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New Type of Fuzzy Algebra Structure Setting Complex Bipolar Neutrosophic Sets of Bisemirings

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Abstract. The notion of complex bipolar neutrosophic subbisemirings (CBNSBSs) is constructed and analyzed. We examine the significant characteristics and homomorphic features of CBNSBSs. We propose the CBNSBS level sets for bisemirings. Suppose that $\Bbbk$ is a subset of $\mathfrak{S}$. Then $R = (\complement_{\Bbbk}^{T^-} \cdot e^{i\omega\daleth_{\Bbbk}^{T^-}}, \complement_{\Bbbk}^{I^-} \cdot e^{i\omega\daleth_{\Bbbk}^{I^-}}, \complement_{\Bbbk}^{F^-} \cdot e^{i\omega\daleth_{\Bbbk}^{F^-}}, \complement_{\Bbbk}^{T^+} \cdot e^{i\omega\daleth_{\Bbbk}^{T^+}}, \complement_{\Bbbk}^{I^+} \cdot e^{i\omega\daleth_{\Bbbk}^{I^+}}, \complement_{\Bbbk}^{F^+} \cdot e^{i\omega\daleth_{\Bbbk}^{F^+}})$ is a CBNSBS of $\mathfrak{S}$ if and only if $\complement^{(\hbar_1,\hbar_2)}$ is a subbisemiring (SBS) of $\mathfrak{S}$ for all $\hbar_1, \hbar_2 \in [-1,0] \times [0,1]$. It is demonstrated that all CBNSBSs have homomorphic images, and all CBNSBSs have homomorphic pre-images. Examples are provided to show how our findings are used.

1. Introduction

Fuzzy set (FS) theory was initially developed by Zadeh [20], and it is the best at dealing with ambiguity and uncertainty. If an element in an FS has a single value inside the interval, it is regarded as a member. The degree of non-membership does not always equal one minus the degree of membership, though, as resistance can occur in real-world circumstances. An increasing number of hybrid fuzzy models are being created as FS theory develops swiftly. The uncertainties have contributed to the development of a number of uncertain theories, such as FS [20], intuitionistic FS (IFS) [3], Pythagorean FS (PFS) [19] and spherical FS (SFS) [2]. MG sets, or sets with grades between 0 and 1, make up an FS. Although the representation made by Atanassov [3] that non-membership grades (NMG) can only have a value of 1, IFS is categorized as MG. The total of MGs and NMGs

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may occasionally exceed 1 throughout a decision-making process. Yager [19] used PFS logic to develop the generalized MG and NMG logic, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs. As the neutral state is neither positive nor negative, these theories are unable to express it. Cuong [4] talked to colleagues about the picture FS. FS used three grading points: positive, neutral, and negative. The sum of these grades could not be greater than 1. It also outperforms PFS and IFS in several situations. It addresses the truth, indeterminacy, and falsity of FS and IFS and is an autonomous generalization of three models. In Lee [7], the idea of bipolar fuzzy sets was presented. The membership degree range in conventional fuzzy sets is $[0, 1]$. FSs that have their membership degree range expanded from the interval $[0, 1]$ to the interval $[-1, 1]$ are called bipolar fuzzy sets. In a bipolar fuzzy set, elements with a membership degree of 0 are not relevant to the corresponding property; elements with a membership degree of $(0, 1]$ indicate that the property is somewhat satisfied; elements with a membership degree of $[-1, 0]$ indicate that the implicit counter property is somewhat satisfied.

To handle conflicting and unclear data, Smarandache [18] developed the neutrosophic set (NS). The degree to which an idea is true, ambiguous, or false is established using this logic. Ramot et al. [16] introduce the concept of the complex fuzzy set (CFS). The membership functions of CFS's transactions can have a very broad range of values. While the unit circle of a fuzzy membership function remains fixed, the unit circle of the complex plane is expanded to $[0, 1]$. Rather than extending exclusively to $[0, 1]$, the membership function $\mu_X(x)$ of the CFS X extends to the unit circle in the complex plane. Hence, $\mu_X(x)$ is a complex-valued function that, for any element x in the discourse universe, provides a grade of membership of the type $\eta_X(x) \cdot e^{i\tau_X(x)}$, where $i = \sqrt{-1}$. The two real-valued variables, $\eta_X(x)$ and $\tau_X(x)$, where $\eta_X(x) \in [0, 1]$, define the value of $\mu_X(x)$. Golan [5] established the concept of semiring logic and its applications. Hussian et al. [6] discussed the concept and use of bisemirings. Fuzzy semirings were studied by Ahsan et al. [1]. Sen et al. [17] introduced the concept of bisemirings. Palanikumar et al. (2019) introduced an intuitionistic fuzzy normal subbisemiring of bisemirings [10]. Palanikumar et al. [14] introduced the concept of bisemirings by using bipolar-valued neutrosophic normal sets. The novel aggregating operator was discussed by Palanikumar et al. [8,9,11–13,15].

We will examine particular elements of the SBS and CBNSBS ideas and draw some inferences. The following five sections make up the article. We introduce semirings and SBSs in Section 1. Information on semirings and SBS preparations is provided in Section 2. The properties of CBNSBS are listed in Section 3. For CBNSBS evaluation, numerical examples are advised. The conclusion and subsequent direction are indicated in Section 4.

2. Preliminaries

Definition 2.1. *[17] An algebraic structure $(\mathfrak{S}, \uplus, \ominus, \odot)$ is a bisemiring if $(\mathfrak{S}, \uplus, \ominus)$ and $(\mathfrak{S}, \ominus, \odot)$ are semirings, i.e., $(\mathfrak{S}, \uplus)$, $(\mathfrak{S}, \ominus)$, and $(\mathfrak{S}, \odot)$ are semigroups and*

(1) $z_1 \ominus (z_2 \uplus z_3) = (z_1 \ominus z_2) \uplus (z_1 \ominus z_3)$,

(2) $(z_2 \uplus z_3) \ominus z_1 = (z_2 \ominus z_1) \uplus (z_3 \ominus z_1)$,
(3) $z_1 \odot (z_2 \ominus z_3) = (z_1 \odot z_2) \ominus (z_1 \odot z_3)$,
(4) $(z_2 \ominus z_3) \odot z_1 = (z_2 \odot z_1) \ominus (z_3 \odot z_1), \forall z_1, z_2, z_3 \in \mathfrak{S}$.

Definition 2.2. *A bipolar fuzzy set* $\complement$ *in a universe* $\mathcal{U}$ *is an object having the form*

$$\complement = \left\{\left\langle x, A^+_\complement(x), A^-_\complement(x)\right\rangle \mid x \in \mathcal{U}\right\},$$

where $A^+_\complement : \mathcal{U} \to [0,1]$ *and* $A^-_\complement : \mathcal{U} \to [-1,0]$.

Here $A^+_\complement(x)$ *represents the degree of satisfaction of the element x to the property of* $A^-_\complement(x)$ *representing the degree of satisfaction of x to some implicit counter property of* $\complement$*. For simplicity, the symbol* $\left\langle A^+_\complement, A^-_\complement\right\rangle$ *is used for the bipolar fuzzy set* $\complement = \left\{\left\langle x, A^+_\complement(x), A^-_\complement(x)\right\rangle \mid x \in \mathcal{U}\right\}$.

Definition 2.3. *For two bipolar fuzzy subsets* $\complement = (\complement^+, \complement^-)$ *and* $\lambda = (\lambda^+, \lambda^-)$*, the product of* $\complement$ *and* λ *is denoted by* $\complement \circ \lambda$ *and is defined as*

$$(\complement^+ \circ \lambda^+)(x) = \begin{cases} \sup\limits_{(s,t)\in A_x} \left\{\complement^+(s) \wedge \lambda^+(t)\right\} \text{ if } A_x \neq 0 \\ 0 \text{ if } A_x = 0 \end{cases}$$

$$(\complement^- \circ \lambda^-)(x) = \begin{cases} \inf\limits_{(s,t)\in A_x} \left\{\lambda^-(s) \vee \lambda^-(t)\right\} \text{ if } A_x \neq 0 \\ -1 \text{ if } A_x = 0 \end{cases}$$

Definition 2.4. *[18] An NS* $\complement$ *in the universe* $\mathcal{U}$ *is* $\complement = \{x, A^T_\complement(x), A^I_\complement(x), A^F_\complement(x) \mid x \in \mathcal{U}\}$*, where* $A^T_\complement(x)$, $A^I_\complement(x)$ $A^F_\complement(x)$ *represents the TD, ID, and FD of v respectively. Consider the mapping* $A^T_\complement : \mathcal{U} \to [0,1], A^I_\complement : \mathcal{U} \to [0,1], A^F_\complement : \mathcal{U} \to [0,1]$*, and* $0 \le A^T_\complement(x) + A^I_\complement(x) + A^F_\complement(x) \le 3$.

Definition 2.5. *[18] Let* $\complement_1 = \langle \chi^T_{\complement_1}, \chi^I_{\complement_1}, \chi^F_{\complement_1}\rangle$, $\complement_2 = \langle \chi^T_{\complement_2}, \chi^I_{\complement_2}, \chi^F_{\complement_2}\rangle$*, and* $\complement_3 = \langle \chi^T_{\complement_3}, \chi^I_{\complement_3}, \chi^F_{\complement_3}\rangle$ *be three neutrosophic numbers over* $\mathcal{U}$*. Then*

(1) $\complement_2 \ominus \complement_3 = \left\langle \max\{\chi^T_{\complement_2}, \chi^T_{\complement_3}\}, \min\{\chi^I_{\complement_2}, \chi^I_{\complement_3}\}, \min\{\chi^F_{\complement_2}, \chi^F_{\complement_3}\}\right\rangle$,
(2) $\complement_2 \uplus \complement_3 = \left\langle \min\{\chi^T_{\complement_2}, \chi^T_{\complement_3}\}, \max\{\chi^I_{\complement_2}, \chi^I_{\complement_3}\}, \max\{\chi^F_{\complement_2}, \chi^F_{\complement_3}\}\right\rangle$,
(3) $\complement_2 \ge \complement_3 \Leftrightarrow \chi^T_{\complement_2} \ge \chi^T_{\complement_3}$ *and* $\chi^I_{\complement_2} \le \chi^I_{\complement_3}$ *and* $\chi^F_{\complement_2} \le \chi^F_{\complement_3}$,
(4) $\complement_2 = \complement_3 \Leftrightarrow \chi^T_{\complement_2} = \chi^T_{\complement_3}$ *and* $\chi^I_{\complement_2} = \chi^I_{\complement_3}$ *and* $\chi^F_{\complement_2} = \chi^F_{\complement_3}$.

Definition 2.6. *[18] For any NS* $\complement = \{x, A^T_\complement(x), A^I_\complement(x), A^F_\complement(x)\}$ *of* $\mathcal{U}$*. Then* (τ, β)*-cut is defined as*

$$\{x \in U \mid A^T_\complement(x) \ge \tau, A^I_\complement(x) \ge \tau, A^F_\complement(x) \le \beta\}.$$

3. Complex bipolar neutrosophic subbisemirings

Here $\mathfrak{S}$ denotes a bisemiring unless otherwise stated, $\complement$ stands for the real part and $\exists$ stands for the imaginary part and $\omega = 2\pi$.

Definition 3.1. *For any complex bipolar neutrosophic set (CBNS)* $\Bbbk$ *in a universal set* U,
$\Bbbk = \{\varkappa, \complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa)}, \complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}}, \complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^-(\varkappa)}}, \complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa)}, \complement_{\Bbbk}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^+(\varkappa)}}, \complement_{\Bbbk}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^+(\varkappa)}} \mid \varkappa \in U\}$, *where* $\complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-(\varkappa)}}, \complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}}, \complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^-(\varkappa)}} : U \to [-1,0] \times [0,1]$ *and* $\complement_{\Bbbk}^{T}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T(\varkappa)}}, \complement_{\Bbbk}^{I}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I(\varkappa)}}, \complement_{\Bbbk}^{F}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F(\varkappa)}} : U \to [-1,0] \times [0,1]$ *represents the truth degree, indeterminacy degree, and false degree, respectively.*

For simplicity, the symbols $\complement_{\Bbbk}^{T^-}, \complement_{\Bbbk}^{I^-}, \complement_{\Bbbk}^{F^-}$ *of the CBNS* $\Bbbk = \{\varkappa, \complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-(\varkappa)}}, \complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}}, \complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^-(\varkappa)}}, \complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^+(\varkappa)}}, \complement_{\Bbbk}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^+(\varkappa)}}, \complement_{\Bbbk}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^+(\varkappa)}} \mid \varkappa \in U\}$.

Definition 3.2. *Let* $\Bbbk = \left\{\varkappa, \complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-(\varkappa)}}, \complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}}, \complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^-(\varkappa)}}, \complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^+(\varkappa)}}, \complement_{\Bbbk}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^+(\varkappa)}}, \complement_{\Bbbk}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^+(\varkappa)}}\right\}$ *and* $\circledS = \left\{\varkappa, \complement_{\circledS}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{T^-(\varkappa)}}, \complement_{\circledS}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{I^-(\varkappa)}}, \complement_{\circledS}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{F^-(\varkappa)}}, \complement_{\circledS}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{T^+(\varkappa)}}, \complement_{\circledS}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{I^+(\varkappa)}}, \complement_{\circledS}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{F^+(\varkappa)}}\right\}$ *be two CBNSs of* U. *Then we define the intersection and union operation as*

(i) $\Bbbk \cap \circledS = \Big\{\Big(\varkappa, \max\{\complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-(\varkappa)}}, \complement_{\circledS}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{T^-(\varkappa)}}\}, \max\{\complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}}, \complement_{\circledS}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{I^-(\varkappa)}}\},$
$\min\{\complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^-(\varkappa)}}, \complement_{\circledS}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{F^-(\varkappa)}}\}, \min\{\complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^+(\varkappa)}}, \complement_{\circledS}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{T^+(\varkappa)}}\},$
$\min\{\complement_{\Bbbk}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^+(\varkappa)}}, \complement_{\circledS}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{I^+(\varkappa)}}\}, \max\{\complement_{\Bbbk}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^+(\varkappa)}}, \complement_{\circledS}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{F^+(\varkappa)}}\}\Big) \Big| \varkappa \in U\Big\}$.

(ii) $\Bbbk \cup \circledS = \Big\{\Big(\varkappa, \min\{\complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-(\varkappa)}}, \complement_{\circledS}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{T^-(\varkappa)}}\}, \min\{\complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}}, \complement_{\circledS}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{I^-(\varkappa)}}\},$
$\max\{\complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^-(\varkappa)}}, \complement_{\circledS}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{F^-(\varkappa)}}\}, \max\{\complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^+(\varkappa)}}, \complement_{\circledS}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{T^+(\varkappa)}}\},$
$\max\{\complement_{\Bbbk}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^+(\varkappa)}}, \complement_{\circledS}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{I^+(\varkappa)}}\}, \min\{\complement_{\Bbbk}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^+(\varkappa)}}, \complement_{\circledS}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\circledS}^{F^+(\varkappa)}}\}\Big) \Big| \varkappa \in U\Big\}$.

Definition 3.3. *For any CBNS* $\Bbbk = \left\{\varkappa, \complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-(\varkappa)}}, \complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}}, \complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^-(\varkappa)}}, \complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^+(\varkappa)}}, \complement_{\Bbbk}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^+(\varkappa)}}, \complement_{\Bbbk}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^+(\varkappa)}}\right\}$ *of a universal set* U. *Then* $(\hbar_1, \hbar_2)$*-cut is defined as* $\Big\{\varkappa \in U \mid \complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-(\varkappa)}} \le \hbar_1, \complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}} \le \hbar_1, \complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^-(\varkappa)}} \ge \hbar_2, \complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^+(\varkappa)}} \ge \hbar_1, \complement_{\Bbbk}^{I^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^+(\varkappa)}} \ge \hbar_1, \complement_{\Bbbk}^{F^+}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^+(\varkappa)}} \le \hbar_2\Big\}$.

