International Journal of Analysis and Applications
ISSN 2291-8639
Volume 7, Number 1 (2015), 1-15
http://www.etamaths.com

p-FRAME MULTIRESOLUTION ANALYSIS RELATED TO THE WALSH FUNCTIONS

F. A. SHAH

Abstract. A generalization of the notion of p-multiresoltion analysis on a half-line, based on the theory of shift-invariant spaces is considered. In contrast to the standard setting, the associated subspace V_0 of $L^2(\mathbb{R}^+)$ has a frame, a collection of translates of the scaling function φ of the form $\{\varphi(\cdot \ominus k)\}_{k\in\mathbb{Z}^+}$, where $\mathbb{Z}^+$ is the set of non-negative integers. We investigate certain properties of multiresolution subspaces which provides the quantitative criteria for the construction of p-frame multiresoltion analysis (p-FMRA) on positive half-line $\mathbb{R}^+$. Finally, we establish a complete characterization of all p-wavelet frames associated with p-FMRA on positive half-line $\mathbb{R}^+$ using the shift-invariant space theory.

1. INTRODUCTION

In the early nineties a general scheme for the construction of wavelets was defined. This scheme is based on the notion of *multiresolution analysis* (MRA) introduced by Mallat [9]. In recent years, the concept of MRA has become an important tool in mathematics and applications. It provides a natural framework for understanding of wavelet bases, bases that consist of the scaled and integer translated versions of a finite number of functions. Mathematically, an MRA is an increasing sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ such that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and which satisfies $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$. Furthermore, there should exist an element $\varphi \in V_0$ such that the collection of integer translates of $\varphi, \{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is a complete orthonormal system for V_0. The dilation factor 2 can be replaced by any integer $M \geq 2$ and in that case one needs $M-1$ wavelets to generate the whole space $L^2(\mathbb{R})$. A similar generalization of multiresolution analysis can be made in higher dimensions by considering matrix dilations (see [2]).

On the other hand, there is considerable interest both in mathematics and its applications in the study of compactly supported orthonormal scaling functions and wavelets with an arbitrary dilation factor $p \in \mathbb{N}, p \geq 2$. The motivation comes partly from signal processing and numerical applications, where such wavelets are useful in image compression and feature extraction because of their small support and multifractal structure. Farkov [4] has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ and proved necessary and

2010 *Mathematics Subject Classification.* 42C15, 42C40, 42A38, 41A17.
Key words and phrases. Frame; wavelet frame; p-multiresoltion analysis; scaling function; shift invariant; Walsh-Fourier transform.

sufficient conditions for scaling filters with p^n many terms (for any integers $p, n \geq 2$) to generate an p-MRA in $L^2(\mathbb{R}^+)$. These studies were continued by Farkov and his colleagues [5, 6] where they have given some new algorithms for constructing the corresponding biorthogonal and non-stationary wavelets related to the Walsh polynomials on the positive half-line $\mathbb{R}^+$. On the other hand, Shah and Debnath [19] have constructed dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ using the Walsh-Fourier transform and have established a necessary condition and a sufficient condition for the system $\left\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x \ominus k) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\right\}$ to be a frame for $L^2(\mathbb{R}^+)$. Further, wavelet packets related to the Walsh functions are discussed in a series of papers by the author [14-17]. In his recent, Shah [18] has given the construction of tight wavelet frames generated by the Walsh polynomials on a half-line $\mathbb{R}^+$ by following the procedure of Daubechies et al. [3] via extension principles. He also provide a sufficient condition for finite number of functions to form a tight wavelet frame and established a general principle for constructing tight wavelet frames on $\mathbb{R}^+$. Recently, Meenaski et al. [10] have introduced the notion of *non-uniform multiresolution analysis* (NUMRA) on a half-line $\mathbb{R}^+$ and have also established the necessary and sufficient condition for the existence of corresponding wavelets on $\mathbb{R}^+$.

Since the use of multiresolution analysis has proven to be a very efficient tool in wavelet theory mainly because of its simplicity, it is of interest to try to generalize this notion as much as possible while preserving its connection with wavelet analysis. In this connection, Benedetto and Li considered the dyadic *semi-orthogonal frame multiresolution analysis* of $L^2(\mathbb{R})$ with a single scaling function and successfully applied the theory in the analysis of narrow band signals [1]. The characterization of the dyadic semi-orthogonal frame multiresolution analysis with a single scaling function admitting a single frame wavelet whose dyadic dilations of the integer translates form a frame for $L^2(\mathbb{R})$ was obtained independently by Benedetto and Treiber by a direct method [2], and by Kim and Lim by using the theory of shift-invariant spaces [8]. Later on, Xiaojiang [21] extended the results of Benedetto and Li's theory of FMRA to higher dimensions with arbitrary integral expansive matrix dilations, and has established the necessary and sufficient conditions to characterize semi-orthogonal multiresolution analysis frames for $L^2(\mathbb{R}^n)$. On the other hand, Zhang [22] has given the characterization of *generalized frame MRA* on $\mathbb{R}$ and has provided a general algorithm for the construction of non-MRA wavelets by means of the Fourier transforms.

In this paper, we introduce the notion of *p-frame multiresolution analysis* (p-FMRA) on positive half-real line $\mathbb{R}^+$ by extending the above described methods. We first investigate the properties of multiresolution subspaces, which will provide the quantitative criteria for the construction of p-FMRAs. We also show that the scaling property of an p-FMRA also holds for the wavelet subspaces and that the space $L^2(\mathbb{R}^+)$ can be decomposed into the orthogonal sum of these wavelet subspaces. Finally, we study the characterization of p-wavelet frames associated with p-FMRA on positive half-line $\mathbb{R}^+$ using the shift-invariant space theory.

