = 2^n[2(2^n-2)] = 2^{n+1}(2^n-2)$.

□

Theorem 3.8. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$. The Total Eccentricity index:*

$$\xi(G) = 2^{n+1}$$

Proof. From (1.10) and $Ecc(u) = 2$ for all $v \in V(G_S)$ we have:
$\xi(G) := \sum_{u\in V(G)} Ecc(u) = 2^{n+1}$. □

Theorem 3.9. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$,The First Zagreb eccentricity index:*

$$E_1(G_S(Q_{2^n})) = 2^{n+2}$$

Proof. From (1.11) and $Ecc(u) = 2$ for all $v \in V(G_S)$ we have:

$$E_1(G_S(Q_{2^n})) = \sum_{u\in V(G)} (Ecc(u))^2 = 2^{n+2}$$

□

Theorem 3.10. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$, The Second Zagreb eccentricity index:*

$$E_2(G_S(Q_{2^n})) = 2^{n+1}(2^n-2)$$

Proof. From (1.12) and $Ecc(u) = 2$ for all $v \in V(G_S)$ we have:

$$E_2(G) := \sum_{\{u,v\}\in E(G)} Ecc(u).Ecc(v) = (\frac{2^n(2^n-2)}{2})2^2 = 2^{n+1}(2^n-2)$$

□

3.4. **Eigenvalue-based topological indices of Inverse graph $G_S(Q_n)$.**

Theorem 3.11. *Let $G_S(Q_{2^n})$ be an inverse graph of the generalized quaternion group Q_{2^n} where $n > 2$. The graph energy index is:*

$$E(G) := 2(2^n - 2)$$

Proof. From (1.14) and Corollary 1.1 we have:

$$E(G) = \sum_{i=1}^{n} |\lambda_i| = 1.(2^n - 2) + 0.(\frac{2^n}{2}) + |-2|.(\frac{2^n}{2} - 1) = (2^n - 2) + 2(\frac{2^n - 2}{2})$$

Hence, $E(G) = 2(2^n - 2)$.

□

Theorem 3.12. *Let $G_S(Q_{2^n})$ be an inverse graph of generalized quaternion group Q_{2^n} where $n > 2$. The Estrada index index is:*

$$EE(G) = e^{2^n - 2} + (\frac{2^n}{2}) + (\frac{2^n}{2} - 1).e^{-2}$$

Proof. From (1.13) and Corollary 2.1 we have:

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i} = 1.e^{2^n - 2} + (\frac{2^n}{2}).e^0 + (\frac{2^n}{2} - 1).e^{-2}$$

$$EE(G) = e^{2^n - 2} + (\frac{2^n}{2}) + (\frac{2^n}{2} - 1).e^{-2}$$

□

The following tablecontains the topological indices given in this section for $G_S(Q_{2^2}); n = 2, 3, 4$.

Topological indices for $G_S(Q_{2^n}); n = 2, 3, 4$			
Index	n=2	n=3	n=4
$H(G_S(Q_{2^n}), x)$	$3x + x^2$	$20x + 4x^2$	$102x + 10x^2$
$W(G_S(Q_{2^n}))$	5	28	122
$WW(G_S(Q_{2^n}))$	2	8	20
$M_1(G_S(Q_{2^n}))$	16	288	3136
$M_2(G_S(Q_{2^n}))$	16	864	21952
$ABC(G_S(Q_{2^n}))$	$2\sqrt{2}$	$4\sqrt{10}$	$8\sqrt{26}$
$\xi^c(G_S(Q_{2^n}))$	16	96	448
$\xi(G_S(Q_{2^n}))$	8	16	32
$E_1(G_S(Q_{2^n}))$	16	32	64
$E_2(G_S(Q_{2^n}))$	16	96	448
$E(G_S(Q_{2^n}))$	4	12	28
$EE(G_S(Q_{2^n}))$	$e^2 + 2 + e^{-2}$	$e^6 + 4 + 3e^{-2}$	$e^{14} + 8 + 7e^{-2}$

4. Discussion

The concept of graphs related to algebraic structures is one of the useful methods to study the properties of these algebraic structures. To describe the structure of a group, Arthur Cayley was the first who introduce in 1878 the concept of a graph for a group called Cayley graph. There are many interesting algebraic problems that arise from the translation of some graph-theoretic parameters such as topological indices. The concept of an Inverse graph of a finite group was first introduced in 2017. It has been studied by many researchers, we mention one of them for example [6], about some of the topological indices for inverse graphs associated with finite cyclic group. In our paper, we studied and proved many results about the topological indices for this graph in case the group is a generalized quternion group. The nature of this group was reflected in the structure of the graph. Our studying has become easy because the graph is a regular graph with a fixed diameter equal to 2. We just compute Hosoya polynomial and many topological indices. In the future, many topics can be studied for the inverse graph concerning different groups.