Definition 3.4. *The Cartesian product of* $\Bbbk$ *and* $\circledS$ *is defined as*
$\Bbbk \times \circledS = \Big\{\complement_{\Bbbk\times\circledS}^{T^-}((\varkappa,\partial)) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{T^-((\varkappa,\partial))}}, \complement_{\Bbbk\times\circledS}^{I^-}((\varkappa,\partial)) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{I^-((\varkappa,\partial))}}, \complement_{\Bbbk\times\circledS}^{F^-}(\varkappa,\partial) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{F^-((\varkappa,\partial))}}, \complement_{\Bbbk\times\circledS}^{T^+}((\varkappa,\partial)) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{T^+((\varkappa,\partial))}}, \complement_{\Bbbk\times\circledS}^{I^+}((\varkappa,\partial)) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{I^+((\varkappa,\partial))}}, \complement_{\Bbbk\times\circledS}^{F^+}(\varkappa,\partial) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{F^+((\varkappa,\partial))}} \mid \varkappa, \partial \in S\Big\}$, *where* $\Bbbk$ *and* $\circledS$ *are CBNSs of* U,

$$\begin{cases} \complement_{\Bbbk\times\circledS}^{T^-}((\varkappa,\partial)) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{T^-((\varkappa,\partial))}} = \max\left\{\complement_{\Bbbk}^{T^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{T^-(\varkappa)}}, \complement_{\circledS}^{T^-}(\partial) \cdot e^{i\omega\daleth_{\circledS}^{T^-(\partial)}}\right\} \\ \complement_{\Bbbk\times\circledS}^{I^-}((\varkappa,\partial)) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{I^-((\varkappa,\partial))}} = \frac{\complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{I^-(\varkappa)}} + \complement_{\circledS}^{I^-}(\partial) \cdot e^{i\omega\daleth_{\circledS}^{I^-(\partial)}}}{2} \\ \complement_{\Bbbk\times\circledS}^{F^-}((\varkappa,\partial)) \cdot e^{i\omega\daleth_{\Bbbk\times\circledS}^{F^-((\varkappa,\partial))}} = \min\left\{\complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega\daleth_{\Bbbk}^{F^+(\varkappa)}}, \complement_{\circledS}^{F^-}(\partial) \cdot e^{i\omega\daleth_{\circledS}^{F^-(\partial)}}\right\} \end{cases}$$

$$\begin{cases} \complement^{T^+}_{\Bbbk\times\circledS}((\varkappa,\partial))\cdot e^{i\omega\daleth^{T^+((\varkappa,\partial))}_{\Bbbk\times\circledS}} = \min\left\{\complement^{T^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{T^+(\varkappa)}_{\Bbbk}}, \complement^{T^+}_{\circledS}(\partial)\cdot e^{i\omega\daleth^{T^+(\partial)}_{\circledS}}\right\} \\ \complement^{I^+}_{\Bbbk\times\circledS}((\varkappa,\partial))\cdot e^{i\omega\daleth^{I^+((\varkappa,\partial))}_{\Bbbk\times\circledS}} = \frac{\complement^{I^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{I^+(\varkappa)}_{\Bbbk}} + \complement^{I^+}_{\circledS}(\partial)\cdot e^{i\omega\daleth^{I^+(\partial)}_{\circledS}}}{2} \\ \complement^{F^+}_{\Bbbk\times\circledS}((\varkappa,\partial))\cdot e^{i\omega\daleth^{F^+((\varkappa,\partial))}_{\Bbbk\times\circledS}} = \max\left\{\complement^{F^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{F^+(\varkappa)}_{\Bbbk}}, \complement^{F^+}_{\circledS}(\partial)\cdot e^{i\omega\daleth^{F^+(\partial)}_{\circledS}}\right\} \end{cases}$$

Definition 3.5. *For any CBNS $\Bbbk$ of $\mathfrak{S}$ is said to be a Q-complex bipolar neutrosophic SBS (CBNSBS) of $\mathfrak{S}$ if*

$$\begin{cases} \complement^{T^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))} \leq \max\{\complement^{T^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}(\varkappa)}, \complement^{T^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}(\partial)}\} \\ \complement^{T^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))} \leq \max\{\complement^{T^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}(\varkappa)}, \complement^{T^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}(\partial)}\} \\ \complement^{T^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))} \leq \max\{\complement^{T^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}(\varkappa)}, \complement^{T^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}(\partial)}\} \end{cases}$$

$$\begin{cases} \complement^{I^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))} \leq \frac{\complement^{I^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\varkappa)} + \complement^{I^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\partial)}}{2} \\ \qquad OR \\ \complement^{I^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))} \leq \frac{\complement^{I^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\varkappa)} + \complement^{I^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\partial)}}{2} \\ \qquad OR \\ \complement^{I^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))} \leq \frac{\complement^{I^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\varkappa)} + \complement^{I^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{T^-}_{\Bbbk}(\partial)}}{2} \end{cases}$$

$$\begin{cases} \complement^{F^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))} \geq \min\{\complement^{F^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\varkappa)}, \complement^{F^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\partial)}\} \\ \complement^{F^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))} \geq \min\{\complement^{F^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\varkappa)}, \complement^{F^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\partial)}\} \\ \complement^{F^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))} \geq \min\{\complement^{F^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\varkappa)}, \complement^{F^-}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\partial)}\} \end{cases}$$

$$\begin{cases} \complement^{T^+}_{\Bbbk}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}((\varkappa\heartsuit_1\partial))} \geq \min\{\complement^{T^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}(\varkappa)}, \complement^{T^+}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}(\partial)}\} \\ \complement^{T^+}_{\Bbbk}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}((\varkappa\heartsuit_2\partial))} \geq \min\{\complement^{T^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}(\varkappa)}, \complement^{T^+}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}(\partial)}\} \\ \complement^{T^+}_{\Bbbk}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}((\varkappa\heartsuit_3\partial))} \geq \min\{\complement^{T^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}(\varkappa)}, \complement^{T^+}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}(\partial)}\} \end{cases}$$

$$\begin{cases} \complement^{I^+}_{\Bbbk}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}((\varkappa\heartsuit_1\partial))} \geq \frac{\complement^{I^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}(\varkappa)} + \complement^{I^+}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}(\partial)}}{2} \\ \qquad OR \\ \complement^{I^+}_{\Bbbk}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}((\varkappa\heartsuit_2\partial))} \geq \frac{\complement^{I^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}(\varkappa)} + \complement^{I^+}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}(\partial)}}{2} \\ \qquad OR \\ \complement^{I^+}_{\Bbbk}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}((\varkappa\heartsuit_3\partial))} \geq \frac{\complement^{I^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}(\varkappa)} + \complement^{I^+}_{\Bbbk}(\partial)\cdot e^{i\omega\daleth^{T^+}_{\Bbbk}(\partial)}}{2} \end{cases}$$

$$\begin{Bmatrix} \complement_{\mathbb{k}}^{F^+}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}((\varkappa\heartsuit_1\partial))} \leq \max\{\complement_{\mathbb{k}}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}(\varkappa)}, \complement_{\mathbb{k}}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}(\partial)}\} \\ \complement_{\mathbb{k}}^{F^+}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}((\varkappa\heartsuit_2\partial))} \leq \max\{\complement_{\mathbb{k}}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}(\varkappa)}, \complement_{\mathbb{k}}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}(\partial)}\} \\ \complement_{\mathbb{k}}^{F^+}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}((\varkappa\heartsuit_3\partial))} \leq \max\{\complement_{\mathbb{k}}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}(\varkappa)}, \complement_{\mathbb{k}}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\mathbb{k}}^{F^+}(\partial)}\} \end{Bmatrix}$$

for all $\varkappa, \partial \in \mathfrak{S}$.

Example 3.1. *Consider the bisemiring* $\mathfrak{S} = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ *with the Cayley tables:*

$\heartsuit_1$	η_1	η_2	η_3	η_4
η_1	η_1	η_1	η_1	η_1
η_2	η_1	η_2	η_1	η_2
η_3	η_1	η_1	η_3	η_3
η_4	η_1	η_2	η_3	η_4

$\heartsuit_2$	η_1	η_2	η_3	η_4
η_1	η_1	η_2	η_3	η_4
η_2	η_2	η_2	η_4	η_4
η_3	η_3	η_4	η_3	η_4
η_4	η_4	η_4	η_4	η_4

$\heartsuit_3$	η_1	η_2	η_3	η_4
η_1	η_1	η_1	η_1	η_1
η_2	η_1	η_2	η_3	η_4
η_3	η_4	η_4	η_4	η_4
η_4	η_4	η_4	η_4	η_4

	$\flat = \eta_1$	$\flat = \eta_2$	$\flat = \eta_3$	$\flat = \eta_4$
$(\complement_{\mathbb{k}}^{T^-}, \Finv_{\mathbb{k}}^{T^-})(\flat)$	$-0.9e^{i2\pi(-0.75)}$	$-0.85e^{i2\pi(-0.70)}$	$-0.75e^{i2\pi(-0.60)}$	$-0.8e^{i2\pi(0-.65)}$
$(\complement_{\mathbb{k}}^{I^-}, \Finv_{\mathbb{k}}^{I^-})(\flat)$	$-1e^{i2\pi(-0.85)}$	$-0.95e^{i2\pi(-0.8)}$	$-0.85e^{i2\pi(-0.7)}$	$-0.9e^{i2\pi(-0.75)}$
$(\complement_{\mathbb{k}}^{F^-}, \Finv_{\mathbb{k}}^{F^-})(\flat)$	$-0.8e^{i2\pi(-0.65)}$	$-0.9e^{i2\pi(-0.75)}$	$-1e^{i2\pi(-0.85)}$	$-0.95e^{i2\pi(-0.8)}$

	$\flat = \eta_1$	$\flat = \eta_2$	$\flat = \eta_3$	$\flat = \eta_4$
$(\complement_{\mathbb{k}}^{T^+}, \Finv_{\mathbb{k}}^{T^+})(\flat)$	$0.8e^{i2\pi(0.7)}$	$0.7e^{i2\pi(0.6)}$	$0.5e^{i2\pi(0.4)}$	$0.6e^{i2\pi(0.5)}$
$(\complement_{\mathbb{k}}^{I^+}, \Finv_{\mathbb{k}}^{I^+})(\flat)$	$1e^{i2\pi(0.9)}$	$0.9e^{i2\pi(0.8)}$	$0.6e^{i2\pi(0.5)}$	$0.7e^{i2\pi(0.6)}$
$(\complement_{\mathbb{k}}^{F^+}, \Finv_{\mathbb{k}}^{F^+})(\flat)$	$0.7e^{i2\pi(0.6)}$	$0.8e^{i2\pi(0.7)}$	$1e^{i2\pi(0.9)}$	$0.9e^{i2\pi(0.8)}$

Hence, $\mathbb{k}$ *is a CBNSBS of* $\mathfrak{S}$.

Theorem 3.1. *The intersection of every CBNSBS is a CBNSBS of* $\mathfrak{S}$.

Proof. Let $\{\sigma_i \mid i \in I\}$ be the family of CBNSBSs of $\mathfrak{S}$ and $\mathbb{k} = \bigwedge_{i\in I} \sigma_i$. Let $\varkappa, \partial \in \mathfrak{S}$. Then

$$\begin{aligned} \complement_{\mathbb{k}}^{T^-}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}((\varkappa\heartsuit_1\partial))} &= \sup_{i\in I} \complement_{\sigma_i}^{T^-}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}((\varkappa\heartsuit_1\partial))} \\ &\leq \sup_{i\in I} \max\{\complement_{\sigma_i}^{T^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma_i}^{T^-}(\varkappa)}, \complement_{\sigma_i}^{T^-}(\partial)\cdot e^{i\omega\Finv_{\sigma_i}^{T^-}(\partial)}\} \\ &= \max\left\{\sup_{i\in I} \complement_{\sigma_i}^{T^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma_i}^{T^-}(\varkappa)}, \sup_{i\in I} \complement_{\sigma_i}^{T^-}(\partial)\cdot e^{i\omega\Finv_{\sigma_i}^{T^-}(\partial)}\right\} \\ &= \max\{\complement_{\mathbb{k}}^{T^-}(\varkappa)\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}(\varkappa)}, \complement_{\mathbb{k}}^{T^-}(\partial)\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}(\partial)}\}. \end{aligned}$$

Similarly,
$\complement_{\mathbb{k}}^{T^-}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}((\varkappa\heartsuit_2\partial))} \leq \max\{\complement_{\mathbb{k}}^{T^-}(\varkappa)\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}(\varkappa)}, \complement_{\mathbb{k}}^{T^-}(\partial)\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}(\partial)}\}$ and
$\complement_{\mathbb{k}}^{T^-}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}((\varkappa\heartsuit_3\partial))} \leq \max\{\complement_{\mathbb{k}}^{T^-}(\varkappa)\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}(\varkappa)}, \complement_{\mathbb{k}}^{T^-}(\partial)\cdot e^{i\omega\Finv_{\mathbb{k}}^{T^-}(\partial)}\}$.
Now,

$$\begin{aligned}
\complement^{I^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))} &= \sup_{i\in I}\complement^{I^-}_{\sigma_i}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))}\\
&\leq \sup_{i\in I}\frac{\complement^{I^-}_{\sigma_i}(\varkappa)\cdot e^{i\omega\dashv^{I^-}_{\sigma_i}(\varkappa)}+\complement^{I^-}_{\sigma_i}(\partial)\cdot e^{i\omega\dashv^{I^-}_{\sigma_i}(\partial)}}{2}\\
&= \frac{\sup_{i\in I}\complement^{I^-}_{\sigma_i}(\varkappa)\cdot e^{i\omega\dashv^{I^-}_{\sigma_i}(\varkappa)}+\sup_{i\in I}\complement^{I^-}_{\sigma_i}(\partial)\cdot e^{i\omega\dashv^{I^-}_{\sigma_i}(\partial)}}{2}\\
&= \frac{\complement^{I^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}(\varkappa)}+\complement^{I^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}(\partial)}}{2}.
\end{aligned}$$

Similarly,
$\complement^{I^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))} \leq \frac{\complement^{I^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}(\varkappa)}+\complement^{I^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}(\partial)}}{2}$ and
$\complement^{I^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))} \leq \frac{\complement^{I^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}(\varkappa)}+\complement^{I^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}(\partial)}}{2}$.
Now,

$$\begin{aligned}
\complement^{F^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))} &= \inf_{i\in I}\complement^{F^-}_{\sigma_i}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}((\varkappa\heartsuit_1\partial))}\\
&\geq \inf_{i\in I}\ \min\{\complement^{F^-}_{\sigma_i}(\varkappa)\cdot e^{i\omega\dashv^{F^-}_{\sigma_i}(\varkappa)},\complement^{F^-}_{\sigma_i}(\partial)\cdot e^{i\omega\dashv^{F^-}_{\sigma_i}(\partial)}\}\\
&= \min\left\{\inf_{i\in I}\complement^{F^-}_{\sigma_i}(\varkappa)\cdot e^{i\omega\dashv^{F^-}_{\sigma_i}(\varkappa)},\inf_{i\in I}\complement^{F^-}_{\sigma_i}(\partial)\cdot e^{i\omega\dashv^{F^-}_{\sigma_i}(\partial)}\right\}\\
&= \min\{\complement^{F^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}(\varkappa)},\complement^{F^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}(\partial)}\}.
\end{aligned}$$