The paper is structured as follows. In Section 2, we introduce some notations and preliminaries related to the operations on positive half-line $\mathbb{R}^+$ including the definition of the Walsh-Fourier transform. The notion of p-FMRA of $L^2(\mathbb{R}^+)$ is introduced in Section 3 and its quantitative criteria is given by means of the Theorem 3.10. In Section 4, we establish a complete characterization of p-wavelet frames generated by a finite number of mother wavelets on $\mathbb{R}^+$.

2. WALSH-FOURIER ANALYSIS

We start this section with certain results on Walsh-Fourier analysis. We present a brief review of generalized Walsh functions, Walsh-Fourier transforms and its various properties.

As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Denote by $[x]$ the integer part of x. Let p be a fixed natural number greater than 1. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x] (\operatorname{mod} p), \qquad x_{-j} = [p^{1-j} x] (\operatorname{mod} p), \tag{2.1}$$

where $x_j, x_{-j} \in \{0, 1, \dots, p-1\}$. It is clear that for each $x \in \mathbb{R}^+$, there exist $k = k(x)$ in $\mathbb{N}$ such that $x_{-j} = 0 \ \forall\, j > k$.

Consider on $\mathbb{R}^+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j},$$

with $\zeta_j = x_j + y_j (\operatorname{mod} p)$, $j \in \mathbb{Z} \setminus \{0\}$, where $\zeta_j \in \{0, 1, \dots, p-1\}$ and x_j, y_j are calculated by (2.1). As usual, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

For $x \in [0, 1)$, let $r_0(x)$ is given by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p) \\ \varepsilon_p^\ell, & \text{if } x \in \left[\ell p^{-1}, (\ell+1)p^{-1}\right), \quad \ell = 1, 2, \dots, p-1, \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is given by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}^+$. Then, the *generalized Walsh functions* $\{w_m(x) : m \in \mathbb{Z}^+\}$ are defined by

$$w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \{0, 1, \dots, p-1\}$, $\mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions.

For $x, y \in \mathbb{R}^+$, let

$$\chi(x, y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{2.2}$$

where x_j, y_j are given by (2.1).

We observe that

$$\chi\left(x, \frac{m}{p^n}\right) = \chi\left(\frac{x}{p^n}, m\right) = w_m\left(\frac{x}{p^n}\right), \qquad \forall\, x \in [0, p^n),\ m, n \in \mathbb{Z}^+,$$

and

$$\chi(x \oplus y, z) = \chi(x, z)\, \chi(y, z), \quad \chi(x \ominus y, z) = \chi(x, z)\, \overline{\chi(y, z)},$$

where $x, y, z \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational. It is well known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$ (See Golubov et al.[7]).

The *Walsh-Fourier transform* of a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x)\, \overline{\chi(x, \xi)}\, dx, \tag{2.3}$$

where $\chi(x, \xi)$ is given by (2.2). The Walsh-Fourier operator $\mathcal{F} : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $\mathcal{F}f = \hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [7, 13]). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\left\|\hat{f}\right\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}.$$

Definition 2.1 Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathbb{H}$ is called a *frame* for $\mathbb{H}$ if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{k\in\mathbb{Z}} \left|\langle f, f_k\rangle\right|^2 \le B\|f\|^2 \tag{2.4}$$

for all $f \in \mathbb{H}$. The largest constant A and the smallest constant B satisfying (2.4) are called the *upper* and the *lower frame bound*, respectively. A frame is said to be *tight* if it is possible to choose $A = B$ and a frame is said to be *exact* if it ceases to be a frame when any one of its elements is removed. An exact frame is also known as a *Riesz basis.*

The following theorem gives us an elementary characterization of frames.

Theorem 2.2. A sequence $\{f_k\}_{k\in\mathbb{Z}}$ in a Hilbert space $\mathbb{H}$ is a frame for $\mathbb{H}$ if and only if there exists a sequence $a = \{a_k\} \in l^2(\mathbb{Z})$ with $\|a\|_{l^2(\mathbb{Z})} \le C\|f\|, C > 0$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} a_k f_k(x)$$

and $\sum_{k\in\mathbb{Z}} \left|\langle f, f_k\rangle\right|^2 < \infty$, for every $f \in \mathbb{H}$.

For any integer $p \ge 2$, let $D_p : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ be the unitary operator defined via $D_p f(x) = p^{1/2} f(px)$. For $k \in \mathbb{Z}^+$, let $\tau_k : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ denotes the unitary translation operator such that $\tau_k f(x) = f(x \ominus k)$.

Our study uses the theory of shift-invariant spaces developed in [11, 12] and the references therein. A closed subspace S of $L^2(\mathbb{R}^+)$ is said to be *shift-invariant* if $\tau_k f \in S$ whenever $f \in S$ and $k \in \mathbb{Z}^+$. A closed shift-invariant subspace S of $L^2(\mathbb{R}^+)$ is said to be *generated* by $\Phi \subset L^2(\mathbb{R}^+)$ if $S = \overline{\text{span}}\,\{\tau_k\varphi(.) := \varphi(. \ominus k) : k \in \mathbb{Z}^+, \varphi \in \Phi\}$. The cardinality of a smallest generating set Φ for S is called the *length* of S which is denoted by $|S|$. If $|S| =$ finite, then S is called a *finite shift-invariant space* (FSI) and if $|S| = 1$, then S is called a *principal shift-invariant space* (PSI). Moreover, the *spectrum* of a shift-invariant space is defined to be $\sigma(S) = \left\{\xi \in [0,1] : \hat{S}(\xi) \neq \{0\}\right\}$, where $\hat{S}(\xi) = \left\{\hat{f}(\xi \oplus k) \in l^2(\mathbb{Z}^+) : f \in S, k \in \mathbb{Z}^+\right\}$.

3. p-FRAME MULTIRESOLUTION ANALYSIS ON A POSITIVE HALF-LINE

We first introduce the notion of a *p-frame multiresolution analysis* (p-FMRA) of $L^2(\mathbb{R}^+)$.

