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Remarks on Cauchy-Riemann Structure

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Abstract. The present paper deals with Cauchy-Riemann structure (CR structure) satisfying relation $F^{2\nu+3} + \lambda^r F^2 = 0$. Certain results with CR structure on distributions, mathematical operators, integrabitity condtions satisfying the above structure are established.

1. Introduction

Properties of parallelism and geodesic over distributions were well studied by Nikic [1]. Lagrangian submersion and its properties were studied by Tastan and Siddiqui [2]. Nivas and Saxena defined and studied the properties of lifts on special manifold [3]. Demetropoulou and Andrew [4] define the properties of tensor field (1, 1) in the specific structure which is extended further by Nivas and Saxena and defined the properties of Horizontal and complete lift [5], [16], [6] . Mishra et. al. [17] studied the aspects of invariant submanifolds. Yano and Ishihara [12] explain the concept of tangent and cotangent bundles in detailed. Numerous investigators studied various geometric structures like complex structure, GF-structure, general quadratic structure, mathematical operators etc. and established certain results on tangent bundle [7]- [11].

Let us define N be a differentiable manifold of class C^∞, with the property that a non-zero (1, 1) tensor field F defined over N satisfying

$$F^{2\nu+3} + \lambda^r F^2 = 0, \tag{1.1}$$

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here ν is consider as a natural number, r be any positive integer, λ be a non-zero complex number and F is of constant rank equals to r. Let us represent the structure defined by (1.1) over N as $F_\lambda(2\nu+3,2)$ structure of rank r.

Now we define the projection operators over differentiable manifold N as

$$l = -\frac{F^{2\nu+1}}{\lambda^r} \text{ and } m = I + \frac{F^{2\nu+1}}{\lambda^r} \tag{1.2}$$

in the above equation (1.2), I is considered as the identity operator. We can easily show that the following conditions over the projection operators l and m holds good

$$\begin{aligned} m^2 = m,\ l^2 = l, \\ m + l = I,\ ml = lm = 0, \end{aligned} \tag{1.3}$$

where mathematical operators l and m are complementary projection operators and l and m, both are also defined over the $T(N)$, the tangent space of N.

Now let us define complementary distribution of projection operator l and m are G_l and G_m respectively. Then, we can easily show that G_m as a null operator and G_l as an almost complex structure operator, further $\frac{F^{\nu-1/2}}{\lambda^{r/2}}$ applicable on complementary distributions defined over the differentiable manifold N.

The Nijenhuise Tensor. Over the differentiable manifold N, for the tensor field F of type (1, 1), the Nijenhuis tensor $\mathcal{N}(U,V)$ is defined as

$$\mathcal{N}(U,V) = [FU,FV] - F[FU,V] - F[U,FV] + F^2[U,V], \tag{1.4}$$

where vector fields U and V are defined over N, further, the requisite condition for integrability of $F_\lambda(2\nu+3,2)$–Hsu structure is $\mathcal{N}\,(U,V) = 0$.

Definition 1.1. *If U and V are two vector fields defined over N, then torsion field Levi Civita connection is represented by Lie bracket $[U,V]$, which satisfy the equation*

$$[U,V] = \nabla_U V - \nabla_V U. \tag{1.5}$$

Above result is defined over CR structure.

2. CR-Structure

Properties of CR submanifold was studied by Blair [13] and further extended by Das [14]. Conditions of integrability and tangent space for the structure was defined by [4]. Das et. al [15] studied the properties of extended structure of F on N. Let N be defined as a differentiable manifold and the complex tangent bundle on the differentiable manifold N is defined as $T_C(N)$. The complex sub bundle is defined as H_p on $T_C(N)$ for the CR structure over N, satisfying the condition

$$H_p \cap \widetilde{H_p} = 0,$$

here H_p is involutive, which means that lie bracket $[U, V]$ is also belongs to H_p, for all complex vector fields U and V belongs to H_p, so H_p forms a CR-submanifold and $\widetilde{H_p}$ is complex conjugate of H_p.