Similarly,
$\complement^{F^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}((\varkappa\heartsuit_2\partial))} \geq \min\{\complement^{F^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}(\varkappa)},\complement^{F^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}(\partial)}\}$ and
$\complement^{F^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}((\varkappa\heartsuit_3\partial))} \geq \min\{\complement^{F^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}(\varkappa)},\complement^{F^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}(\partial)}\}$.
Let $\{\sigma_i \mid i\in I\}$ be the family of CBNSBSs of $\mathfrak{S}$ and $\Bbbk = \bigwedge_{i\in I}\sigma_i$. Let $\varkappa,\partial\in\mathfrak{S}$. Then

$$\begin{aligned}
\complement^{T^+}_{\Bbbk}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}((\varkappa\heartsuit_1\partial))} &= \inf_{i\in I}\complement^{T^+}_{\sigma_i}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}((\varkappa\heartsuit_1\partial))}\\
&\geq \inf_{i\in I}\ \min\{\complement^{T^+}_{\sigma_i}(\varkappa)\cdot e^{i\omega\dashv^{T^+}_{\sigma_i}(\varkappa)},\complement^{T^+}_{\sigma_i}(\partial)\cdot e^{i\omega\dashv^{T^+}_{\sigma_i}(\partial)}\}\\
&= \min\left\{\inf_{i\in I}\complement^{T^+}_{\sigma_i}(\varkappa)\cdot e^{i\omega\dashv^{T^+}_{\sigma_i}(\varkappa)},\inf_{i\in I}\complement^{T^+}_{\sigma_i}(\partial)\cdot e^{i\omega\dashv^{T^+}_{\sigma_i}(\partial)}\right\}\\
&= \min\{\complement^{T^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}(\varkappa)},\complement^{T^+}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}(\partial)}\}.
\end{aligned}$$

Similarly,
$\complement^{T^+}_{\Bbbk}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}((\varkappa\heartsuit_2\partial))} \geq \min\{\complement^{T^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}(\varkappa)},\complement^{T^+}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}(\partial)}\}$ and
$\complement^{T^+}_{\Bbbk}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}((\varkappa\heartsuit_3\partial))} \geq \min\{\complement^{T^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}(\varkappa)},\complement^{T^+}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}(\partial)}\}$.
Now,

$$
\begin{aligned}
\complement_{\Bbbk}^{I^+}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}((\varkappa\heartsuit_1\partial))} &= \inf_{i\in I}\complement_{\sigma_i}^{I^+}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}((\varkappa\heartsuit_1\partial))}\\
&\geq \inf_{i\in I}\frac{\complement_{\sigma_i}^{I^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma_i}^{I^+}(\varkappa)}+\complement_{\sigma_i}^{I^+}(\partial)\cdot e^{i\omega\Finv_{\sigma_i}^{I^+}(\partial)}}{2}\\
&= \frac{\inf_{i\in I}\complement_{\sigma_i}^{I^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma_i}^{I^+}(\varkappa)}+\inf_{i\in I}\complement_{\sigma_i}^{I^+}(\partial)\cdot e^{i\omega\Finv_{\sigma_i}^{I^+}(\partial)}}{2}\\
&= \frac{\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\varkappa)}+\complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\partial)}}{2}.
\end{aligned}
$$

Similarly,
$\complement_{\Bbbk}^{I^+}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}((\varkappa\heartsuit_2\partial))}\geq\frac{\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\varkappa)}+\complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\partial)}}{2}$ and
$\complement_{\Bbbk}^{I^+}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}((\varkappa\heartsuit_3\partial))}\geq\frac{\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\varkappa)}+\complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\partial)}}{2}$.
Now,

$$
\begin{aligned}
\complement_{\Bbbk}^{F^+}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa\heartsuit_1\partial))} &= \sup_{i\in I}\complement_{\sigma_i}^{F^+}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa\heartsuit_1\partial))}\\
&\leq \sup_{i\in I}\ \max\{\complement_{\sigma_i}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma_i}^{F^+}(\varkappa)},\complement_{\sigma_i}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\sigma_i}^{F^+}(\partial)}\}\\
&= \max\left\{\sup_{i\in I}\complement_{\sigma_i}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma_i}^{F^+}(\varkappa)},\sup_{i\in I}\complement_{\sigma_i}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\sigma_i}^{F^+}(\partial)}\right\}\\
&= \max\{\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa)},\complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\partial)}\}.
\end{aligned}
$$

Similarly,
$\complement_{\Bbbk}^{F^+}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa\heartsuit_2\partial))}\leq\max\{\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa)},\complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\partial)}\}$ and
$\complement_{\Bbbk}^{F^+}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa\heartsuit_3\partial))}\leq\max\{\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa)},\complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\partial)}\}$.
Thus, $\Bbbk$ is a CBNSBS of $\mathfrak{S}$.

Theorem 3.2. *If $\Bbbk$ and $\circledS$ be the CBNSBSs of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ respectively, then $\Bbbk\times\circledS$is a CBNSBS of $\mathfrak{S}_1\times\mathfrak{S}_2$.*

Proof. Let $\varkappa_1,\varkappa_2\in\mathfrak{S}_1$ and $\partial_1,\partial_2\in\mathfrak{S}_2$. Then $(\varkappa_1,\partial_1)$ and $(\varkappa_2,\partial_2)$ are in $\mathfrak{S}_1\times\mathfrak{S}_2$. Now,

$$
\begin{aligned}
&\complement_{\Bbbk\times\circledS}^{T^-}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]\cdot e^{i\omega\Finv_{\Bbbk\times\circledS}^{T^-}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]}\\
=\ &\complement_{\Bbbk\times\circledS}^{T^-}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega\Finv_{\Bbbk\times\circledS}^{T^-}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))}\\
=\ &\max\{\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\varkappa_2))},\complement_{\circledS}^{T^-}((\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega\Finv_{\circledS}^{T^-}((\partial_1\heartsuit_1\partial_2))}\}\\
\leq\ &\max\{\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\varkappa_2)}\},\\
&\max\{\complement_{\circledS}^{T^-}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\partial_1)},\complement_{\circledS}^{T^-}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\partial_2)}\}\}\\
=\ &\max\{\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\circledS}^{T^-}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\partial_1)}\},\\
&\max\{\complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\varkappa_2)},\complement_{\circledS}^{T^-}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\partial_2)}\}\}\\
=\ &\max\{\complement_{\Bbbk\times\circledS}^{T^-}((\varkappa_1,\partial_1))\cdot e^{i\omega\Finv_{\Bbbk\times\circledS}^{T^-}((\varkappa_1,\partial_1))},\complement_{\Bbbk\times\circledS}^{T^-}((\varkappa_2,\partial_2))\cdot e^{i\omega\Finv_{\Bbbk\times\circledS}^{T^-}((\varkappa_2,\partial_2))}\}.
\end{aligned}
$$

Also, $\complement^{T^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]}$
$\leq \max\{\complement^{T^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))}, \complement^{T^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))}\}$ and
$\complement^{T^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]}$
$\leq \max\{\complement^{T^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))}, \complement^{T^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))}\}$.
Now,

$$\begin{aligned}
&\complement^{I^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]}\\
=\;&\complement^{I^-}_{\Bbbk\times ⓢ}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))}\\
=\;&\frac{\complement^{I^-}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}((\varkappa_1\heartsuit_1\varkappa_2))}+\complement^{I^-}_{ⓢ}((\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\partial_1\heartsuit_1\partial_2))}}{2}\\
\leq\;&\frac{1}{2}\left[\frac{\complement^{I^-}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\varkappa_1)}+\complement^{I^-}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\varkappa_2)}}{2}+\frac{\complement^{I^-}_{ⓢ}(\partial_1)\cdot e^{i\omega\daleth^{I^-}_{ⓢ}(\partial_1)}+\complement^{I^-}_{ⓢ}(\partial_2)\cdot e^{i\omega\daleth^{I^-}_{ⓢ}(\partial_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\frac{\complement^{I^-}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\varkappa_1)}+\complement^{I^-}_{ⓢ}(\partial_1)\cdot e^{i\omega\daleth^{I^-}_{ⓢ}(\partial_1)}}{2}+\frac{\complement^{I^-}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}(\varkappa_2)}+\complement^{I^-}_{ⓢ}(\partial_2)\cdot e^{i\omega\daleth^{I^-}_{ⓢ}(\partial_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\complement^{I^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))}+\complement^{I^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))}\right].
\end{aligned}$$

Also,
$\complement^{I^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]} \leq \frac{1}{2}\Big[\complement^{I^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))} + \complement^{I^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))}\Big]$ and
$\complement^{I^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]} \leq \frac{1}{2}\Big[\complement^{I^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))} + \complement^{I^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))\cdot e^{i\omega\daleth^{I^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))}\Big]$.
Now,

$$\begin{aligned}
&\complement^{F^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]\cdot e^{i\omega\daleth^{F^-}_{\Bbbk\times ⓢ}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]}\\
=\;&\complement^{F^-}_{\Bbbk\times ⓢ}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega\daleth^{F^-}_{\Bbbk\times ⓢ}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))}\\
=\;&\min\{\complement^{F^-}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\daleth^{F^-}_{\Bbbk\times ⓢ}((\varkappa_1\heartsuit_1\varkappa_2))},\complement^{F^-}_{ⓢ}((\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega\daleth^{F^-}_{\Bbbk\times ⓢ}((\partial_1\heartsuit_1\partial_2))}\}\\
\geq\;&\min\{\min\{\complement^{F^-}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\varkappa_1)},\complement^{F^-}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\varkappa_2)}\},\\
&\min\{\complement^{F^-}_{ⓢ}(\partial_1)\cdot e^{i\omega\daleth^{F^-}_{ⓢ}(\partial_1)},\complement^{F^-}_{ⓢ}(\partial_2)\cdot e^{i\omega\daleth^{F^-}_{ⓢ}(\partial_2)}\}\}\\
=\;&\min\{\min\{\complement^{F^-}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\varkappa_1)},\complement^{F^-}_{ⓢ}(\partial_1)\cdot e^{i\omega\daleth^{F^-}_{ⓢ}(\partial_1)}\},\\
&\min\{\complement^{F^-}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\daleth^{F^-}_{\Bbbk}(\varkappa_2)},\complement^{F^-}_{ⓢ}(\partial_2)\cdot e^{i\omega\daleth^{F^-}_{ⓢ}(\partial_2)}\}\}\\
=\;&\min\{\complement^{F^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\varkappa_1,\partial_1))},\complement^{F^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))\cdot e^{i\omega\daleth^{T^-}_{\Bbbk\times ⓢ}((\varkappa_2,\partial_2))}\}.
\end{aligned}$$

Also, $\complement^{F^-}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]\cdot e^{i\omega \daleth^{T^-}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]}$
$\geq \min\{\complement^{F^-}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega \daleth^{T^-}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))}, \complement^{F^-}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega \daleth^{T^-}_{\Bbbk\times \circledS}}((\varkappa_2,\partial_2))\}$ and
$\complement^{F^-}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]\cdot e^{i\omega \daleth^{T^-}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]}$
$\geq \min\{\complement^{F^-}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega \daleth^{T^-}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))}, \complement^{F^-}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega \daleth^{T^-}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))}\}$.
Let $\varkappa_1,\varkappa_2\in \mathfrak{S}_1$ and $\partial_1,\partial_2\in\mathfrak{S}_2$. Then $(\varkappa_1,\partial_1)$ and $(\varkappa_2,\partial_2)$ are in $\mathfrak{S}_1\times\mathfrak{S}_2$. Now,

$$\begin{aligned}
&\complement^{T^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]}\\
=\;&\complement^{T^+}_{\Bbbk\times \circledS}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))}\\
=\;&\min\{\complement^{T^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2))}, \complement^{T^+}_{\circledS}((\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega \daleth^{T^+}_{\circledS}((\partial_1\heartsuit_1\partial_2))}\}\\
\geq\;&\min\{\min\{\complement^{T^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}(\varkappa_1)}, \complement^{T^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}(\varkappa_2)}\}, \min\{\complement^{T^+}_{\circledS}(\partial_1)\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}(\partial_1)}, \complement^{T^+}_{\circledS}(\partial_2)\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}(\partial_2)}\}\}\\
=\;&\min\{\min\{\complement^{T^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}(\varkappa_1)}, \complement^{T^+}_{\circledS}(\partial_1)\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}(\partial_1)}\}, \min\{\complement^{T^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}(\varkappa_2)}, \complement^{T^+}_{\circledS}(\partial_2)\cdot e^{i\omega \daleth^{T^+}_{\Bbbk}(\partial_2)}\}\}\\
=\;&\min\{\complement^{T^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))}, \complement^{T^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))}\}.
\end{aligned}$$

Also, $\complement^{T^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]}$
$\geq \min\{\complement^{T^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))}, \complement^{T^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2)) e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))}\}$ and
$\complement^{T^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]}$
$\geq \min\{\complement^{T^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))}, \complement^{T^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))}\}$.
Now,

$$\begin{aligned}
&\complement^{I^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]\cdot e^{i\omega \daleth^{I^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]}\\
=\;&\complement^{I^+}_{\Bbbk\times \circledS}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega \daleth^{I^+}_{\Bbbk\times \circledS}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))}\\
=\;&\frac{\complement^{I^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega \daleth^{I^+}_{\Bbbk\times \circledS}((\varkappa_1\heartsuit_1\varkappa_2))}+\complement^{I^+}_{\circledS}((\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\partial_1\heartsuit_1\partial_2))}}{2}\\
\geq\;&\frac{1}{2}\left[\frac{\complement^{I^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega \daleth^{I^+}_{\Bbbk}(\varkappa_1)}+\complement^{I^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega \daleth^{I^+}_{\Bbbk}(\varkappa_2)}}{2}+\frac{\complement^{I^+}_{\circledS}(\partial_1)\cdot e^{i\omega \daleth^{I^+}_{\circledS}(\partial_1)}+\complement^{I^+}_{\circledS}(\partial_2)\cdot e^{i\omega \daleth^{I^+}_{\circledS}(\partial_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\frac{\complement^{I^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega \daleth^{I^+}_{\Bbbk}(\varkappa_1)}+\complement^{I^+}_{\circledS}(\partial_1)\cdot e^{i\omega \daleth^{I^+}_{\circledS}(\partial_1)}}{2}+\frac{\complement^{I^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega \daleth^{I^+}_{\Bbbk}(\varkappa_2)}+\complement^{I^+}_{\circledS}(\partial_2)\cdot e^{i\omega \daleth^{I^+}_{\circledS}(\partial_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\complement^{I^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))}+\complement^{I^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega \daleth^{T^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))}\right].
\end{aligned}$$

Also,
$\complement^{I^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]\cdot e^{i\omega \daleth^{I^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]} \geq \frac{1}{2}\Big[\complement^{I^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega \daleth^{I^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))} +$
$\complement^{I^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega \daleth^{I^+}_{\Bbbk\times \circledS}((\varkappa_2,\partial_2))}\Big]$ and
$\complement^{I^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]\cdot e^{i\omega \daleth^{I^+}_{\Bbbk\times \circledS}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]} \geq \frac{1}{2}\Big[\complement^{I^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega \daleth^{I^+}_{\Bbbk\times \circledS}((\varkappa_1,\partial_1))} +$

$\complement^{I^+}_{\Bbbk\times\circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega\dashv^{I^+}_{\Bbbk\times\circledS}((\varkappa_2,\partial_2))}\Big]$.