Definition 3.1. A p-frame multiresolution analysis of $L^2(\mathbb{R}^+)$ is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ such that

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iii) $f(\cdot) \in V_j$ if and only if $f(p\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(iv) The function f lies in V_0 implies that the collection $f(\cdot \ominus k)$ lies in V_0, for all $k \in \mathbb{Z}^+$.

(v) The sequence $\{\tau_k\varphi := \varphi(\cdot \ominus k) : k \in \mathbb{Z}^+\}$ is a frame for the subspace V_0.

The function φ is known as the *scaling function* while the subspaces V_j's are known as *approximation spaces* or *multiresolution subspaces*. An p-FMRA is said to be *non-exact* and respectively *exact* if the frame for the subspace V_0 is *non-exact* and respectively *exact*. In p-MRA's studied in [4], the frame condition is replaced by that of an orthonormal basis or an exact frame.

Next, we establish several properties of multiresolution subspaces that will help in the construction of p-FMRA's. The following proposition shows that for every $j \in \mathbb{Z}$, the sequence $\{\varphi_{j,k}\}_{k\in\mathbb{Z}^+}$, where

$$\varphi_{j,k}(x) = p^{j/2}\varphi(p^j x \ominus k), \tag{3.1}$$

is a frame for V_j.

Proposition 3.2. *Let $\{\tau_k\varphi\}$ be a frame for $V_0 = \overline{span}\{\tau_k\varphi : k \in \mathbb{Z}^+\}$ and*

$$V_j = \left\{f \in L^2(\mathbb{R}^+) : f(p^{-j}.) \in V_0\right\}, \quad j \in \mathbb{Z}. \tag{3.2}$$

Then, the sequence $\{\varphi_{j,k} : k \in \mathbb{Z}^+\}$ defined in (3.1) *is a frame for V_j with the same bounds as those for V_0.*

Proof. For any $f \in V_j$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^+} \left|\langle D^{-j} f, \tau_k\varphi\rangle\right|^2 &= \sum_{k\in\mathbb{Z}^+} \left|\int_{\mathbb{R}^+} p^{-j} f(p^{-j}x)\, \overline{p^{j/2}\varphi(x\ominus k)}\, dx\right|^2 \\
&= \sum_{k\in\mathbb{Z}^+} \left|\int_{\mathbb{R}^+} f(x)\, \overline{p^{j/2}\varphi(p^j x\ominus k)}\, dx\right|^2 \\
&= \sum_{k\in\mathbb{Z}^+} |\langle f, \varphi_{j,k}\rangle|^2.
\end{aligned}$$

Since $\{\tau_k\varphi\}_{k\in\mathbb{Z}^+}$ be a frame for V_0, therefore, we have

$$A\|f\|_2^2 = A\|D^{-j} f\|_2^2 \le \sum_{k\in\mathbb{Z}^+} |\langle f, \varphi_{j,k}\rangle|^2 \le B\|D^{-j} f\|_2^2 = B\|f\|_2^2.$$

This completes the proof of the Proposition.

Next, we characterize all functions of FSI space in terms of its Walsh-Fourier transform.

Proposition 3.3. *Let $\{\tau_k\varphi : k \in \mathbb{Z}^+, \varphi \in \Omega\}$ be a frame for its closed linear span V, where $\Omega = \{\varphi_1, \varphi_2, ..., \varphi_L\} \subset L^2(\mathbb{R}^+)$. Then, $f \in L^2(\mathbb{R}^+)$ lies in V if and only if there exist periodic functions $h_\ell \in L^2[0,1], \ell = 1, ..., L$ such that*

$$\hat{f}(\xi) = \sum_{\ell=1}^{L} h_\ell(\xi)\hat{\varphi}_\ell(\xi). \tag{3.3}$$

Proof. Since the system $\{\tau_k\varphi : k \in \mathbb{Z}^+, \varphi \in \Omega\}$ is a frame for V, then by Theorem 2.2, there exist a sequence $\left\{a_k^\ell\right\} \in l^2(\mathbb{Z}^+)$, for $\ell = 1, ..., L$ such that

$$f(x) = \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^+} a_k^\ell \varphi_\ell(x\ominus k). \tag{3.4}$$

Taking Walsh-Fourier transform on both sides of (3.4), we obtain

$$\hat{f}(\xi) = \sum_{\ell=1}^{L} h_\ell(\xi)\hat{\varphi}_\ell(\xi),$$

where $h_\ell(\xi) = \sum_{k\in\mathbb{Z}^+} a_k^\ell \chi_k(\xi)$ are the periodic functions in $L^2[0,1]$. The converse is established by taking h_ℓ as above and applying the inverse Walsh-Fourier transform on both sides of (3.3). □

We now study some properties of the multiresolution subspaces V_j of the form (3.2) by means of the Walsh-Fourier transform.

Proposition 3.4. *Let $\{\tau_k\varphi\}$ be a frame for $V_0 = \overline{span}\{\tau_k\varphi : k \in \mathbb{Z}^+\}$ and for $j \in \mathbb{Z}$, define V_j by (3.2). Then for any function $\psi \in V_1$, there exist periodic* *functions* $G \in L^2[0,1]$ *such that*

$$\hat{\psi}(p\xi) = p^{-1/2} G(\xi)\hat{\varphi}(\xi). \tag{3.5}$$

Proof. By the definition of V_j, it follows that $\psi(p^{-1}x) \in V_0$. By Proposition 3.3, there exists a periodic function $G \in L^2[0,1]$ such that $\big(\psi(p^{-1}x)\big)^\wedge = \hat{\psi}(p\xi) = p^{-1/2}G(\xi)\hat{\varphi}(\xi)$ lies in $L^2(\mathbb{R}^+)$. □

The following theorem establishes a sufficient condition to ensure that the nesting property holds for the subspaces V_j's.