Let $F_\lambda(2\nu+3,2)$ is a integrable structure satisfying equation (1.1), defined over the differentiable manifold N. Complex subbundle H_p of $T_c(N)$ is defined as

$$H_p = U - \sqrt{-1}FV;\ \forall\, U,\ V \in \chi(G_l), \tag{2.1}$$

where $\chi(G_l)$ is the $F(G_l)$ part of all the differentiable sections of G_l, which gives us

$$R_e(H_p) = G_l \text{ and } H_p \cap \widetilde{H_p} = 0.$$

Theorem 2.1. *H_p is the defined as complex subbundle and mathematical operators i. e. distributions P and Q belongs to H_p, then the following relation holds*

$$[P,Q] = [U,V] - [FU,FV] - \sqrt{-1}(-1)([FU,V] + [U,FV]). \tag{2.2}$$

Proof: To prove the result defined by (2.2) in the theorem, first of all we have to consider P and Q as follows

$$P = U - \sqrt{-1}FU;\ Q = V - \sqrt{-1}FV. \tag{2.3}$$

Using the property of differentiable manifold and structure we have

$$\begin{aligned}[P,Q] &= [U - \sqrt{-1}FU,\ V - \sqrt{-1}FV],\\ &= [U,V] - \sqrt{-1}(-1)[U,FV] - \sqrt{-1}(-1)[FU,V] - [FU,FV],\\ &= [U,V] - [FU,FV] - \sqrt{-1}(-1)([U,FV] + [FU,V]).\end{aligned}$$

Theorem 2.2. *If $F_\lambda(2\nu+3,2)$-Hsu structure is the CR structure which satisfy (1.1) is integrable, then*

$$-\frac{F^{2\nu}}{\lambda^r}(F[FU,FV] + F^2[U,V]) = l([FU,V] + [U,FV]). \tag{2.4}$$

Proof: From equation (1.4), which states

$$\mathrm{N}(U,V) = [FU,FV] - F[FU,V] - F[U,FV] + F^2[U,V].$$

If $F_\lambda(2\nu+3,2)$ is integrable then $\mathrm{N}(U,V) = 0$, which gives

$$[FU,FV] + F^2[U,V] = F[U,FV] + F[FU,V]. \tag{2.5}$$

Operating (2.5) by $-\dfrac{F^{2\nu}}{\lambda^r}$, we get

$$\begin{aligned}-\frac{F^{2\nu}}{\lambda^r}([FU,FV] + F^2[U,V]) &= -\frac{F^{2\nu}}{\lambda^r}F([U,FV] + [FU,V]),\\ &= -\frac{F^{2\nu+1}}{\lambda^r}([FU,V] + [U,FV]).\end{aligned} \tag{2.6}$$

Making use (1.2), (2.5) be in the form

$$-\frac{F^{2\nu}}{\lambda^r}(F[FU,FV] + F^2[U,V]) = l([FU,V] + [U,FV])$$

above equation concludes that theorem 2.2 holds.

Proposition 2.1. *For $F_\lambda(2\nu+3,2)$-Hsu structure defined over the manifold N, the following identities hold*

$$m\underset{,}{N}(U,V) = 0, \tag{2.7}$$

$$m\underset{,}{N}(\frac{F^{2\nu}}{\lambda^r}U,V) = m[\frac{F^{2\nu}}{\lambda^r}U,FV]. \tag{2.8}$$

Theorem 2.3. *In a $F_\lambda(2\nu+3,2)$-Hsu structure manifold N. Vector fields U and V defined over N, then the following results holds.*

$$m\underset{,}{N}(U,V) = 0, \tag{2.9}$$

$$m[FU,FV] = 0, \tag{2.10}$$

$$m\underset{,}{N}(\frac{F^{2\nu}}{\lambda^r}U,V) = 0, \tag{2.11}$$

$$m\underset{,}{N}(\frac{F^{2\nu+1}}{\lambda^r}U,FV) = 0, \tag{2.12}$$

$$m\underset{,}{N}(\frac{F^{2\nu-1}}{\lambda^r}lU,FV) = 0. \tag{2.13}$$

To prove the equations (2.9)–(2.13), we use the equations (1.1), (1.2), (1.4) and (2.4).