Now,

$$
\begin{aligned}
&\complement^{F^+}_{\Bbbk\times\circledS}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]\cdot e^{i\omega\dashv^{F^+}_{\Bbbk\times\circledS}[((\varkappa_1,\partial_1)\heartsuit_1(\varkappa_2,\partial_2))]}\\
=\;&\complement^{F^+}_{\Bbbk\times\circledS}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega\dashv^{F^+}_{\Bbbk\times\circledS}((\varkappa_1\heartsuit_1\varkappa_2,\partial_1\heartsuit_1\partial_2))}\\
=\;&\max\{\complement^{F^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\dashv^{F^+}_{\Bbbk\times\circledS}((\varkappa_1\heartsuit_1\varkappa_2))},\complement^{F^+}_{\circledS}((\partial_1\heartsuit_1\partial_2))\cdot e^{i\omega\dashv^{F^+}_{\Bbbk\times\circledS}((\partial_1\heartsuit_1\partial_2))}\}\\
\leq\;&\max\{\max\{\complement^{F^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\dashv^{F^+}_{\Bbbk}(\varkappa_1)},\complement^{F^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\dashv^{F^+}_{\Bbbk}(\varkappa_2)}\},\\
&\max\{\complement^{F^+}_{\circledS}(\partial_1)\cdot e^{i\omega\dashv^{F^+}_{\circledS}(\partial_1)},\complement^{F^+}_{\circledS}(\partial_2)\cdot e^{i\omega\dashv^{F^+}_{\circledS}(\partial_2)}\}\}\\
=\;&\max\{\max\{\complement^{F^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\dashv^{F^+}_{\Bbbk}(\varkappa_1)},\complement^{F^+}_{\circledS}(\partial_1)\cdot e^{i\omega\dashv^{F^+}_{\circledS}(\partial_1)}\},\\
&\max\{\complement^{F^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\dashv^{F^+}_{\Bbbk}(\varkappa_2)},\complement^{F^+}_{\circledS}(\partial_2)\cdot e^{i\omega\dashv^{F^+}_{\circledS}(\partial_2)}\}\}\\
=\;&\max\{\complement^{F^+}_{\Bbbk\times\circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk\times\circledS}((\varkappa_1,\partial_1))},\complement^{F^+}_{\Bbbk\times\circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk\times\circledS}((\varkappa_2,\partial_2))}\}.
\end{aligned}
$$

Also, $\complement^{F^+}_{\Bbbk\times\circledS}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]\cdot e^{i\omega\dashv^{T^+}_{\Bbbk\times\circledS}[((\varkappa_1,\partial_1)\heartsuit_2(\varkappa_2,\partial_2))]}$
$\leq\max\{\complement^{F^+}_{\Bbbk\times\circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk\times\circledS}((\varkappa_1,\partial_1))},\complement^{F^+}_{\Bbbk\times\circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk\times\circledS}}((\varkappa_2,\partial_2))\}$ and
$\complement^{F^+}_{\Bbbk\times\circledS}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]\cdot e^{i\omega\dashv^{T^+}_{\Bbbk\times\circledS}[((\varkappa_1,\partial_1)\heartsuit_3(\varkappa_2,\partial_2))]}$
$\leq\max\{\complement^{F^+}_{\Bbbk\times\circledS}((\varkappa_1,\partial_1))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk\times\circledS}((\varkappa_1,\partial_1))},\complement^{F^+}_{\Bbbk\times\circledS}((\varkappa_2,\partial_2))\cdot e^{i\omega\dashv^{T^+}_{\Bbbk\times\circledS}((\varkappa_2,\partial_2))}\}$.
Thus, $\Bbbk\times\circledS$ is a CBNSBS of $\mathfrak{S}$.

Corollary 3.1. *If $\Bbbk_1,\Bbbk_2,...,\Bbbk_n$ be the finite collection of CBNSBSs of $\mathfrak{S}_1,\mathfrak{S}_2,...,\mathfrak{S}_n$ respectively. Then $\Bbbk_1\times\Bbbk_2\times...\times\Bbbk_n$ is a CBNSBS of $\mathfrak{S}_1\times\mathfrak{S}_2\times...\times\mathfrak{S}_n$.*

Definition 3.6. *Let $\Bbbk\subseteq\mathfrak{S}$. The strongest CBN relation on $\mathfrak{S}$ is*

$$
\begin{Bmatrix}
\complement^{T^-}_{\sigma}((\varkappa,\partial))\cdot e^{i\omega\dashv^{T^-}_{\sigma}((\varkappa,\partial))}=\min\{\complement^{T^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{T^-}_{\Bbbk}(\varkappa)},\complement^{T^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{T^-}_{\Bbbk}(\partial)}\}\\
\complement^{I^-}_{\sigma}((\varkappa,\partial))\cdot e^{i\omega\dashv^{I^-}_{\sigma}((\varkappa,\partial))}=\frac{\complement^{I^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}(\varkappa)}+\complement^{I^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{I^-}_{\Bbbk}(\partial)}}{2}\\
\complement^{F^-}_{\sigma}((\varkappa,\partial))\cdot e^{i\omega\dashv^{F^-}_{\sigma}((\varkappa,\partial))}=\max\{\complement^{F^-}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}(\varkappa)},\complement^{F^-}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{F^-}_{\Bbbk}(\partial)}\}
\end{Bmatrix}
$$

$$
\begin{Bmatrix}
\complement^{T^+}_{\sigma}((\varkappa,\partial))\cdot e^{i\omega\dashv^{T^+}_{\sigma}((\varkappa,\partial))}=\max\{\complement^{T^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}(\varkappa)},\complement^{T^+}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{T^+}_{\Bbbk}(\partial)}\}\\
\complement^{I^+}_{\sigma}((\varkappa,\partial))\cdot e^{i\omega\dashv^{I^+}_{\sigma}((\varkappa,\partial))}=\frac{\complement^{I^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{I^+}_{\Bbbk}(\varkappa)}+\complement^{I^+}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{I^+}_{\Bbbk}(\partial)}}{2}\\
\complement^{F^+}_{\sigma}((\varkappa,\partial))\cdot e^{i\omega\dashv^{F^+}_{\sigma}((\varkappa,\partial))}=\min\{\complement^{F^+}_{\Bbbk}(\varkappa)\cdot e^{i\omega\dashv^{F^+}_{\Bbbk}(\varkappa)},\complement^{F^+}_{\Bbbk}(\partial)\cdot e^{i\omega\dashv^{F^+}_{\Bbbk}(\partial)}\}
\end{Bmatrix}
$$

Theorem 3.3. *Let $\Bbbk$ be a CBNSBS of $\mathfrak{S}$ and σ be the strongest complex bipolar neutrosophic relation of $\mathfrak{S}$. Then $\Bbbk$ is a CBNSBS of $\mathfrak{S}\times\mathfrak{S}$ if and only if σ is a CBNSBS of $\mathfrak{S}\times\mathfrak{S}$.*

Proof. Suppose that $\Bbbk$ is a CBNSBS of $\mathfrak{S}\times\mathfrak{S}$ and σ be the strongest complex bipolar neutrosophic relation of $\mathfrak{S}$. For any $\varkappa=(\varkappa_1,\varkappa_2),\partial=(\partial_1,\partial_2)\in S\times\mathfrak{S}$. Now,

$$\begin{aligned}
&\complement_{\sigma}^{T^-}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\sigma}^{T^-}((\varkappa\heartsuit_1\partial))}\\
=\;&\complement_{\sigma}^{T^-}[((((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]\cdot e^{i\omega\Finv_{\sigma}^{T^-}[((((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]}\\
=\;&\complement_{\sigma}^{T^-}(\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2)\cdot e^{i\omega\Finv_{\sigma}^{T^-}(\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2)}\\
=\;&\max\{\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))},\complement_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_1\partial_2))}\}\\
\leq\;&\max\{\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\partial_1)}\},\\
&\max\{\complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\varkappa_2)},\complement_{\Bbbk}^{T^-}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\partial_2)}\}\}\\
=\;&\max\{\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\varkappa_2)}\},\\
&\max\{\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\partial_1)},\complement_{\Bbbk}^{T^-}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}(\partial_2)}\}\}\\
=\;&\max\{\complement_{\sigma}^{T^-}((\varkappa_1,\varkappa_2))\cdot e^{i\omega\Finv_{\sigma}^{T^-}((\varkappa_1,\varkappa_2))},\complement_{\sigma}^{T^-}((\partial_1,\partial_2))\cdot e^{i\omega\Finv_{\sigma}^{T^-}((\partial_1,\partial_2))}\}\\
=\;&\max\{\complement_{\sigma}^{T^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{T^-}(\varkappa)},\complement_{\sigma}^{T^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{T^-}(\partial)}\}.
\end{aligned}$$

Also, $\complement_{\sigma}^{T^-}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\sigma}^{T^-}((\varkappa\heartsuit_2\partial))}\leq\max\{\complement_{\sigma}^{T^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{T^-}(\varkappa)},\complement_{\sigma}^{T^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{T^-}(\partial)}\}$ and
$\complement_{\sigma}^{T^-}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\sigma}^{T^-}((\varkappa\heartsuit_3\partial))}\leq\max\{\complement_{\sigma}^{T^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{T^-}(\varkappa)},\complement_{\sigma}^{T^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{T^-}(\partial)}\}$.
Now,

$$\begin{aligned}
&\complement_{\sigma}^{I^-}(\varkappa\heartsuit_1\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^-}(\varkappa\heartsuit_1\partial)}\\
&=\complement_{\sigma}^{I^-}[((((\varkappa_1,\varkappa_2))\heartsuit_1(\partial_1,\partial_2))]\cdot e^{i\omega\Finv_{\sigma}^{I^-}[((((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2)))]}\\
&=\complement_{\sigma}^{I^-}((\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\Finv_{\sigma}^{I^-}((\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2))}\\
&=\frac{\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\partial_1))}+\complement_{\Bbbk}^{I^-}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}((\varkappa_2\heartsuit_1\partial_2))}{2}\\
&\leq\frac{1}{2}\left[\frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}(\varkappa_1)}+\complement_{\Bbbk}^{I^-}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}(\partial_1)}}{2}+\frac{\complement_{\Bbbk}^{I^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}(\varkappa_2)}+\complement_{\Bbbk}^{I^-}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}(\partial_2)}}{2}\right]\\
&=\frac{1}{2}\left[\frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}(\varkappa_1)}+\complement_{\Bbbk}^{I^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}(\varkappa_2)}}{2}+\frac{\complement_{\Bbbk}^{I^-}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}(\partial_1)}+\complement_{\Bbbk}^{I^-}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}(\partial_2)}}{2}\right]\\
&=\frac{\complement_{\sigma}^{I^-}((\varkappa_1,\varkappa_2))\cdot e^{i\omega\Finv_{\sigma}^{I^-}((\varkappa_1,\varkappa_2))}+\complement_{\sigma}^{I^-}((\partial_1,\partial_2))\cdot e^{i\omega\Finv_{\sigma}^{I^-}((\partial_1,\partial_2))}}{2}\\
&=\frac{\complement_{\sigma}^{I^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{I^-}(\varkappa)}+\complement_{\sigma}^{I^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^-}(\varkappa)}}{2}.
\end{aligned}$$

Also, $\complement_{\sigma}^{I^-}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\sigma}^{I^-}((\varkappa\heartsuit_2\partial))}\leq\frac{\complement_{\sigma}^{I^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{I^-}(\varkappa)}+\complement_{\sigma}^{I^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^-}(\partial)}}{2}$ and
$\complement_{\sigma}^{I^-}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\sigma}^{I^-}((\varkappa\heartsuit_3\partial))}\leq\frac{\complement_{\sigma}^{I^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{I^-}(\varkappa)}+\complement_{\sigma}^{I^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^-}(\partial)}}{2}$.
Similarly, $\complement_{\sigma}^{F^-}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\sigma}^{F^-}((\varkappa\heartsuit_1\partial))}\geq\min\{\complement_{\sigma}^{F^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{F^-}(\varkappa)\cdot},\complement_{\sigma}^{F^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{F^-}(\partial)}\}$,
$\complement_{\sigma}^{F^-}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\sigma}^{F^-}((\varkappa\heartsuit_2\partial))}\geq\min\{\complement_{\sigma}^{F^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{F^-}(\varkappa)},\complement_{\sigma}^{F^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{F^-}(\partial)}\}$ and

$\complement_{\sigma}^{F^-}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\sigma}^{F^-}((\varkappa\heartsuit_3\partial))} \geq \min\{\complement_{\sigma}^{F^-}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{F^-}(\varkappa)}, \complement_{\sigma}^{F^-}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^-}(\partial)}\}$.
For any $\varkappa = (\varkappa_1, \varkappa_2), \partial = (\partial_1, \partial_2) \in \mathfrak{S}\times\mathfrak{S}$. Now,

$$
\begin{aligned}
& \complement_{\sigma}^{T^+}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\sigma}^{T^+}((\varkappa\heartsuit_1\partial))} \\
= \; & \complement_{\sigma}^{T^+}[((((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]\cdot e^{i\omega\Finv_{\sigma}^{T^+}[((((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]} \\
= \; & \complement_{\sigma}^{T^+}(\varkappa_1\heartsuit_1\partial_1, \varkappa_2\heartsuit_1\partial_2)\cdot e^{i\omega\Finv_{\sigma}^{T^+}(\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2)} \\
= \; & \min\{\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))}, \complement_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_1\partial_2))}\} \\
\geq \; & \min\{\min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\varkappa_1)}, \complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\partial_1)}\}, \\
& \min\{\complement_{\Bbbk}^{T^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\varkappa_2)}, \complement_{\Bbbk}^{T^+}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\partial_2)}\}\} \\
= \; & \min\{\min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\varkappa_1)}, \complement_{\Bbbk}^{T^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\varkappa_2)}\}, \\
& \min\{\complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\partial_1)}, \complement_{\Bbbk}^{T^+}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\partial_2)}\}\} \\
= \; & \min\{\complement_{\sigma}^{T^+}((\varkappa_1,\varkappa_2))\cdot e^{i\omega\Finv_{\sigma}^{T^+}((\varkappa_1,\varkappa_2))}, \complement_{\sigma}^{T^+}((\partial_1,\partial_2))\cdot e^{i\omega\Finv_{\sigma}^{T^+}((\partial_1,\partial_2))}\} \\
= \; & \min\{\complement_{\sigma}^{T^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{T^+}(\varkappa)}, \complement_{\sigma}^{T^+}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{T^+}(\partial)}\}.
\end{aligned}
$$

Also, $\complement_{\sigma}^{T^+}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\sigma}^{T^+}((\varkappa\heartsuit_2\partial))} \geq \min\{\complement_{\sigma}^{T^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{T^+}(\varkappa)}, \complement_{\sigma}^{T^+}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{T^+}(\partial)}\}$ and
$\complement_{\sigma}^{T^+}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\sigma}^{T^+}((\varkappa\heartsuit_3\partial))} \geq \min\{\complement_{\sigma}^{T^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{T^+}(\varkappa)}, \complement_{\sigma}^{T^+}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{T^+}(\partial)}\}$.
Now,

$$
\begin{aligned}
& \complement_{\sigma}^{I^+}(\varkappa\heartsuit_1\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^+}(\varkappa\heartsuit_1\partial)} \\
& = \complement_{\sigma}^{I^+}[((((\varkappa_1,\varkappa_2))\heartsuit_1(\partial_1,\partial_2))]\cdot e^{i\omega\Finv_{\sigma}^{I^+}[((((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2)))]} \\
& = \complement_{\sigma}^{I^+}((\varkappa_1\heartsuit_1\partial_1, \varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\Finv_{\sigma}^{I^+}((\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2))} \\
& = \frac{\complement_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_1\partial_1))} + \complement_{\Bbbk}^{I^+}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}}((\varkappa_2\heartsuit_1\partial_2))}{2} \\
& \geq \frac{1}{2}\left[\frac{\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\varkappa_1)} + \complement_{\Bbbk}^{I^+}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\partial_1)}}{2} + \frac{\complement_{\Bbbk}^{I^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\varkappa_2)} + \complement_{\Bbbk}^{I^+}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\partial_2)}}{2}\right] \\
& = \frac{1}{2}\left[\frac{\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\varkappa_1)} + \complement_{\Bbbk}^{I^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\varkappa_2)}}{2} + \frac{\complement_{\Bbbk}^{I^+}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\partial_1)} + \complement_{\Bbbk}^{I^+}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}(\partial_2)}}{2}\right] \\
& = \frac{\complement_{\sigma}^{I^+}((\varkappa_1,\varkappa_2))\cdot e^{i\omega\Finv_{\sigma}^{I^+}((\varkappa_1,\varkappa_2))} + \complement_{\sigma}^{I^+}((\partial_1,\partial_2))\cdot e^{i\omega\Finv_{\sigma}^{I^+}((\partial_1,\partial_2))}}{2} \\
& = \frac{\complement_{\sigma}^{I^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{I^+}(\varkappa)} + \complement_{\sigma}^{I^+}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^+}(\varkappa)}}{2}.
\end{aligned}
$$