Theorem 3.5. *Let* $\{\tau_k\varphi\}$ *be a frame for* $V_0 = \overline{span}\{\tau_k\varphi : k \in \mathbb{Z}^+\}$ *and for* $j \in \mathbb{Z}$, *define* V_j *by* (3.2). *Assume that there exists a periodic function* $H \in L^\infty[0,1]$ *such that*

$$\hat{\varphi}(\xi) = p^{-1/2} H(p^{-1}\xi)\hat{\varphi}(p^{-1}\xi). \tag{3.6}$$

Then, $V_j \subseteq V_{j+1}$, *for every* $j \in \mathbb{Z}$.

Proof. Given any $f \in V_j$, there exist as sequence $\{a_k\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+)$ such that

$$f(x) = \sum_{k\in\mathbb{Z}^+} p^{j/2} a_k \varphi(p^j x \ominus k). \tag{3.7}$$

Let $m_0(\xi) = \sum_{k\in\mathbb{Z}^+} a_k\chi_k(\xi) \in L^2[0,1]$, $m_1(p^{-1}\xi) = m_0(\xi)H(p^{-1}\xi)$. Then, clearly m_1 lies in $L^2[0,1]$ as H lies in $L^\infty[0,1]$. Therefore, by Parsevals identity, there exist a sequence $\{b_k\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+)$ such that $m_1(\xi) = \sum_{k\in\mathbb{Z}^+} b_k\chi_k(\xi)$ lies in $L^2(\mathbb{R}^+)$.

Applying the Walsh-Fourier transform to (3.7), we obtain

$$\begin{aligned}
\hat{f}(\xi) &= p^{\frac{-j}{2}} m_0(p^{-j}\xi)\,\hat{\varphi}(p^{-j}\xi) \\
&= p^{\frac{-j-1}{2}} m_0(p^{-j}\xi) H(p^{-j-1}\xi)\,\hat{\varphi}(p^{-j-1}\xi) \\
&= p^{\frac{-j-1}{2}} m_1(p^{-j-1}\xi)\,\hat{\varphi}(p^{-j-1}\xi).
\end{aligned} \tag{3.8}$$

Implementing inverse Walsh-Fourier transform to (3.8), we obtain

$$f(x) = p^{\frac{j+1}{2}} \sum_{k\in\mathbb{Z}^+} b_k\,\varphi(p^{j+1}x \ominus k). \tag{3.9}$$

Thus the function f lies in V_{j+1} by Proposition 3.2. Moreover, it is easy to verify that the function H in (3.6) is not unique. □

The following theorem is a converse to Theorem 3.5.

Theorem 3.6. *Let* $\{\tau_k\varphi\}$ *be a frame for* $V_0 = \overline{span}\{\tau_k\varphi : k \in \mathbb{Z}^+\}$ *and for* $j \in \mathbb{Z}$, *define* V_j *by* (3.2). *Assume that* $V_0 \subseteq V_1$ *and* $\Phi(\xi) = \|\hat{\varphi}(\xi \oplus k)\|^2_{l^2(\mathbb{Z}^+)}$. *Then there exists periodic function* $H \in L^\infty[0,1]$ *such that* (3.6) *holds.*

Proof. Since $\{\tau_k\varphi\}_{k\in\mathbb{Z}^+}$ is a frame for V_0, therefore, there exist positive constants A and B such that

$$A \leq \Phi(\xi) \leq B \ a.e \text{ on } \sigma(V_0).$$

Since $V_0 \subseteq V_1$, we have $\varphi \in V_1$. By Proposition 3.4, there exists a periodic function $H_0 \in L^2[0,1]$ such that

$$\hat{\varphi}(p\xi) = p^{-1/2} H_0(\xi)\hat{\varphi}(\xi).$$

Therefore, we have

$$|\hat{\varphi}(\xi)|^2 = p^{-1} \left|H_0(p^{-1}\xi)\right|^2 \left|\hat{\varphi}(p^{-1}\xi)\right|^2 \ a.e. \tag{3.10}$$

Let $S = [0,1)\backslash\sigma(V_0)$ and $H \in L^2[0,1]$ be a periodic function such that $H = H_0$, $a.e$ on $\sigma(V_0)$ and H is bounded on S by a positive constant C. Then, it follows from the above fact that H is not unique so that (3.10) also holds for H, i.e.,

$$|\hat{\varphi}(\xi)|^2 = p^{-1}|H(p^{-1}\xi)|^2|\hat{\varphi}(p^{-1}\xi)|^2 \ a.e.$$

Taking $n = kp + r$, where $k \in \mathbb{Z}^+$ and $r = 0, 1, \ldots, p-1$, we have

$$|\hat{\varphi}(\xi \oplus n)|^2 = p^{-1}|H(p^{-1}\xi \oplus p^{-1}r)|^2|\hat{\varphi}(p^{-1}\xi \oplus rp^{-1} \oplus k|^2 \ a.e. \tag{3.11}$$

Summing up (3.11) for all $k \in \mathbb{Z}^+$ and $r = 0, 1, ..., p-1$, we have

$$\sum_{n\in\mathbb{Z}^+} |\hat{\varphi}(\xi \oplus n)|^2 = p^{-1} \sum_{r=0}^{p-1} |H(p^{-1}\xi \oplus p^{-1}r)|^2 \sum_{k\in\mathbb{Z}^+} |\hat{\varphi}(p^{-1}\xi \oplus rp^{-1} \oplus k|^2 \ a.e,$$

which is equivalent to

$$\Phi(\xi) = p^{-1} \sum_{r=0}^{p-1} |H(p^{-1}\xi \oplus p^{-1}r)|^2 \, \Phi(p^{-1}\xi \oplus p^{-1}r) \ a.e,$$

or

$$\Phi(p\xi) = p^{-1} \sum_{r=0}^{p-1} |H(\xi \oplus p^{-1}r)|^2 \, \Phi(\xi \oplus p^{-1}r) \ a.e. \tag{3.12}$$