5. Conclusion

In conclusion, we achieved our goal from this study. We compute and proved many theories about topological indices; Hosoya polynomial (Theorem 3.1), Wiener index (Theorem 3.2), Hyper-Wiener index (Theorem 3.3), First Zagreb index (Theorem 3.4), Second Zagreb index (Theorem 3.5), ABC index (Theorem 3.6), Eccentric-Connectivity index (Theorem 3.7), Total Eccentric index (Theorem 3.8), First Zagreb Eccentric index (Theorem 3.9), Second Zagreb Eccentric index (Theorem 3.10), Graph Energy index (Theorem 3.11), and Estrada index (Theorem 3.12) for the inverse graph.These results underscore the need for continued research in this area, as well as the development of this concept of graphs for different groups.

Acknowledgements: The author acknowledges Umm Al-Qura University for supporting this research.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] M.R. Alfuraidan, Y.F. Zakariya, Inverse Graphs Associated with Finite Groups, Electron. J. Graph Theory Appl. 5 (2017), 142–154. https://doi.org/10.5614/ejgta.2017.5.1.14.

[2] A.A. Ali, A.M. Ali, Hosoya Polynomials of Pentachains, MATCH Commun. Math. Comput. Chem. 65 (2011), 807–819.

[3] E. Estrada, L. Torres, L. Rodriguez, I. Gutman, An Atom-Bond Connectivity Index: Modeling the Enthalpy of Formation of Alkanes, Indian J. Chem., Sect. A 37 (1998), 849–855.

[4] R. Farooq, M.A. Malik, On Some Eccentricity Based Topological Indices of Nanostar Dendrimers, Optoelectron. Adv. Mater. Rapid Commun. 9 (2015), 842–849.

[5] M. Ghorbani, M. Hosseinzadeh, A New Version of Zagreb Indices, Filomat 26 (2012), 93–100.

[6] S. Gopinath, A.R.P. Doss, G. Kalaimurugan, Topological Indices for Inverse Graphs Associated With Finite Cyclic Group, Commun. Math. Appl. 14 (2023), 415–427. https://doi.org/10.26713/cma.v14i1.2026.

[7] I. Gutman, The Energy of a Graph, Ber. Math. Stat. Sekt. Forschungsz Graz. 103 (1978), 1–22.

[8] I. Gutman, N. Trinajstić, Graph Theory and Molecular Orbitals. Total ϕ-Electron Energy of Alternant Hydrocarbons, Chem. Phys. Lett. 17 (1972), 535–538. https://doi.org/10.1016/0009-2614(72)85099-1.

[9] I. Gutman, B. Ruščić, N. Trinajstić, C.F. Wilcox, Graph Theory and Molecular Orbitals. XII. Acyclic Polyenes, J. Chem. Phys. 62 (1975), 3399–3405. https://doi.org/10.1063/1.430994.

[10] H. Hosoya, On Some Counting Polynomials in Chemistry, Discrete Appl. Math. 19 (1988), 239–257.

[11] J.-B. Liu, I. Khalid, M.T. Rahim, M.U. Rehman, F. Ali, M. Salman, Eccentric Topological Properties of a Graph Associated to a Finite Dimensional Vector Space, Main Group Met. Chem. 43 (2020), 164–176. https://doi.org/10.1515/mgmc-2020-0020.

[12] M.S. Makki, On the Spectrum and Connectivity of Inverse Graph of a Generalized Quaternion Group, Adv. Appl. Discrete Math. 41 (2024), 565–580. https://doi.org/10.17654/0974165824037.

[13] N. De, Sk.Md. Abu Nayeem, A. Pal, Total Eccentricity Index of the Generalized Hierarchical Product of Graphs, Int. J. Appl. Comput. Math. 1 (2015), 503–511. https://doi.org/10.1007/s40819-014-0016-4.

[14] Nilanjan De, Anita Pal, Sk. Md. Abu Nayeem, Total Eccentricity Index of Some Composite Graphs, Malaya J. Mat. 3 (2015), 523–529. https://doi.org/10.26637/mjm304/010.

[15] M. Randic, Characterization of Molecular Branching, J. Amer. Chem. Soc. 97 (1975), 6609–6615. https://doi.org/10.1021/ja00856a001.

[16] M. Randić, Novel Molecular Descriptor for Structure–Property Studies, Chem. Phys. Lett. 211 (1993), 478–483. https://doi.org/10.1016/0009-2614(93)87094-J.

[17] V. Sharma, R. Goswami, A.K. Madan, Eccentric Connectivity Index: A Novel Highly Discriminating Topological Descriptor for Structure–Property and Structure–Activity Studies, J. Chem. Inf. Comput. Sci. 37 (1997), 273–282. https://doi.org/10.1021/ci960049h.

[18] D. Vukičević, B. Furtula, Topological Index Based on the Ratios of Geometrical and Arithmetical Means of End-Vertex Degrees of Edges, J. Math. Chem. 46 (2009), 1369–1376. https://doi.org/10.1007/s10910-009-9520-x.

[19] H. Wiener, Structural Determination of Paraffin Boiling Points, J. Amer. Chem. Soc. 69 (1947), 17–20. https://doi.org/10.1021/ja01193a005.

[20] S. Yancheshmeh, R. Modabbernia, M. Jahandideh, The Topological Indices of the Cayley Graphs of Dihedral Groups D_{2n} and the Generalized Quaternion Groups Q_{2^n}, Ital. J. Pure Appl. Math. 40 (2018), 424–433.