Also, $\complement_{\sigma}^{I^+}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\Finv_{\sigma}^{I^+}((\varkappa\heartsuit_2\partial))} \geq \frac{\complement_{\sigma}^{I^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{I^+}(\varkappa)} + \complement_{\sigma}^{I^+}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^+}(\partial)}}{2}$ and
$\complement_{\sigma}^{I^+}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\Finv_{\sigma}^{I^+}((\varkappa\heartsuit_3\partial))} \geq \frac{\complement_{\sigma}^{I^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{I^+}(\varkappa)} + \complement_{\sigma}^{I^+}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{I^+}(\partial)}}{2}$.
Similarly, $\complement_{\sigma}^{F^+}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\Finv_{\sigma}^{F^+}((\varkappa\heartsuit_1\partial))} \leq \max\{\complement_{\sigma}^{F^+}(\varkappa)\cdot e^{i\omega\Finv_{\sigma}^{F^+}(\varkappa)\cdot}, \complement_{\sigma}^{F^+}(\partial)\cdot e^{i\omega\Finv_{\sigma}^{F^+}(\partial)}\}$,

$\complement_{\sigma}^{F^+}((\varkappa\heartsuit_2\partial))\cdot e^{i\omega\daleth_{\sigma}^{F^+}((\varkappa\heartsuit_2\partial))}\leq\max\{\complement_{\sigma}^{F^+}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{F^+}(\varkappa)},\complement_{\sigma}^{F^+}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{F^+}(\partial)}\}$ and
$\complement_{\sigma}^{F^+}((\varkappa\heartsuit_3\partial))\cdot e^{i\omega\daleth_{\sigma}^{F^+}((\varkappa\heartsuit_3\partial))}\leq\max\{\complement_{\sigma}^{F^+}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{F^+}(\varkappa)},\complement_{\sigma}^{F^+}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^+}(\partial)}\}$.
Therefore, σ is a CBNSBS of $\mathfrak{S}\times\mathfrak{S}$.

Conversely, suppose that σ is a CBNSBS of $\mathfrak{S}\times\mathfrak{S}$. Let $\varkappa=((\varkappa_1,\varkappa_2)),\partial=((\partial_1,\partial_2))\in\mathfrak{S}\times\mathfrak{S}$. Now,

$$\begin{aligned}
&\max\{\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))},\complement_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_1\partial_2))}\}\\
&=\complement_{\sigma}^{T^-}(\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2)\cdot e^{i\omega\daleth_{\sigma}^{T^-}(\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2)}\\
&=\complement_{\sigma}^{T^-}[((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]\cdot e^{i\omega\daleth_{\upsilon}^{T^+}[((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]}\\
&=\complement_{\sigma}^{T^-}(\varkappa\heartsuit_1\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^-}(\varkappa\heartsuit_1\partial)}\\
&\leq\max\{\complement_{\sigma}^{T^-}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}(\varkappa)},\complement_{\sigma}^{T^-}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^-}(\partial)}\}\\
&=\max\{\complement_{\sigma}^{T^-}((\varkappa_1,\varkappa_2))\}\cdot e^{i\omega\daleth_{\sigma}^{T^-}((\varkappa_1,\varkappa_2))\}},\complement_{\sigma}^{T^-}((\partial_1,\partial_2))\cdot e^{i\omega\daleth_{\sigma}^{T^-}((\partial_1,\partial_2))}\}\\
&=\max\{\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_2)}\},\max\{\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_1)},\complement_{\Bbbk}^{T^-}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_2)}\}\}.
\end{aligned}$$

If $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))}\geq\complement_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_1\partial_2))}$, then $\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_1)}\geq\complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_2)}$ and $\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_1)}\geq\complement_{\Bbbk}^{T^-}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_2)}$. We get $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))}\leq\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_1)}\}$ for all $\varkappa_1,\partial_1\in\mathfrak{S}$,and
$\max\{\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_2\partial_1))},\complement_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_2\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_2\partial_2))}\}\leq\max\{\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_2)}\},\max\{\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_1)},\complement_{\Bbbk}^{T^-}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_2)}\}\}$.
If $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_2\partial_1))}\geq\complement_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_2\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_2\partial_2))}$, then $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_2\partial_1))}\leq\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_1)}\}$.
$\max\{\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_3\partial_1))},\complement_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_3\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_3\partial_2))}\}\leq\max\{\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_2)}\},\max\{\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_1)},\complement_{\Bbbk}^{T^-}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_2)}\}\}$.
If $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_3\partial_1))}\geq\complement_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_3\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_3\partial_2))}$, then $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_3\partial_1))}\leq\max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa_1)},\complement_{\Bbbk}^{T^-}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial_1)}\}$.
Now,

$$\begin{aligned}
&\frac{1}{2}\Big[\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\partial_1))}+\complement_{\Bbbk}^{I^-}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{I^-((\varkappa_2\heartsuit_1\partial_2))}}\Big]\\
&=\complement_{\sigma}^{I^-}(\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2)\cdot e^{i\omega\daleth_{\sigma}^{I^-}(\varkappa_1\heartsuit_1\partial_1,\varkappa_2\heartsuit_1\partial_2)}\\
&=\complement_{\sigma}^{I^-}[((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]\cdot e^{i\omega\daleth_{\sigma}^{I^-}[((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]}\\
&=\complement_{\sigma}^{I^-}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\daleth_{\sigma}^{I^-}((\varkappa\heartsuit_1\partial))}\\
&\leq\frac{\complement_{\sigma}^{I^-}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{I^-(\varkappa)}}+\complement_{\sigma}^{I^-}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^-}(\partial)}}{2}\\
&=\frac{\complement_{\sigma}^{I^-}((\varkappa_1,\varkappa_2))\cdot e^{i\omega\daleth_{\sigma}^{I^-}((\varkappa_1,\varkappa_2))}+\complement_{\sigma}^{I^-}((\partial_1,\partial_2))\cdot e^{i\omega\daleth_{\sigma}^{I^-}((\partial_1,\partial_2))}}{2}
\end{aligned}$$

$$= \frac{1}{2}\left[\frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa_1)} + \complement_{\Bbbk}^{I^-}(\varkappa_2)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa_2)}}{2} + \frac{\complement_{\Bbbk}^{I^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial_1)} + \complement_{\Bbbk}^{I^-}(\partial_2)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial_2)}}{2}\right].$$

If $\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\partial_1))} \geq \complement_{\Bbbk}^{I^-}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}((\varkappa_2\heartsuit_1\partial_2))}$, then $\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa_1)} \geq \complement_{\Bbbk}^{I^-}(\varkappa_2)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa_2)}$ and $\complement_{\Bbbk}^{I^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial_1)} \geq \complement_{\Bbbk}^{I^-}(\partial_2)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial_2)}$.
We get $\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\partial_1))} \leq \frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa_1)} + \complement_{\Bbbk}^{I^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}(\partial_1)}}{2}$.
Similarly, $\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_2\partial_1))} \leq \frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa_1)} + \complement_{\Bbbk}^{I^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial_1)}}{2}$ and
$\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_3\partial_1))} \leq \frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa_1)} + \complement_{\Bbbk}^{I^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial_1)}}{2}$.
Similarly, to prove that
$\min\{\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\partial_1))}, \complement_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_1\partial_2))}\} \geq \min\{\min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_1)}, \complement_{\Bbbk}^{F^-}(\varkappa_2)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_2)}\}, \min\{\complement_{\Bbbk}^{F^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_1)}, \complement_{\Bbbk}^{F^-}(\partial_2)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_2)}\}\}$.
If $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\partial_1))} \leq \complement_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_1\partial_2))}$, then $\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_1)} \leq \complement_{\Bbbk}^{F^-}(\varkappa_2)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_2)}$ and $\complement_{\Bbbk}^{F^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_1)} \leq \complement_{\Bbbk}^{F^-}(\partial_2)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_2)}$.
We get $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\partial_1))} \geq \min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_1)}, \complement_{\Bbbk}^{F^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_1)}\}$.
$\min\{\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_2\partial_1))}, \complement_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_2\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_2\partial_2))}\}$
$\geq \min\{\min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_1)}, \complement_{\Bbbk}^{F^-}(\varkappa_2)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_2)}\}, \min\{\complement_{\Bbbk}^{F^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_1)}, \complement_{\Bbbk}^{F^-}(\partial_2)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}}\}\}$.
If $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_2\partial_1))} \leq \complement_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_2\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_2\partial_2))}$, then $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_2\partial_1))} \geq \min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_1)}, \complement_{\Bbbk}^{F^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_1)}\}$.
$\min\{\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_3\partial_1))}, \complement_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_3\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_3\partial_2))}\} \geq \min\{\min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_1)}, \complement_{\Bbbk}^{F^-}(\varkappa_2)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_2)}\}, \min\{\complement_{\Bbbk}^{F^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_1)}, \complement_{\Bbbk}^{F^-}(\partial_2)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial_2)}\}\}$.
If $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_3\partial_1))} \leq \complement_{\Bbbk}^{F^-}((\varkappa_2\heartsuit_3\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}((\varkappa_2\heartsuit_3\partial_2))}$, then $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_3\partial_1))} \geq \min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa_1)}, \complement_{\Bbbk}^{F^-}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}(\partial_1)}\}$.
Let $\varkappa = ((\varkappa_1, \varkappa_2)), \partial = ((\partial_1, \partial_2)) \in \mathfrak{S}\times\mathfrak{S}$. Now,

$$\begin{aligned}
&\min\{\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))}, \complement_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_1\partial_2))}\}\\
&= \complement_{\sigma}^{T^+}(\varkappa_1\heartsuit_1\partial_1, \varkappa_2\heartsuit_1\partial_2)\cdot e^{i\omega \daleth_{\sigma}^{T^+}(\varkappa_1\heartsuit_1\partial_1, \varkappa_2\heartsuit_1\partial_2)}\\
&= \complement_{\sigma}^{T^+}[((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]\cdot e^{i\omega \daleth_{v}^{T^+}[((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]}\\
&= \complement_{\sigma}^{T^+}(\varkappa\heartsuit_1\partial)\cdot e^{i\omega \daleth_{\sigma}^{T^+}(\varkappa\heartsuit_1\partial)}\\
&\geq \min\{\complement_{\sigma}^{T^+}(\varkappa)\cdot e^{i\omega \daleth_{\sigma}^{T^+}(\varkappa)}, \complement_{\sigma}^{T^+}(\partial)\cdot e^{i\omega \daleth_{\sigma}^{T^+}(\partial)}\}\\
&= \min\{\complement_{\sigma}^{T^+}((\varkappa_1,\varkappa_2))\}\cdot e^{i\omega \daleth_{\sigma}^{T^+}((\varkappa_1,\varkappa_2))\}}, \complement_{\sigma}^{T^+}((\partial_1,\partial_2))\cdot e^{i\omega \daleth_{\sigma}^{T^+}((\partial_1,\partial_2))}\}\\
&= \min\{\min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\varkappa_1)}, \complement_{\Bbbk}^{T^+}(\varkappa_2)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\varkappa_2)}\}, \min\{\complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\partial_1)}, \complement_{\Bbbk}^{T^+}(\partial_2)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\partial_2)}\}\}.
\end{aligned}$$

If $\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))} \leq \complement_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_1\partial_2))}$, then $\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\varkappa_1)} \leq \complement_{\Bbbk}^{T^+}(\varkappa_2)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\varkappa_2)}$ and $\complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\partial_1)} \leq \complement_{\Bbbk}^{T^+}(\partial_2)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\partial_2)}$. We get $\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))\cdot$

$e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))} \geq \min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa_1)}, \complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial_1)}\}$ for all $\varkappa_1, \partial_1 \in \mathfrak{S}$, and

$\min\{\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_2\partial_1))}, \complement_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_2\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_2\partial_2))}\} \geq \min\{\min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa_1)}, \complement_{\Bbbk}^{T^+}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa_2)}\}, \min\{\complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial_1)}, \complement_{\Bbbk}^{T^+}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial_2)}\}\}$.

If $\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_2\partial_1))} \leq \complement_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_2\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_2\partial_2))}$, then $\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_2\partial_1))} \geq \min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa_1)}, \complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial_1)}\}$.

$\min\{\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_3\partial_1))}, \complement_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_3\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_3\partial_2))}\} \geq \min\{\min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa_1)}, \complement_{\Bbbk}^{T^+}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa_2)}\}, \min\{\complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial_1)}, \complement_{\Bbbk}^{T^+}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial_2)}\}\}$.

If $\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_3\partial_1))} \leq \complement_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_3\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_3\partial_2))}$, then $\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_3\partial_1))} \geq \min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa_1)}, \complement_{\Bbbk}^{T^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial_1)}\}$.

Now,

$$\begin{aligned}
&\frac{1}{2}\Big[\complement_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_1\partial_1))} + \complement_{\Bbbk}^{I^+}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+((\varkappa_2\heartsuit_1\partial_2))}}\Big]\\
&= \complement_{\sigma}^{I^+}(\varkappa_1\heartsuit_1\partial_1, \varkappa_2\heartsuit_1\partial_2)\cdot e^{i\omega\daleth_{\sigma}^{I^+}(\varkappa_1\heartsuit_1\partial_1, \varkappa_2\heartsuit_1\partial_2)}\\
&= \complement_{\sigma}^{I^+}[((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]\cdot e^{i\omega\daleth_{\sigma}^{I^+}[((\varkappa_1,\varkappa_2))\heartsuit_1((\partial_1,\partial_2))]}\\
&= \complement_{\sigma}^{I^+}((\varkappa\heartsuit_1\partial))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\varkappa\heartsuit_1\partial))}\\
&\geq \frac{\complement_{\sigma}^{I^+}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{I^+(\varkappa)}} + \complement_{\sigma}^{I^+}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^+}(\partial)}}{2}\\
&= \frac{\complement_{\sigma}^{I^+}((\varkappa_1,\varkappa_2))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\varkappa_1,\varkappa_2))} + \complement_{\sigma}^{I^+}((\partial_1,\partial_2))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\partial_1,\partial_2))}}{2}\\
&= \frac{1}{2}\Bigg[\frac{\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa_1)} + \complement_{\Bbbk}^{I^+}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa_2)}}{2} + \frac{\complement_{\Bbbk}^{I^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial_1)} + \complement_{\Bbbk}^{I^+}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial_2)}}{2}\Bigg].
\end{aligned}$$

If $\complement_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_1\partial_1))} \leq \complement_{\Bbbk}^{I^+}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}((\varkappa_2\heartsuit_1\partial_2))}$, then $\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa_1)} \leq \complement_{\Bbbk}^{I^+}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa_2)}$ and $\complement_{\Bbbk}^{I^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial_1)} \leq \complement_{\Bbbk}^{I^+}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial_2)}$.

We get $\complement_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\partial_1))} \geq \frac{\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa_1)} + \complement_{\Bbbk}^{I^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial_1)}}{2}$.

Similarly, $\complement_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_2\partial_1))} \geq \frac{\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa_1)} + \complement_{\Bbbk}^{I^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial_1)}}{2}$

and $\complement_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_3\partial_1))} \geq \frac{\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa_1)} + \complement_{\Bbbk}^{I^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial_1)}}{2}$.