Note that $\Phi(p\,\xi) \leq B$ a.e and hence, (3.12) becomes

$$\sum_{r=0}^{p-1} |H(\xi \oplus p^{-1}r)|^2 \, \Phi(\xi \oplus p^{-1}r) \leq pB \ a.e.$$

This implies that for almost every $\xi \in [0, \frac{1}{p})$ and $r = 0, 1, ..., p-1$, we have

$$|H(\xi \oplus p^{-1}r)|^2 \, \Phi(\xi \oplus p^{-1}r) \leq pB.$$

Further, if $\Phi(\xi \oplus p^{-1}r) = 0$, then $|H(\xi \oplus p^{-1}r)| \leq C$ and if $\Phi(\xi \oplus p^{-1}r) > 0$, then we may assume that $A \leq \Phi(\xi \oplus p^{-1}r) \leq B$. Thus, for almost every $\xi \in [0, \frac{1}{p})$ and $r = 0, 1, \ldots, p-1$, we have

$$|H(\xi \oplus p^{-1}r)|^2 \leq \max\left\{C^2, pBA^{-1}\right\}.$$

Hence H is essentially bounded on the interval $[0, 1)$. This proves the theorem completely.

The following two propositions are proved in [4]:

Proposition 3.7. *Suppose* $V_0 = \overline{span}\{\tau_k\varphi : k \in \mathbb{Z}^+\}$ *and for each* $j \in \mathbb{Z}$, *define* V_j *by* (3.2) *such that* $V_0 \subseteq V_1$. *Assume that* $|\hat{\varphi}| > 0$, *a.e on a neighborhood of zero. Then, the union* $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R}^+)$.

Proposition 3.8. *Let* $\varphi \in L^2(\mathbb{R}^+)$ *and define* $V_0 = \overline{span}\{\tau_k\varphi : k \in \mathbb{Z}^+\}$. *For each* $j \in \mathbb{Z}$, *define* V_j *by* (3.2). *Then, we have* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Lemma 3.9. *Let* V_j *be the family of subspaces defined by* (3.2) *with* $V_j \subseteq V_{j+1}$, *for each* $j \in \mathbb{Z}$. *Suppose* $\varphi \in L^2(\mathbb{R}^+)$ *be a non-zero function with* $V_0 = \overline{span}\{\tau_k\varphi : k \in \mathbb{Z}^+\}$. *Then, for every* $j \in \mathbb{Z}$, V_j *is a proper subspace of* V_{j+1}.

Proof. Suppose that $V_\ell = V_{\ell+1}$ for some $\ell \in \mathbb{Z}$. Let $f \in V_{j+1}$, then for any given $j \in \mathbb{Z}$, we have $f(p^{-j-1+\ell+1}x) \in V_{j+1}$. Since $f(p^{-j+\ell}x) \in V_\ell$, therefore f lies in V_j and $V_j = V_{j+1}$. Hence, $\bigcap_{j\in\mathbb{Z}} V_j = V_0$. Therefore, it follows from Proposition 3.8 that $V_j = \{0\}$, which is a contradiction. □

Combining all our results so far, we have the following theorem.

Theorem 3.10. *Let* $\varphi \in L^2(\mathbb{R}^+)$ *and define* $V_0 = \overline{span}\{\tau_k\varphi : k \in \mathbb{Z}^+\}$. *For each* $j \in \mathbb{Z}$, *define* V_j *by* (3.2) *and* $\Phi(\xi) = \|\hat{\varphi}(\xi \oplus k)\|^2_{l^2(\mathbb{Z}^+)}$. *Suppose that the following hold:*

(i) $A \leq \Phi(\xi) \leq B$ *a.e on* $\sigma(V_0)$

(ii) *There exists a periodic function* $H \in L^\infty[0, 1]$ *such that*

$$\hat{\varphi}(\xi) = p^{-1/2} H(p^{-1}\xi)\hat{\varphi}(p^{-1}\xi), \ \text{a.e.}$$

(iii) $|\hat{\varphi}| > 0$, *a.e on a neighborhood of zero.*

Then, $\{V_j\}_{j\in\mathbb{Z}}$ *defines a p-frame multiresolution analysis of* $L^2(\mathbb{R}^+)$.

Proof. Since V_0 is a shift-invariant subspace of $L^2(\mathbb{R}^+)$. Therefore, the system $\{\tau_k\varphi\}$ forms a frame for V_0 with frame bounds A and B. Then, it follows from Theorem 3.5 and Lemma 3.9 that $V_j \subset V_{j+1}$, for every $j \in \mathbb{Z}$. Now, by the definition of V_j, f lies in V_j if and only if $f(p^{-j}.)$ lies in V_0, while $f(p.)$ lies in V_{j+1} if and only if $f(p^{-j-1}(p.))$ lies in V_0. Thus, f lies in V_j if and only if $f(p.)$ lies in V_{j+1}. Further, by our assumption (iii) and Proposition 3.8, it follows that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. Therefore, the sequence $\{V_j\}_{j\in\mathbb{Z}}$ satisfies all the conditions to be an p-FMRA of $L^2(\mathbb{R}^+)$. □

In order to construct *p-wavelet frames* associated with p-FMRA on a positive half-line $\mathbb{R}^+$, we introduce the orthogonal complement subspaces $\{W_j\}_{j\in\mathbb{Z}}$ of V_j in V_{j+1}. Further, it is easy verify that the sequence of subspaces $\{W_j\}_{j\in\mathbb{Z}}$ also satisfies the scaling property, i.e.,

$$W_j = \left\{ f \in L^2(\mathbb{R}^+) : f(p^{-j}\cdot) \in W_0 \right\}, \quad j \in \mathbb{Z}. \tag{3.13}$$