Theorem 2.4. *If $\frac{F^{2\nu}}{\lambda^r}$ apply on G_l as an almost complex structure, then*

$$m(\frac{F^{2\nu+1}}{\lambda^r}lU,FV) = m[-FU,FV].$$

Proof: We have

$$\begin{aligned} m(\frac{F^{2\nu+1}}{\lambda^r}lU,FV) &= m(\frac{F^{2\nu}}{\lambda^r}FlU,FV), \\ &= m[-lFlU,FV], \\ &= m[-FU,FV]. \end{aligned}$$

Theorem 2.5. *For the vector fields U, $V \in \chi(G_l)$, following relation holds*

$$l([FU,V]+[U,FV]) = [FU,V]+[U,FV]. \tag{2.14}$$

Proof: Since $[FU,V]$ and $[U,FV] \in \chi(G_l)$,
then as

$$Fl = lF \text{ and } Fm = mF = 0.$$

Using equation (1.5), we have

$$\begin{aligned} ([FU,V]+[U,FV]) &= lU.FV - FV.U - V.FU \\ &= U.FV - FV.U + FVUV - V.FU, \\ &= [FU,V]+[U,FV]. \end{aligned}$$

Theorem 2.6. *The $F_\lambda(2\nu+3,2)$–Hsu structure satisfying equation (1.1) on N such that $R_eH = G_l$, where H is a complex sub bundle, then $F_\lambda(2\nu+3,2)$ structure is defines as a CR-structure over the differential manifold N.*

Proof: Let $[U, FV]$ and $[FU, V] \in \chi(G_l)$, using (2.2), (2.4) and theorem (2.5), we get

$$\begin{aligned} l[P,Q] &= l[U,V] - l[FU,FV] - \sqrt{-1}(-1)l([FU,V] + [U,FV]), \\ &= [U,V] - [FU,FV] - \sqrt{-1}(-1)([FU,V] + [U,FV]), \\ &= [P,Q]. \end{aligned}$$

Hence $[P,Q] \in \chi(G_l)$. Then $F_\lambda(2\nu+3,2)$–Hsu structure defines a CR-Structure satisfying equation (1.1) on N.

Proposition 2.2. *For $\widetilde{k}$ be the complementary distribution of R_eH_p over N. Then their exist a morphism of vector bundles $F\colon T(N) \rightarrowtail T(N)$ given by $F(U) = 0$ for all $U \in \chi(\widetilde{k})$ such that*

$$FU = 1/2\sqrt{-1}((\widetilde{P}) - P), \tag{2.15}$$

$\forall P = U + \sqrt{-1}V \in \chi(H_p)$ and $\widetilde{P}$ is defined as complex conjugate of P

Proposition 2.3. *If P and $\widetilde{P}$ are complex conjugate with their values as $P = U + iV$. $\widetilde{P} = U - iV$ for all $P, \widetilde{P} \in H_p$ and*

$$\begin{aligned} F(U) &= \frac{1}{2}i(P - \widetilde{P}), \\ F(V) &= \frac{1}{2}(P + \widetilde{P}), \\ F(-V) &= -\frac{1}{2}(P - \widetilde{P}). \end{aligned}$$

then we can easily conclude that $F(-V) = -U$, $F(U) = -V$ and $F^2(U) = -U$.

Theorem 2.7. *If H is CR-structure over manifold N, $F_\lambda(2\nu+3,2)$-Hsu structure is defined over N such that both distribution G_l and G_m coincide with $R_3(H)$ and k respectively.*

Proof: If H is a CR-structure over manifold N. Then in view of Proposition (2.1), (2.2), we get

$$F(U) = -V. \tag{2.16}$$

Multiply (2.12) by $(F^{2\nu+2} + \lambda^r F)$, we have

$$(F^{2\nu+2} + \lambda^r F)F(U) = (F^{2\nu+2} + \lambda^r F)(-V).$$

Using Proposition (2.2), the above equation can be redefined as

$$\begin{aligned} (F^{2\nu+3} + \lambda^r F^2)(U) &= (F^{2\nu+1} + \lambda^r)F^2(U), \\ &= (F^{2\nu+1} + \lambda^r)(-U), \\ &= -(F^{2\nu+1} + \lambda^r)(U). \end{aligned}$$

Applying the same process we get

$$(F^{2\nu+3} + \lambda^r F^2)(U) = 0,$$

which is indeed

$$F^{2\nu+3}(U) + \lambda^r F^2(U) = 0.$$

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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