Similarly, to prove that

$\max\{\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_1\partial_1))}, \complement_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_1\partial_2))}\}$

$\leq \max\{\max\{\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa_1)}, \complement_{\Bbbk}^{F^+}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa_2)}\}, \max\{\complement_{\Bbbk}^{F^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\partial_1)}, \complement_{\Bbbk}^{F^+}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\partial_2)}\}\}$.

If $\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_1\partial_1))} \geq \complement_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_1\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_1\partial_2))}$, then $\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa_1)} \geq \complement_{\Bbbk}^{F^+}(\varkappa_2)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa_2)}$ and $\complement_{\Bbbk}^{F^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\partial_1)} \geq \complement_{\Bbbk}^{F^+}(\partial_2)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\partial_2)}$.

We get $\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_1\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_1\partial_1))} \leq \max\{\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa_1)}, \complement_{\Bbbk}^{F^+}(\partial_1)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\partial_1)}\}$.

$\max\{\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_2\partial_1))}, \complement_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_2\partial_2))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_2\partial_2))}\}$

$\leq \max\{\max\{\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa_1)}, \complement_{\Bbbk}^{F^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa_2)}\}, \max\{\complement_{\Bbbk}^{F^+}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\partial_1)}, \complement_{\Bbbk}^{F^+}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}}\}\}$.

If $\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_2\partial_1))} \geq \complement_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_2\partial_2))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_2\partial_2))}$, then $\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_2\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_2\partial_1))} \leq \max\{\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa_1)}, \complement_{\Bbbk}^{F^+}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\partial_1)}\}$.

$\max\{\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_3\partial_1))}, \complement_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_3\partial_2))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_3\partial_2))}\}$

$\leq \max\{\max\{\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa_1)}, \complement_{\Bbbk}^{F^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa_2)}\}, \max\{\complement_{\Bbbk}^{F^+}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\partial_1)}, \complement_{\Bbbk}^{F^+}(\partial_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\partial_2)}\}\}$.

If $\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_3\partial_1))} \geq \complement_{\Bbbk}^{F^+}((\varkappa_2\heartsuit_3\partial_2))\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}((\varkappa_2\heartsuit_3\partial_2))}$, then $\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_3\partial_1))\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_3\partial_1))} \leq \max\{\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}(\varkappa_1)}, \complement_{\Bbbk}^{F^+}(\partial_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}(\partial_1)}\}$.

Therefore, $\Bbbk$ is a CBNSBS of $\mathfrak{S}$.

Theorem 3.4. *Suppose that* $\Bbbk$ *is a subset of* $\mathfrak{S}$. *Then* $R = (\complement_{\Bbbk}^{T^-}\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}, \complement_{\Bbbk}^{I^-}\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}, \complement_{\Bbbk}^{F^-}\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}, \complement_{\Bbbk}^{T^+}\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}}, \complement_{\Bbbk}^{I^+}\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}}, \complement_{\Bbbk}^{F^+}\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}})$ *is a CBNSBS of* $\mathfrak{S}$ *if and only if* $\complement^{(\hbar_1,\hbar_2)}$ *is an SBS of* $\mathfrak{S}$ *for all* $\hbar_1,\hbar_2 \in [-1,0]\times[0,1]$.

Proof. Assume that $\complement$ is a CBNSBS of $\mathfrak{S}$. For each $\hbar_1,\hbar_2 \in [-1,0]\times[0,1]$ and $\varkappa_1,\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$. Now, $\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}(\varkappa_1) \leq \hbar_1, \complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}(\varkappa_2) \leq \hbar_1$ and $\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}(\varkappa_1) \leq \hbar_1, \complement_{\Bbbk}^{I^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}(\varkappa_2) \leq \hbar_1$ and $\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}(\varkappa_1) \geq \hbar_2, \complement_{\Bbbk}^{F^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}(\varkappa_2) \geq \hbar_2$. Now, $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}((\varkappa_1\heartsuit_1\varkappa_2)) \leq \max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}(\varkappa_1), \complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}(\varkappa_2)\} \leq \hbar_1$ and $\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}((\varkappa_1\heartsuit_1\varkappa_2)) \leq \frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}(\varkappa_1)+\complement_{\Bbbk}^{I^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}(\varkappa_2)}{2} \leq \frac{\hbar_1+\hbar_1}{2} = \hbar_1$ and $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}((\varkappa_1\heartsuit_1\varkappa_2)) \geq \min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}(\varkappa_1), \complement_{\Bbbk}^{F^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}(\varkappa_2)\} \geq \hbar_2$. This implies that $\varkappa_1\heartsuit_1\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$. Similarly, $\varkappa_1\heartsuit_2\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$ and $\varkappa_1\heartsuit_3\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$. Hence, $\complement^{(\hbar_1,\hbar_2)}$ is an SBS of $\mathfrak{S}$, for all $\hbar_1,\hbar_2 \in [-1,0]\times[0,1]$.

For each $\hbar_1,\hbar_2 \in [0,1]$ and $\varkappa_1,\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$. Now, $\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}}(\varkappa_1) \geq \hbar_1, \complement_{\Bbbk}^{T^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}}(\varkappa_2) \geq \hbar_1$ and $\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}}(\varkappa_1) \geq \hbar_1, \complement_{\Bbbk}^{I^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}}(\varkappa_2) \geq \hbar_1$ and $\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}}(\varkappa_1) \leq \hbar_2, \complement_{\Bbbk}^{F^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}}(\varkappa_2) \leq \hbar_2$. Now, $\complement_{\Bbbk}^{T^+}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}}((\varkappa_1\heartsuit_1\varkappa_2)) \geq \min\{\complement_{\Bbbk}^{T^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}}(\varkappa_1), \complement_{\Bbbk}^{T^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^+}}(\varkappa_2)\} \geq \hbar_1$ and $\complement_{\Bbbk}^{I^+}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}}((\varkappa_1\heartsuit_1\varkappa_2)) \geq \frac{\complement_{\Bbbk}^{I^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}}(\varkappa_1)+\complement_{\Bbbk}^{I^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^+}}(\varkappa_2)}{2} \geq \frac{\hbar_1+\hbar_1}{2} = \hbar_1$ and $\complement_{\Bbbk}^{F^+}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}}((\varkappa_1\heartsuit_1\varkappa_2)) \leq \max\{\complement_{\Bbbk}^{F^+}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}}(\varkappa_1), \complement_{\Bbbk}^{F^+}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^+}}(\varkappa_2)\} \leq \hbar_2$. This implies that $\varkappa_1\heartsuit_1\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$. Similarly, $\varkappa_1\heartsuit_2\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$ and $\varkappa_1\heartsuit_3\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$. Hence, $\complement^{(\hbar_1,\hbar_2)}$ is an SBS of $\mathfrak{S}$,for all $\hbar_1,\hbar_2 \in [-1,0]\times[0,1]$.

Conversely, assume that $\complement^{(\hbar_1,\hbar_2)}$ is an SBS of $\mathfrak{S}$ and $\hbar_1,\hbar_2 \in [-1,0]\times[0,1]$. Suppose if there exist $\varkappa_1,\varkappa_2 \in \mathfrak{S}$ such that $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}((\varkappa_1\heartsuit_1\varkappa_2)) > \max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}(\varkappa_1), \complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}(\varkappa_2)\}$, $\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}((\varkappa_1\heartsuit_1\varkappa_2)) > \frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}(\varkappa_1)+\complement_{\Bbbk}^{I^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}(\varkappa_2)}{2}$ and $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}((\varkappa_1\heartsuit_1\varkappa_2)) < \min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}(\varkappa_1), \complement_{\Bbbk}^{F^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}(\varkappa_2)\}$. For $\hbar_1,\hbar_2 \in [-1,0]\times[0,1]$ such that $\complement_{\Bbbk}^{T^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}((\varkappa_1\heartsuit_1\varkappa_2)) > \hbar_1 \geq \max\{\complement_{\Bbbk}^{T^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}(\varkappa_1), \complement_{\Bbbk}^{T^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{T^-}}(\varkappa_2)\}$ and $\complement_{\Bbbk}^{I^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}((\varkappa_1\heartsuit_1\varkappa_2)) > \hbar_1 \geq \frac{\complement_{\Bbbk}^{I^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}(\varkappa_1)+\complement_{\Bbbk}^{I^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{I^-}}(\varkappa_2)}{2}$ and $\complement_{\Bbbk}^{F^-}((\varkappa_1\heartsuit_1\varkappa_2))\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}((\varkappa_1\heartsuit_1\varkappa_2)) < \hbar_2 \leq \min\{\complement_{\Bbbk}^{F^-}(\varkappa_1)\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}(\varkappa_1), \complement_{\Bbbk}^{F^-}(\varkappa_2)\cdot e^{i\omega\Finv_{\Bbbk}^{F^-}}(\varkappa_2)\}$. Thus, $\varkappa_1,\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$,but $\varkappa_1\heartsuit_1\varkappa_2 \notin \complement^{(\hbar_1,\hbar_2)}$. This contradicts, $\complement^{(\hbar_1,\hbar_2)}$ is an SBS of $\mathfrak{S}$.

Therefore, $\complement^{T^-}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{T^-}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) \leq \max\{\complement^{T^-}_{\Bbbk}(\varkappa_1) \cdot e^{i\omega\daleth^{T^-}_{\Bbbk}}(\varkappa_1), \complement^{T^-}_{\Bbbk}(\varkappa_2) \cdot e^{i\omega\daleth^{T^-}_{\Bbbk}}(\varkappa_2)\}$, $\complement^{I^-}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{I^-}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) \leq \frac{\complement^{I^-}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}}(\varkappa_1)+\complement^{I^-}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\daleth^{I^-}_{\Bbbk}}(\varkappa_2)}{2}$ and $\complement^{F^-}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{F^-}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) \geq \min\{\complement^{F^-}_{\Bbbk}(\varkappa_1) \cdot e^{i\omega\daleth^{F^-}_{\Bbbk}}(\varkappa_1), \complement^{F^-}_{\Bbbk}(\varkappa_2) \cdot e^{i\omega\daleth^{F^-}_{\Bbbk}}(\varkappa_2)\}$. Similarly, $\heartsuit_2$ and $\heartsuit_3$ cases.
Hence, $\complement = (\complement^{T^-}_{\Bbbk} \cdot e^{i\omega\daleth^{T^-}_{\Bbbk}}, \complement^{I^-}_{\Bbbk} \cdot e^{i\omega\daleth^{I^-}_{\Bbbk}}, \complement^{F^-}_{\Bbbk} \cdot e^{i\omega\daleth^{F^-}_{\Bbbk}})$ is a CBNSBS of $\mathfrak{S}$.

Let us assume that $\complement^{(\hbar_1,\hbar_2)}$ is an SBS of $\mathfrak{S}$ and $\hbar_1,\hbar_2 \in [-1,0]\times[0,1]$. Suppose if there exist $\varkappa_1,\varkappa_2 \in \mathfrak{S}$ such that $\complement^{T^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) > \min\{\complement^{T^+}_{\Bbbk}(\varkappa_1) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}(\varkappa_1), \complement^{T^+}_{\Bbbk}(\varkappa_2) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}(\varkappa_2)\}$, $\complement^{I^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) > \frac{\complement^{I^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}(\varkappa_1)+\complement^{I^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}(\varkappa_2)}{2}$ and $\complement^{F^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) < \max\{\complement^{F^+}_{\Bbbk}(\varkappa_1) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}(\varkappa_1), \complement^{F^+}_{\Bbbk}(\varkappa_2) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}(\varkappa_2)\}$. For $\hbar_1,\hbar_2 \in [-1,0]\times[0,1]$ such that $\complement^{T^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) > \hbar_1 \leq \min\{\complement^{T^+}_{\Bbbk}(\varkappa_1) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}(\varkappa_1), \complement^{T^+}_{\Bbbk}(\varkappa_2) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}(\varkappa_2)\}$ and
$\complement^{I^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\daleth^{I^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) > \hbar_1 \leq \frac{\complement^{I^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}(\varkappa_1)+\complement^{I^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}(\varkappa_2)}{2}$ and
$\complement^{F^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) < \hbar_2 \geq \max\{\complement^{F^+}_{\Bbbk}(\varkappa_1) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}(\varkappa_1), \complement^{F^+}_{\Bbbk}(\varkappa_2) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}(\varkappa_2)\}$.
Thus, $\varkappa_1,\varkappa_2 \in \complement^{(\hbar_1,\hbar_2)}$,but $\varkappa_1\heartsuit_1\varkappa_2 \notin \complement^{(\hbar_1,\hbar_2)}$. This contradicts,$\complement^{(\hbar_1,\hbar_2)}$ is an SBS of $\mathfrak{S}$.
Therefore, $\complement^{T^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) \geq \min\{\complement^{T^+}_{\Bbbk}(\varkappa_1) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}(\varkappa_1), \complement^{T^+}_{\Bbbk}(\varkappa_2) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}(\varkappa_2)\}$, $\complement^{I^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) \geq \frac{\complement^{I^+}_{\Bbbk}(\varkappa_1)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}(\varkappa_1)+\complement^{I^+}_{\Bbbk}(\varkappa_2)\cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}(\varkappa_2)}{2}$ and $\complement^{F^+}_{\Bbbk}((\varkappa_1\heartsuit_1\varkappa_2)) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}((\varkappa_1\heartsuit_1\varkappa_2)) \leq \max\{\complement^{F^+}_{\Bbbk}(\varkappa_1) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}(\varkappa_1), \complement^{F^+}_{\Bbbk}(\varkappa_2) \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}(\varkappa_2)\}$. Similarly, $\heartsuit_2$ and $\heartsuit_3$ cases. Hence, $\complement = (\complement^{T^+}_{\Bbbk} \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}, \complement^{I^+}_{\Bbbk} \cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}, \complement^{F^+}_{\Bbbk} \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}})$ is a CBNSBS of $\mathfrak{S}$.