Theorem 3.11. *Let $\{V_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of closed subspaces of $L^2(\mathbb{R}^+)$ such that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. Let W_j be the orthogonal complement of V_j in V_{j+1}, for each $j \in \mathbb{Z}$. Then, the subspaces W_j are pairwise orthogonal and*

$$L^2(\mathbb{R}^+) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Proof. Assume that $i < j$, then $\langle f_i, f_j \rangle = 0$, for any $f_i \in W_j$ as $W_i \subset V_{i+1} \subset V_j$. Let P_j be the orthogonal projection operators from $L^2(\mathbb{R}^+)$ onto V_j, then $\lim_{j\to\infty} P_j f = f$, $\lim_{j\to-\infty} P_j f = 0$ and $W_j = \{f - P_j f : f \in V_{j+1}\}$. Therefore, for any $f \in L^2(\mathbb{R}^+)$, we have

$$f = \sum_{j\in\mathbb{Z}} (P_{j+1} f - P_j f).$$

Thus, the result of the direct sum follows since $P_{j+1} - P_j$ is the orthogonal projector from $L^2(\mathbb{R}^+)$ onto W_j. □

4. CHARACTERIZATION OF p-WAVELET FRAMES

In this section, we give the characterization all p-wavelet frames associated with p-FMRA on a half-line $\mathbb{R}^+$. First, we shall characterize the existence of a function ψ in W_0, where W_0 is the orthogonal complement of V_0 in V_1, by virtue of the *analysis filters* G and H.

Theorem 4.1. *Let H be a periodic function associated with an p-FMRA $\{V_j : j \in \mathbb{Z}\}$ such that* (3.6) *holds. Define W_0 as the orthogonal complement of V_0 in V_1. Let $\psi \in V_1$ such that*

$$\hat{\varphi}(\xi) = p^{-1/2} G(\xi/p) \hat{\varphi}(\xi/p). \tag{4.1}$$

where G is a periodic function in $L^2[0,1]$. Then ψ lies in W_0 if and only if

$$\sum_{r=0}^{p-1} H\left(p^{-1}\xi \oplus p^{-1}r\right) \Phi\left(p^{-1}\xi \oplus p^{-1}r\right) \overline{G\left(p^{-1}\xi \oplus p^{-1}r\right)} = 0 \ a.e. \tag{4.2}$$

Proof. We note that ψ lies in W_0 if and only if

$$\langle \psi, \tau_k \psi \rangle = \langle \psi, \psi(. \ominus k) \rangle = 0, \text{ for all } k \in \mathbb{Z}^+. \tag{4.3}$$

Define

$$F(\xi) = \sum_{k \in \mathbb{Z}^+} \hat{\varphi}(\xi \oplus k) \, \overline{\hat{\psi}(\xi \oplus k)}.$$

Then, it is easy to verify that F lies in $L^1[0,1]$ by using Monotonic Convergence Theorem and the Plancherel Theorem as

$$\begin{aligned}
\int_0^1 |F(\xi)| d\xi &\le \int_0^1 \sum_{k \in \mathbb{Z}^+} \left| \hat{\varphi}(\xi \oplus k) \hat{\psi}(\xi \oplus k) \right| d\xi \\
&= \sum_{k \in \mathbb{Z}^+} \int_0^1 \left| \hat{\varphi}(\xi \oplus k) \hat{\psi}(\xi \oplus k) \right| d\xi \\
&= \int_{\mathbb{R}^+} \left| \hat{\varphi}(\xi) \hat{\psi}(\xi) \right| d\xi \\
&\le \|\hat{\varphi}\|_2 \left\| \hat{\psi} \right\|_2 = \|\varphi\|_2 \|\psi\|_2.
\end{aligned}$$

Now, for a fixed $n \in \mathbb{Z}^+$, let F_M be the function defined by

$$F_M(\xi) = \sum_{k=0}^{M} \hat{\varphi}(\xi \oplus k) \, \overline{\hat{\psi}(\xi \oplus k)} \, \chi_n(\xi).$$

Then, in view of (3.6) and (4.1), we have

$$F_M(\xi) = \sum_{r=0}^{p-1} \sum_{|pk+r| \le M} H\left(p^{-1}\xi \oplus p^{-1} r\right) |\hat{\varphi}\left(p^{-1}\xi \oplus p^{-1} r \oplus k\right)|^2 \, \overline{G\left(p^{-1}\xi \oplus p^{-1} r\right)} \, \chi_n(\xi). \tag{3.4}$$

Implementation of Monotonic Convergence Theorem and the Cauchy-Schwartz inequality yields

$$\begin{aligned}
\left\| F_M - F\chi_n \right\|_{L^2[0,1]} &\le \int_0^1 \sum_{|k| \ge M+1} \left| \hat{\varphi}(\xi \oplus k) \hat{\psi}(\xi \oplus k) \right| d\xi \\
&= \sum_{|k| \ge M+1} \int_0^1 \left| \hat{\varphi}(\xi \oplus k) \hat{\psi}(\xi \oplus k) \right| d\xi \\
&= \sum_{|k| \ge M+1} \int_k^{k+1} \left| \hat{\varphi}(\xi) \hat{\psi}(\xi) \right| d\xi
\end{aligned}$$

$$\le \int_{|\xi| > M} \left| \hat{\varphi}(\xi) \hat{\psi}(\xi) \right| d\xi$$

$$\leq \left\{ \int_{|\xi|>M} |\hat{\varphi}(\xi)|^2 \, d\xi \right\}^{1/2} \left\{ \int_{|\xi|>M} \left| \hat{\psi}(\xi) \right| |^2 d\xi \right\}^{1/2} \to 0 \text{ as } M \to \infty.$$