Definition 3.7. *Let $(\mathfrak{S}_1,\square_1,\square_2,\square_3)$ and $(\mathfrak{S}_2,\varnothing_1,\varnothing_2,\varnothing_3)$ be any two bisemirings. The mapping $\mathcal{F}:\mathfrak{S}_1 \to \mathfrak{S}_2$ and $\Bbbk$ be any CBNSBS in $\mathfrak{S}_1$, σ be any CBNSBS in $\mathcal{F}(\mathfrak{S}_1) = \mathfrak{S}_2$. If $\complement_{\Bbbk} \cdot e^{i\omega\daleth_{\Bbbk}} = [\complement^{T^-}_{\Bbbk} \cdot e^{i\omega\daleth^{T^-}_{\Bbbk}}, \complement^{I^-}_{\Bbbk} \cdot e^{i\omega\daleth^{I^-}_{\Bbbk}}, \complement^{F^-}_{\Bbbk} \cdot e^{i\omega\daleth^{F^-}_{\Bbbk}}, \complement_{\Bbbk} \cdot e^{i\omega\daleth_{\Bbbk}}, \complement^{T^+}_{\Bbbk} \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}, \complement^{I^+}_{\Bbbk} \cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}, \complement^{F^+}_{\Bbbk} \cdot e^{i\omega\daleth^{F^+}_{\Bbbk}}$ is a CBNS in $\mathfrak{S}_1$, then $\complement_\sigma$ is a CBNS in $\mathfrak{S}_2$, defined by*

$$\complement^{T^-}_{\sigma}(\partial) \cdot e^{i\omega\daleth^{T^-}_{\Bbbk}}(\partial) = \begin{cases} \inf \complement^{T^-}_{\Bbbk}(\varkappa) \cdot e^{i\omega\daleth^{T^-}_{\Bbbk}}(\varkappa) & if \ \ \varkappa \in \mathcal{F}^{-1}\partial \\ 0 & otherwise \end{cases}$$

$$\complement^{I^-}_{\sigma}(\partial) \cdot e^{i\omega\daleth^{I^-}_{\Bbbk}}(\partial) = \begin{cases} \inf \complement^{I^-}_{\Bbbk}(\varkappa) \cdot e^{i\omega\daleth^{I^-}_{\Bbbk}}(\varkappa) & if \ \ \varkappa \in \mathcal{F}^{-1}\partial \\ 0 & otherwise \end{cases}$$

$$\complement^{F^-}_{\sigma}(\partial) \cdot e^{i\omega\daleth^{F^-}_{\Bbbk}}(\partial) = \begin{cases} \sup \complement^{F^-}_{\Bbbk}(\varkappa) \cdot e^{i\omega\daleth^{F^-}_{\Bbbk}}(\varkappa) & if \ \ \varkappa \in \mathcal{F}^{-1}\partial \\ -1 & otherwise \end{cases}$$

$$\complement^{T^+}_{\sigma}(\partial) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}(\partial) = \begin{cases} \sup \complement^{T^+}_{\Bbbk}(\varkappa) \cdot e^{i\omega\daleth^{T^+}_{\Bbbk}}(\varkappa) & if \ \ \varkappa \in \mathcal{F}^{-1}\partial \\ 0 & otherwise \end{cases}$$

$$\complement^{I^+}_{\sigma}(\partial) \cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}(\partial) = \begin{cases} \sup \complement^{I^+}_{\Bbbk}(\varkappa) \cdot e^{i\omega\daleth^{I^+}_{\Bbbk}}(\varkappa) & if \ \ \varkappa \in \mathcal{F}^{-1}\partial \\ 0 & otherwise \end{cases}$$

$$\complement_{\sigma}^{F^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}}(\partial)=\begin{cases}\inf \complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}}(\varkappa) & if\ \ \varkappa\in\mathcal{F}^{-1}\partial\\ 1 & otherwise\end{cases}$$

for all $\varkappa \in \mathfrak{S}_1$ and $\partial \in \mathfrak{S}_2$ it represents the image of $R_{\Bbbk}$ under $\mathcal{F}$.
Similarly, if $\complement_{\sigma}\cdot e^{i\omega\daleth_{\sigma}} = [\complement_{\sigma}^{T^-}\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}}, \complement_{\sigma}^{I^-}\cdot e^{i\omega\daleth_{\sigma}^{I^-}}, \complement_{\sigma}^{F^-}\cdot e^{i\omega\daleth_{\sigma}^{F^-}}]$, $\complement_{\sigma}\cdot e^{i\omega\daleth_{\sigma}}, \complement_{\sigma}^{T^+}\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}}, \complement_{\sigma}^{I^+}\cdot e^{i\omega\daleth_{\sigma}^{I^+}}, \complement_{\sigma}^{F^+}\cdot e^{i\omega\daleth_{\sigma}^{F^+}}$ is a CBNS in $\mathfrak{S}_2$, then CBNS $\complement_{\Bbbk} = \mathcal{F}\circ\complement_{\sigma}$ in $\mathfrak{S}_1$ i.e., the CBNS defined by $\complement_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}(\varkappa)}, \complement_{\sigma}(\mathcal{F}(\varkappa))\cdot e^{i\omega\daleth_{\Bbbk}(\mathcal{F}(\varkappa))}$, $\complement_{\Bbbk} = \mathcal{F}\circ\complement_{\sigma}$ in $\mathfrak{S}_1$ [i.e., the CBNS defined by $\complement_{\Bbbk}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}(\varkappa)} = \complement_{\sigma}(\mathcal{F}(\varkappa))\cdot e^{i\omega\daleth_{\Bbbk}(\mathcal{F}(\varkappa))}$ it represents the preimage of $\complement_{\sigma}$ under $\mathcal{F}$.

Theorem 3.5. *The homomorphic image of every CBNSBS is a CBNSBS.*

Proof. The mapping $\mathcal{F}:\mathfrak{S}_1\to\mathfrak{S}_2$ be any homomorphism. Now, $\mathcal{F}((\varkappa\square_1\partial)) = \mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial), \mathcal{F}((\varkappa\square_2\partial)) = \mathcal{F}(\varkappa)\oslash_2\mathcal{F}(\partial)$ and $\mathcal{F}((\varkappa\square_3\partial)) = \mathcal{F}(\varkappa)\oslash_3\mathcal{F}(\partial)$ for all $\varkappa,\partial\in\mathfrak{S}_1$. Let $\sigma = \mathcal{F}(\Bbbk)$, $\Bbbk$ is any CBNSBS of $\mathfrak{S}_1$. Let $\mathcal{F}(\varkappa),\mathcal{F}(\partial)\in\mathfrak{S}_2$. Let $\varkappa\in u\mathcal{F}^{-1}(\mathcal{F}(\varkappa))$ and $\partial\in\mathcal{F}^{-1}(\mathcal{F}(\partial))$ be such that $\complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa)} = \inf\limits_{\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa))}\complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa)}$ and $\complement_{\Bbbk}^{T^-}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial)} = \inf\limits_{\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\partial))}\complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa)}$.
Now,

$$\begin{aligned}
&\complement_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))}\\
&=\inf_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial))}\complement_{\Bbbk}^{T^-}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa')}\\
&=\inf_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}((\varkappa\square_1\partial))}\complement_{\Bbbk}^{T^-}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa')}\\
&=\complement_{\Bbbk}^{T^-}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa\square_1\partial))}\\
&\le\max\{\complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa)},\complement_{\Bbbk}^{T^-}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial)}\}\\
&=\max\{\complement_{\sigma}^{T^-}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\varkappa)},\complement_{\sigma}^{T^-}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\partial)}\}.
\end{aligned}$$

Thus, $\complement_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))}\le\max\{\complement_{\sigma}^{T^-}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\varkappa)},\complement_{\sigma}^{T^-}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\partial)}\}$.
Similarly, $\complement_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_2\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_2\mathcal{F}(\partial)))}\le\max\{\complement_{\sigma}^{T^-}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\varkappa)},\complement_{\sigma}^{T^-}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\partial)}\}$ and
$\complement_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_3\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_3\mathcal{F}(\partial)))}\le\max\{\complement_{\sigma}^{T^-}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\varkappa)},\complement_{\sigma}^{T^-}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\partial)}\}$.
Let $\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa))$ and $\partial\in\mathcal{F}^{-1}(\mathcal{F}(\partial))$ be such that $\complement_{\Bbbk}^{I^-}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}(\varkappa)} = \inf\limits_{\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa))}\complement_{\Bbbk}^{I^-}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}(\varkappa)}$ and $\complement_{\Bbbk}^{I^-}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^-}(\partial)} = \inf\limits_{\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\partial))}\complement_{\Bbbk}^{I^-}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\varkappa)}$.
Now,

$$\begin{aligned}
&\complement_{\sigma}^{I^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))}\\
&=\inf_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial))}\complement_{\Bbbk}^{I^-}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}(\varkappa')}\\
&=\inf_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}((\varkappa\square_1\partial))}\complement_{\Bbbk}^{I^-}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}(\varkappa')}
\end{aligned}$$

$$
\begin{aligned}
&= \complement_{\Bbbk}^{I^-}((\varkappa \square_1 \partial)) \cdot e^{i\omega \dashv_{\Bbbk}^{I^-}((\varkappa \square_1 \partial))} \\
&\leq \frac{\complement_{\Bbbk}^{I^-}(\varkappa) \cdot e^{i\omega \dashv_{\Bbbk}^{I^-}(\varkappa)} + \complement_{\Bbbk}^{I^-}(\partial) \cdot e^{i\omega \dashv_{\Bbbk}^{I^-}(\partial)}}{2} \\
&= \frac{\complement_{\sigma}^{I^-}\mathcal{F}(\varkappa) \cdot e^{i\omega \dashv_{\sigma}^{I^-}\mathcal{F}(\varkappa)} + \complement_{\sigma}^{I^-}\mathcal{F}(\partial) \cdot e^{i\omega \dashv_{\sigma}^{I^-}\mathcal{F}(\partial)}}{2}.
\end{aligned}
$$

Thus,
$\complement_{\sigma}^{I^-}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial))) \cdot e^{i\omega \dashv_{\sigma}^{I^-}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial)))} \leq \frac{\complement_{\sigma}^{I^-}\mathcal{F}(\varkappa) \cdot e^{i\omega \dashv_{\sigma}^{I^-}\mathcal{F}(\varkappa)} + \complement_{\sigma}^{I^-}\mathcal{F}(\partial) \cdot e^{i\omega \dashv_{\sigma}^{I^-}\mathcal{F}(\partial)}}{2}$.
Similarly,
$\complement_{\sigma}^{I^-}((\mathcal{F}(\varkappa) \oslash_2 \mathcal{F}(\partial))) \cdot e^{i\omega \dashv_{\sigma}^{I^-}((\mathcal{F}(\varkappa) \oslash_2 \mathcal{F}(\partial)))} \leq \frac{\complement_{\sigma}^{I^-}\mathcal{F}(\varkappa) \cdot e^{i\omega \dashv_{\sigma}^{I^-}\mathcal{F}(\varkappa)} + \complement_{\sigma}^{I^-}\mathcal{F}(\partial) \cdot e^{i\omega \dashv_{\sigma}^{I^-}\mathcal{F}(\partial)}}{2}$ and
$\complement_{\sigma}^{I^-}((\mathcal{F}(\varkappa) \oslash_3 \mathcal{F}(\partial))) \cdot e^{i\omega \dashv_{\sigma}^{I^-}((\mathcal{F}(\varkappa) \oslash_3 \mathcal{F}(\partial)))} \leq \frac{\complement_{\sigma}^{I^-}\mathcal{F}(\varkappa) \cdot e^{i\omega \dashv_{\sigma}^{I^-}\mathcal{F}(\varkappa)} + \complement_{\sigma}^{I^-}\mathcal{F}(\partial) \cdot e^{i\omega \dashv_{\sigma}^{I^-}\mathcal{F}(\partial)}}{2}$.
Let $\mathcal{F}(\varkappa), \mathcal{F}(\partial) \in \mathfrak{S}_2$. Let $\varkappa \in \mathcal{F}^{-1}(\mathcal{F}(\varkappa))$ and $\partial \in \mathcal{F}^{-1}(\mathcal{F}(\partial))$ be such that $\complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}(\varkappa)} = \sup\limits_{\varkappa \in \mathcal{F}^{-1}(\mathcal{F}(\varkappa))} \complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}(\varkappa)}$ and $\complement_{\Bbbk}^{F^-}(\partial) \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}(\partial)} = \sup\limits_{\varkappa \in \mathcal{F}^{-1}(\mathcal{F}(\partial))} \complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}(\varkappa)}$. Now,

$$
\begin{aligned}
&\complement_{\sigma}^{F^-}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial))) \cdot e^{i\omega \dashv_{\sigma}^{F^-}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial)))} \\
&= \sup_{(\varkappa') \in \mathcal{F}^{-1}(\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial))} \complement_{\Bbbk}^{F^-}(\varkappa') \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}(\varkappa')} \\
&= \sup_{(\varkappa') \in \mathcal{F}^{-1}(\mathcal{F}((\varkappa \square_1 \partial))} \complement_{\Bbbk}^{F^-}(\varkappa') \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}(\varkappa')} \\
&= \complement_{\Bbbk}^{F^-}((\varkappa \square_1 \partial)) \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}((\varkappa \square_1 \partial))} \\
&\geq \min\{\complement_{\Bbbk}^{F^-}(\varkappa) \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}(\varkappa)}, \complement_{\Bbbk}^{F^-}(\partial) \cdot e^{i\omega \dashv_{\Bbbk}^{F^-}(\partial)}\} \\
&= \min\{\complement_{\sigma}^{F^-}\mathcal{F}(\varkappa) \cdot e^{i\omega \dashv_{\sigma}^{F^-}\mathcal{F}(\varkappa)}, \complement_{\sigma}^{F^-}\mathcal{F}(\partial) \cdot e^{i\omega \dashv_{\sigma}^{F^-}\mathcal{F}(\partial)}\}.
\end{aligned}
$$

Thus, $\complement_{\sigma}^{F^-}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial))) \cdot e^{i\omega \dashv_{\sigma}^{F^-}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial)))} \geq \min\{\complement_{\sigma}^{F^-}\mathcal{F}(\varkappa) \cdot e^{i\omega \dashv_{\sigma}^{F^-}\mathcal{F}(\varkappa)}, \complement_{\sigma}^{F^-}\mathcal{F}(\partial) \cdot e^{i\omega \dashv_{\sigma}^{F^-}\mathcal{F}(\partial)}\}$.
Similarly,
$\complement_{\sigma}^{F^-}((\mathcal{F}(\varkappa) \oslash_2 \mathcal{F}(\partial))) \cdot e^{i\omega \dashv_{\sigma}^{F^-}((\mathcal{F}(\varkappa) \oslash_2 \mathcal{F}(\partial)))} \geq \min\{\complement_{\sigma}^{F^-}\mathcal{F}(\varkappa) \cdot e^{i\omega \dashv_{\sigma}^{F^-}\mathcal{F}(\varkappa)}, \complement_{\sigma}^{F^-}\mathcal{F}(\partial) \cdot e^{i\omega \dashv_{\sigma}^{F^-}\mathcal{F}(\partial)}\}$ and
$\complement_{\sigma}^{F^-}((\mathcal{F}(\varkappa) \oslash_3 \mathcal{F}(\partial))) \cdot e^{i\omega \dashv_{\sigma}^{F^-}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial)))} \geq \min\{\complement_{\sigma}^{F^-}\mathcal{F}(\varkappa) \cdot e^{i\omega \dashv_{\sigma}^{F^-}\mathcal{F}(\partial)}, \complement_{\sigma}^{F^-}\mathcal{F}(\partial) \cdot e^{i\omega \dashv_{\sigma}^{F^-}\mathcal{F}(\partial)}\}$.

The mapping $\mathcal{F} : \mathfrak{S}_1 \rightarrow \mathfrak{S}_2$ be any homomorphism. Now, $\mathcal{F}((\varkappa \square_1 \partial)) = \mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial), \mathcal{F}((\varkappa \square_2 \partial)) = \mathcal{F}(\varkappa) \oslash_2 \mathcal{F}(\partial)$ and $\mathcal{F}((\varkappa \square_3 \partial)) = \mathcal{F}(\varkappa) \oslash_3 \mathcal{F}(\partial)$ for all $\varkappa, \partial \in \mathfrak{S}_1$. Let $\sigma = \mathcal{F}(\Bbbk)$, $\Bbbk$ is any CBNSBS of $\mathfrak{S}_1$. Let $\mathcal{F}(\varkappa), \mathcal{F}(\partial) \in \mathfrak{S}_2$. Let $\varkappa \in \mathcal{F}^{-1}(\mathcal{F}(\varkappa))$ and $\partial \in \mathcal{F}^{-1}(\mathcal{F}(\partial))$ be such that $\complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega \dashv_{\Bbbk}^{T^+}(\varkappa)} = \sup\limits_{\varkappa \in \mathcal{F}^{-1}(\mathcal{F}(\varkappa))} \complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega \dashv_{\Bbbk}^{T^+}(\varkappa)}$ and $\complement_{\Bbbk}^{T^+}(\partial) \cdot e^{i\omega \dashv_{\Bbbk}^{T^+}(\partial)} = \sup\limits_{\varkappa \in \mathcal{F}^{-1}(\mathcal{F}(\partial))} \complement_{\Bbbk}^{T^+}(\varkappa) \cdot e^{i\omega \dashv_{\Bbbk}^{T^+}(\varkappa)}$. Now,

$$
\begin{aligned}
&\complement_{\sigma}^{T^+}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial))) \cdot e^{i\omega \dashv_{\Bbbk}^{T^+}((\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial)))} \\
&= \sup_{(\varkappa') \in \mathcal{F}^{-1}(\mathcal{F}(\varkappa) \oslash_1 \mathcal{F}(\partial))} \complement_{\Bbbk}^{T^+}(\varkappa') \cdot e^{i\omega \dashv_{\Bbbk}^{T^+}(\varkappa')}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}((\varkappa\square_1\partial))} \complement_{\Bbbk}^{T^+}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa')}\\
&= \complement_{\Bbbk}^{T^+}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa\square_1\partial))}\\
&\geq \min\{\complement_{\Bbbk}^{T^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa)}, \complement_{\Bbbk}^{T^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial)}\}\\
&= \min\{\complement_{\sigma}^{T^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\varkappa)}, \complement_{\sigma}^{T^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\partial)}\}.
\end{aligned}
$$

Thus, $\complement_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \geq \min\{\complement_{\sigma}^{T^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\varkappa)}, \complement_{\sigma}^{T^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\partial)}\}$. Similarly, $\complement_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\varnothing_2\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\varnothing_2\mathcal{F}(\partial)))} \geq \min\{\complement_{\sigma}^{T^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\varkappa)}, \complement_{\sigma}^{T^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\partial)}\}$ and
$\complement_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\varnothing_3\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\varnothing_3\mathcal{F}(\partial)))} \geq \min\{\complement_{\sigma}^{T^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\varkappa)}, \complement_{\sigma}^{T^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\partial)}\}$.
Let $\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa))$ and $\partial\in\mathcal{F}^{-1}(\mathcal{F}(\partial))$ be such that $\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^+}(\varkappa)} = \sup_{\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa))}\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^+}(\varkappa)}$ and $\complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^+}(\partial)} = \sup_{\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\partial))}\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\varkappa)}$.