Thus,

$$\lim_{M\to\infty} \left\| F_M - F\chi_n \right\|_{L^2[0,1]} = 0. \tag{4.5}$$

Therefore, there exists a subsequence $\left\{ F_{M_j} \right\}$ such that

$$\lim_{j\to\infty} \left\| F_{M_j} - F\chi_n \right\|_{L^2[0,1]} = 0, \ a.e.$$

Hence

$$F(\xi) = \sum_{r=0}^{p-1} p^{-1} H\left(p^{-1}\xi \oplus p^{-1}r\right) \Phi\left(p^{-1}\xi \oplus p^{-1}r\right) \overline{G\left(p^{-1}\xi \oplus p^{-1}r\right)} \ a.e.$$

Using (4.5) and the Dominated Convergence Theorem, we have for all $n \in \mathbb{Z}^+$,

$$\begin{aligned} \langle \psi, \tau_{-n}\varphi \rangle &= \int_{\mathbb{R}^+} \hat{\psi}(\xi)\overline{\hat{\varphi}(\xi)}\chi_n(\xi)d\xi \\ &= \sum_{k\in\mathbb{Z}^+} \int_k^{k+1} \hat{\psi}(\xi)\overline{\hat{\varphi}(\xi)}\chi_n(\xi)d\xi \\ &= \lim_{M\to\infty} \sum_{k=0}^{M} \int_0^1 \hat{\psi}(\xi \oplus k)\overline{\hat{\varphi}(\xi \oplus k)}\chi_n(\xi)\chi_k(\xi)d\xi \\ &= \lim_{M\to\infty} \int_0^1 F_M(\xi)d\xi \\ &= \int_0^1 F(\xi)\chi_n(\xi)d\xi. \end{aligned}$$

Consequently, $F = 0$ *a.e*, is the necessary and sufficient condition for (4.3) to hold for all $n \in \mathbb{Z}^+$. □

Lemma 4.2. *Let $\{W_j : j \in \mathbb{Z}\}$ be a sequence of pairwise orthogonal closed subspaces of $L^2(\mathbb{R}^+)$ such that $L^2(\mathbb{R}^+) = \bigoplus_{j\in\mathbb{Z}} W_j$. Then, for every $f \in L^2(\mathbb{R}^+)$, there exist $f_j \in W_j, j \in \mathbb{Z}$ such that $f(x) = \sum_{j\in\mathbb{Z}} f_j(x)$. Furthermore,*

$$\left\| f \right\|_2^2 = \sum_{j\in\mathbb{Z}} \left\| f_j \right\|_2^2.$$

Proof. For any arbitrary function $f \in L^2(\mathbb{R}^+)$, we have

$$\lim_{n\to\infty} \left\| f - \sum_{j=-n}^{n} f_j \right\|_2 = 0,$$

where $f_j \in W_j$, for each $j \in \mathbb{Z}$. Moreover, for a fixed $n \in \mathbb{N}$, we have

$$\Big\| \sum_{j=-n}^{n} f_j \Big\|_2^2 = \sum_{j=-n}^{n} \|f_j\|_2^2.$$

Since the norm $\|.\|_2$ is continuous, hence the desired result is obtained by taking $n \to \infty$ on both sides of the above equality. □

Theorem 4.3. *Let φ be the scaling function for an p-FMRA $\{V_j : j \in \mathbb{Z}\}$ and suppose that W_j be the orthogonal complement of V_j in V_{j+1}. Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subset W_0$. Then, the collection*

$$\mathcal{F}_\Psi = \Big\{ \psi^\ell_{j,k}(x) := p^{j/2}\psi^\ell(p^j x \ominus k), j \in \mathbb{Z}, k \in \mathbb{Z}^+, \ell = 1, \dots, L \Big\} \tag{4.6}$$

forms a p-wavelet frame for $L^2(\mathbb{R}^+)$ with frame bounds A and B if and only if

$$\big\{ \tau_k \psi^\ell : k \in \mathbb{Z}^+, \ell = 1, \dots, L \big\}$$

is a frame for W_0 with frame bounds A and B.

Proof. Suppose that the system $\mathcal{F}_\Psi$ given by (4.6) is a p-wavelet frame for $L^2(\mathbb{R}^+)$ with bounds A and B. Therefore, it follows from (3.13) that the family of functions $\psi^\ell_{j,k}$ lies in W_j, for $\ell = 1, \dots, L, j \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$.

By applying Theorem 3.11 to an arbitrary function $f \in W_0$, we have

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^+} |\langle f, \psi^\ell_{j,k} \rangle|^2 = \sum_{k \in \mathbb{Z}^+} |\langle f, \tau_k \psi^\ell \rangle|^2.$$

Consequently, using the p-wavelet frame condition of the system $\mathcal{F}_\Psi$, we have

$$A\|f\|_2^2 \leq \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^+} |\langle f, \tau_k \psi^\ell \rangle|^2 \leq B\|f\|_2^2,$$

and it follows that the collection $\big\{ \tau_k \psi^\ell : k \in \mathbb{Z}^+, \ell = 1, \dots, L \big\}$ is a frame for W_0.

Conversely, suppose that the collection $\big\{ \tau_k \psi^\ell : k \in \mathbb{Z}^+, \ell = 1, \dots, L \big\}$ is a frame for W_0 with bounds A and B. For any fixed $j \in \mathbb{Z}$ and $f \in W_j$, we have from equation (3.13) that $f(p^{-j}\cdot) \in W_0$. Further, by making use of the fact that

$$\big\langle f, \psi^\ell_{j,k} \big\rangle = \int_{\mathbb{R}^+} f(x)\, \overline{p^{j/2}\psi^\ell(p^j x \ominus k)}\, dx$$

and

$$\Big\| p^{-j/2} f(p^{-j}.) \Big\|_2^2 = p^{-j} \int_{\mathbb{R}^+} \big| f(p^{-j}x) \big|^2 dx = \big\| f \big\|_2^2,$$

we have

$$A\left\|p^{-j/2}f(p^{-j}\cdot)\right\|_2^2 \le \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+}\left|\langle f,\psi_{j,k}^{\ell}\rangle\right|^2 \le B\left\|p^{-j/2}f(p^{-j}\cdot)\right\|_2^2. \tag{4.7}$$

Therefore, for a given $j\in\mathbb{Z}$, the collection $\left\{\psi_{j,k}^{\ell} : k\in\mathbb{Z}^+, \ell=1,\ldots,L\right\}$ is a frame for W_j with frame bounds A and B.

Let f be an arbitrary function in $L^2(\mathbb{R}^+)$, then by Theorem 3.11 and Lemma 4.2, there exist $f_j\in W_j$ such that

$$f=\sum_{j\in\mathbb{Z}}f_j, \quad \text{and} \quad \left\langle f_i,\psi_{j,k}^{\ell}\right\rangle=0, \quad i\neq j.$$

Thus, we have

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{j,k}^{\ell}\right\rangle\right|^2=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\sum_{i\in\mathbb{Z}}\left\langle f_i,\psi_{j,k}^{\ell}\right\rangle\right|^2$$

$$=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f_j,\psi_{j,k}^{\ell}\right\rangle\right|^2. \tag{4.8}$$

It follows from (4.7) that

$$A\sum_{j\in\mathbb{Z}}\left\|f_j\right\|_2^2 \le \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f_j,\psi_{j,k}^{\ell}\right\rangle\right|^2 \le B\sum_{j\in\mathbb{Z}}\left\|f_j\right\|_2^2. \tag{4.9}$$

By combining (4.8), (4.9) and Lemma 4.2, we obtain

$$A\left\|f_j\right\|_2^2 \le \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f_j,\psi_{j,k}^{\ell}\right\rangle\right|^2 \le B\left\|f_j\right\|_2^2.$$

This completes the proof of the theorem.

Acknowledgement

The author was partially supported by the University of Kashmir, under seed money grant scheme, letter no. F(Seed Money Grant)Res/KU; dated: March 26, 2013.

References

[1] J. J. Benedetto and S. Li, The theory of multiresolution analysis frames and applications to filter banks, *Appl. Comput. Harmon. Anal.,* **5** (1998), 398-427.

[2] J. J. Benedetto and O. M. Treiber, Wavelet frames: multiresolution analysis and extension principle, in *Wavelet Transforms and Time-Frequency Signal Analysis*, L. Debnath, Editor, Birkhaiiser, Boston, (2000), 3-36.

[3] I. Daubechies, B. Han, A. Ron, and Z. Shen, Framelets: MRA-based constructions of wavelet frames, *Appl. Comput. Harmon. Anal.,* **14** (2003), 1-46.

[4] Yu. A. Farkov, On wavelets related to Walsh series, *J. Approx. Theory*, **161** (2009), 259-279.

[5] Yu. A. Farkov, Yu. A. Maksimov and S. A Stoganov, On biorthogonal wavelets related to the Walsh functions, *Int. J. Wavelets Multiresolut. Inf. Process.,* **9**(3) (2011), 485-499.
[6] Yu. A. Farkov, and E. A. Rodionov, Nonstationary wavelets related to the Walsh functions, *Amer. J. Comput. Math.,* **2** (2012) 82-87.
[7] B. I. Golubov, A. V. Efimov and V. A. Skvortsov, *Walsh Series and Transforms: Theory and Applications*, Kluwer, Dordrecht, 1991.
[8] H. O. Kim and J. K. Lim, On frame wavelets associated frame multiresolution analysis, *Appl. Comput. Harmon. Anal.,* **10** (2001) 61-70.
[9] S. Mallat, Multiresolution approximations and wavelets. *Trans. Amer. Math. Soc.,* **315** (1989), 69-87.
[10] Meenakshi, P. Manchanda and A. H. Siddiqi, Wavelets associated with nonuniform multiresolution analysis on positive half-line, *Int. J. Wavelets Multiresolut. Inf. Process.,* **10**(2) (2012), 1250018.
[11] A. Ron and Z. Shen, Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.,* **148** (1997), 408-447.
[12] A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$, *Canad. J. Math.,* **47** (1995), 1051-1094.
[13] F. Schipp, W. R. Wade and P. Simon, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Bristol and New York, 1990.
[14] F. A. Shah, Construction of wavelet packets on p-adic field, *Int. J. Wavelets Multiresolut. Inf. Process.,* **7**(5) (2009), 553-565.
[15] F. A. Shah, Non-orthogonal p-wavelet packets on a half-line, *Anal. Theory Appl.,* **28**(4) (2012), 385-396.
[16] F. A. Shah, Biorthogonal wavelet packets related to the Walsh polynomials, *J. Classical Anal.,* **1** (2012), 135-146.
[17] F. A. Shah, On some properties of p-wavelet packets via the Walsh-Fourier transform, *J. Nonlinear Anal. Optimiz.,* **3** (2012), 185-193.
[18] F. A. Shah, Tight wavelet frames generated by the Walsh polynomials, *Int. J. Wavelets, Multiresolut. Inf. Process.,* **11**(6) (2013), 1350042.
[19] F. A. Shah and L. Debnath, Dyadic wavelet frames on a half-line using the Walsh-Fourier transform, *Integ. Transf. Special Funct.,* **22**(7) (2011), 477-486.
[20] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge University Press, 1997.
[21] Y. Xiaojiang, Semiorthogonal multiresolution analysis frames in higher dimensions, *Acta Appl. Math.,* **111** (2010), 257-286.
[22] Z. H. Zhang, A characterization of generalized frame MRA's deriving orthonormal wavelets, *Acta Math. Sinica, English Series,* **22**(4) (2006), 1251-1260.

Department of Mathematics, University of Kashmir, South Campus, Anantnag-192101, Jammu and Kashmir, India