Now,

$$
\begin{aligned}
&\complement_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))}\\
&= \sup_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial))} \complement_{\Bbbk}^{I^+}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa')}\\
&= \sup_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}((\varkappa\square_1\partial))} \complement_{\Bbbk}^{I^+}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa')}\\
&= \complement_{\Bbbk}^{I^+}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}((\varkappa\square_1\partial))}\\
&\geq \frac{\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa)} + \complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial)}}{2}\\
&= \frac{\complement_{\sigma}^{I^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\varkappa)} + \complement_{\sigma}^{I^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\partial)}}{2}.
\end{aligned}
$$

Thus, $\complement_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \geq \frac{\complement_{\sigma}^{I^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\varkappa)} + \complement_{\sigma}^{I^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\partial)}}{2}$. Similarly,
$\complement_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_2\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_2\mathcal{F}(\partial)))} \geq \frac{\complement_{\sigma}^{I^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\varkappa)} + \complement_{\sigma}^{I^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\partial)}}{2}$ and
$\complement_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_3\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_3\mathcal{F}(\partial)))} \geq \frac{\complement_{\sigma}^{I^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\varkappa)} + \complement_{\sigma}^{I^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\partial)}}{2}$.
Let $\mathcal{F}(\varkappa), \mathcal{F}(\partial)\in\mathfrak{S}_2$. Let $\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa))$ and $\partial\in\mathcal{F}^{-1}(\mathcal{F}(\partial))$ be such that $\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa)} = \inf_{\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa))}\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa)}$ and $\complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\partial)} = \inf_{\varkappa\in\mathcal{F}^{-1}(\mathcal{F}(\partial))}\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa)}$.
Now,

$$
\begin{aligned}
&\complement_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))}\\
&= \inf_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial))} \complement_{\Bbbk}^{F^+}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa')}\\
&= \inf_{(\varkappa')\in\mathcal{F}^{-1}(\mathcal{F}((\varkappa\square_1\partial))} \complement_{\Bbbk}^{F^+}(\varkappa')\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa')}
\end{aligned}
$$

$$= \complement_{\Bbbk}^{F^+}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa\square_1\partial))}$$
$$\leq \max\{\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\varkappa)},\complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}(\partial)}\}$$
$$= \max\{\complement_{\sigma}^{F^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\varkappa)},\complement_{\sigma}^{F^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\partial)}\}.$$

Thus, $\complement_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))} \leq \max\{\complement_{\sigma}^{F^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\varkappa)},\complement_{\sigma}^{F^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\partial)}\}$. Similarly, $\complement_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\oslash_2\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\oslash_2\mathcal{F}(\partial)))} \leq \max\{\complement_{\sigma}^{F^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\varkappa)},\complement_{\sigma}^{F^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\partial)}\}$ and
$\complement_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\oslash_3\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))} \leq \max\{\complement_{\sigma}^{F^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\partial)},\complement_{\sigma}^{F^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\partial)}\}$. Thus, σ is a CBNSBS of $\mathfrak{S}_2$.

Theorem 3.6. *The homomorphic preimage of every CBNSBS is a CBNSBS.*

Proof. The mapping $\mathcal{F}:\mathfrak{S}_1\to\mathfrak{S}_2$ be a homomorphism. Now, $\mathcal{F}((\varkappa\square_1\partial)) = \mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial), \mathcal{F}((\varkappa\square_2\partial)) = \mathcal{F}(\varkappa)\oslash_2\mathcal{F}(\partial)$ and $\mathcal{F}((\varkappa\square_3\partial)) = \mathcal{F}(\varkappa)\oslash_3\mathcal{F}(\partial)$ for all $\varkappa,\partial\in\mathfrak{S}_1$. Let $\sigma=\mathcal{F}(\Bbbk)$, σ is a CBNSBS of $\mathfrak{S}_2$. Let $\varkappa,\partial\in\mathfrak{S}_1$. Now, $\complement_{\Bbbk}^{T^-}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa\square_1\partial))} = \complement_{\sigma}^{T^-}(\mathcal{F}((\varkappa\square_1\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^-}(\mathcal{F}((\varkappa\square_1\partial)))} = \complement_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))} \leq \max\{\complement_{\sigma}^{T^-}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\varkappa)},\complement_{\sigma}^{T^-}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^-}\mathcal{F}(\partial)}\} = \max\{\complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa)},\complement_{\Bbbk}^{T^-}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial)}\}$. Thus, $\complement_{\Bbbk}^{T^-}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}((\varkappa\square_1\partial))} \leq \max\{\complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\varkappa)},\complement_{\Bbbk}^{T^-}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{T^-}(\partial)}\}$.
Now, $\complement_{\Bbbk}^{I^-}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}((\varkappa\square_1\partial))} = \complement_{\sigma}^{I^-}(\mathcal{F}((\varkappa\square_1\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^-}((\varkappa\square_1\partial))} = \complement_{\sigma}^{I^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^-}((\varkappa\square_1\partial))} \leq \frac{\complement_{\sigma}^{I^-}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{I^-}\mathcal{F}(\varkappa)}+\complement_{\sigma}^{I^-}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^-}\mathcal{F}(\partial)}}{2} = \frac{\complement_{\Bbbk}^{I^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}(\varkappa)}+\complement_{\Bbbk}^{I^-}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}(\partial)}}{2}$.
Thus, $\complement_{\Bbbk}^{I^-}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}((\varkappa\square_1\partial))} \leq \frac{\complement_{\Bbbk}^{I^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}(\varkappa)}+\complement_{\Bbbk}^{I^-}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{I^-}(\partial)}}{2}$.
Now, $\complement_{\Bbbk}^{F^-}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{F^-}((\varkappa\square_1\partial))} = \complement_{\sigma}^{F^-}(\mathcal{F}((\varkappa\square_1\partial)))\cdot e^{i\omega\daleth_{\sigma}^{F^-}(\mathcal{F}((\varkappa\square_1\partial)))} = \complement_{\sigma}^{F^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{F^-}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))} \geq \min\{\complement_{\sigma}^{F^-}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{F^-}\mathcal{F}(\varkappa)},\complement_{\sigma}^{F^-}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{F^-}\mathcal{F}(\partial)}\} = \min\{\complement_{\Bbbk}^{F^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{F^-}(\varkappa)},\complement_{\Bbbk}^{F^-}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{F^-}(\partial)}\}$.
Thus, $\complement_{\Bbbk}^{F^-}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{F^-}((\varkappa\square_1\partial))} \geq \min\{\complement_{\Bbbk}^{F^-}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{F^-}(\varkappa)},\complement_{\Bbbk}^{F^-}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{F^-}(\partial)}\}$.

The mapping $\mathcal{F}:\mathfrak{S}_1\to\mathfrak{S}_2$ be a homomorphism. Now, $\mathcal{F}((\varkappa\square_1\partial)) = \mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial), \mathcal{F}((\varkappa\square_2\partial)) = \mathcal{F}(\varkappa)\oslash_2\mathcal{F}(\partial)$ and $\mathcal{F}((\varkappa\square_3\partial)) = \mathcal{F}(\varkappa)\oslash_3\mathcal{F}(\partial)$ for all $\varkappa,\partial\in\mathfrak{S}_1$. Let $\sigma=\mathcal{F}(\Bbbk)$, σ is a CBNSBS of $\mathfrak{S}_2$. Let $\varkappa,\partial\in\mathfrak{S}_1$. Now, $\complement_{\Bbbk}^{T^+}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa\square_1\partial))} = \complement_{\sigma}^{T^+}(\mathcal{F}((\varkappa\square_1\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^+}(\mathcal{F}((\varkappa\square_1\partial)))} = \complement_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))} \geq \min\{\complement_{\sigma}^{T^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\varkappa)},\complement_{\sigma}^{T^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{T^+}\mathcal{F}(\partial)}\} = \min\{\complement_{\Bbbk}^{T^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa)},\complement_{\Bbbk}^{T^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial)}\}$. Thus, $\complement_{\Bbbk}^{T^+}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}((\varkappa\square_1\partial))} \geq \min\{\complement_{\Bbbk}^{T^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\varkappa)},\complement_{\Bbbk}^{T^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{T^+}(\partial)}\}$.
Now, $\complement_{\Bbbk}^{I^+}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}((\varkappa\square_1\partial))} = \complement_{\sigma}^{I^+}(\mathcal{F}((\varkappa\square_1\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\varkappa\square_1\partial))} = \complement_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{I^+}((\varkappa\square_1\partial))} \geq \frac{\complement_{\sigma}^{I^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\varkappa)}+\complement_{\sigma}^{I^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{I^+}\mathcal{F}(\partial)}}{2} = \frac{\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa)}+\complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial)}}{2}$.
Thus, $\complement_{\Bbbk}^{I^+}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}((\varkappa\square_1\partial))} \geq \frac{\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\varkappa)}+\complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega\daleth_{\Bbbk}^{I^+}(\partial)}}{2}$.
Now, $\complement_{\Bbbk}^{F^+}((\varkappa\square_1\partial))\cdot e^{i\omega\daleth_{\Bbbk}^{F^+}((\varkappa\square_1\partial))} = \complement_{\sigma}^{F^+}(\mathcal{F}((\varkappa\square_1\partial)))\cdot e^{i\omega\daleth_{\sigma}^{F^+}(\mathcal{F}((\varkappa\square_1\partial)))}$
$= \complement_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))\cdot e^{i\omega\daleth_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\oslash_1\mathcal{F}(\partial)))} \leq \max\{\complement_{\sigma}^{F^+}\mathcal{F}(\varkappa)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\varkappa)},\complement_{\sigma}^{F^+}\mathcal{F}(\partial)\cdot e^{i\omega\daleth_{\sigma}^{F^+}\mathcal{F}(\partial)}\}$

$= \max\{\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\varkappa)}, \complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\partial)}\}$.

Thus, $\complement_{\Bbbk}^{F^+}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}((\varkappa\Box_1\partial))} \leq \max\{\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\varkappa)}, \complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\partial)}\}$.

Theorem 3.7. *If $\mathcal{F}: \mathfrak{S}_1 \to \mathfrak{S}_2$ is a homomorphism, then $\mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$ is an SBS of CBNSBS σ of $\mathfrak{S}_2$.*

Proof. The mapping $\mathcal{F}: \mathfrak{S}_1 \to \mathfrak{S}_2$ be a homomorphism. Now, $\mathcal{F}((\varkappa\Box_1\partial)) = \mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial), \mathcal{F}((\varkappa\Box_2\partial)) = \mathcal{F}(\varkappa)\varnothing_2\mathcal{F}(\partial)$ and $\mathcal{F}((\varkappa\Box_3\partial)) = \mathcal{F}(\varkappa)\varnothing_3\mathcal{F}(\partial)$ for all $\varkappa, \partial \in \mathfrak{S}_1$. Let $\sigma = \mathcal{F}(\Bbbk)$, $\Bbbk$ is a CBNSBS of $\mathfrak{S}_1$. By Theorem 3.5, σ is a CBNSBS of $\mathfrak{S}_2$. Let $\Bbbk_{(\hbar_1,\hbar_2)}$ be any SBS of $\Bbbk$. Suppose that $\varkappa, \partial \in \Bbbk_{(\hbar_1,\hbar_2)}$. Then $\varkappa\Box_1\partial, \varkappa\Box_2\partial$ and $\varkappa\Box_3\partial \in \Bbbk_{(\hbar_1,\hbar_2)}$. Now, $\complement_{\sigma}^{T^-}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{T^-}(\mathcal{F}(\varkappa))} = \complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}(\varkappa)} \leq \hbar_1, \complement_{\sigma}^{T^-}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{T^-}(\mathcal{F}(\partial))} = \complement_{\Bbbk}^{T^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}(\partial)} \leq \hbar_1$. Thus, $\complement_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega \daleth_{\sigma}^{T^-}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \leq \complement_{\Bbbk}^{T^-}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}((\varkappa\Box_1\partial))} \leq \hbar_1$. Now, $\complement_{\sigma}^{I^-}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{I^-}(\mathcal{F}(\varkappa))} = \complement_{\Bbbk}^{I^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa)} \leq \hbar_1, \complement_{\sigma}^{I^-}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{I^-}(\mathcal{F}(\partial))} = \complement_{\Bbbk}^{I^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial)} \leq \hbar_1$. Thus, $\complement_{\sigma}^{I^-}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega \daleth_{\sigma}^{I^-}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \leq \complement_{\Bbbk}^{I^-}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}((\varkappa\Box_1\partial))} \leq \hbar_1$. Now, $\complement_{\sigma}^{F^-}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{F^-}(\mathcal{F}(\varkappa))} = \complement_{\Bbbk}^{F^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa)} \geq \hbar_2, \complement_{\sigma}^{F^-}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{F^-}(\mathcal{F}(\partial))} = \complement_{\Bbbk}^{F^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial)} \geq \hbar_2$. Thus, $\complement_{\sigma}^{F^-}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega \daleth_{\sigma}^{F^-}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \geq \complement_{\Bbbk}^{F^-}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa\Box_1\partial))} \geq \hbar_2$,for all $\mathcal{F}(\varkappa), \mathcal{F}(\partial) \in \mathfrak{S}_2$. Similarly other operations, $\mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$ is an SBS of CBNSBS σ of $\mathfrak{S}_2$.

The mapping $\mathcal{F}: \mathfrak{S}_1 \to \mathfrak{S}_2$ be a homomorphism. Now, $\mathcal{F}((\varkappa\Box_1\partial)) = \mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial), \mathcal{F}((\varkappa\Box_2\partial)) = \mathcal{F}(\varkappa)\varnothing_2\mathcal{F}(\partial)$ and $\mathcal{F}((\varkappa\Box_3\partial)) = \mathcal{F}(\varkappa)\varnothing_3\mathcal{F}(\partial)$ for all $\varkappa, \partial \in \mathfrak{S}_1$. Let $\sigma = \mathcal{F}(\Bbbk)$, $\Bbbk$ is a CBNSBS of $\mathfrak{S}_1$. By Theorem 3.5, σ is a CBNSBS of $\mathfrak{S}_2$. Let $\Bbbk_{(\hbar_1,\hbar_2)}$ be any SBS of $\Bbbk$. Suppose that $\varkappa, \partial \in \Bbbk_{(\hbar_1,\hbar_2)}$. Then $\varkappa\Box_1\partial, \varkappa\Box_2\partial$ and $\varkappa\Box_3\partial \in \Bbbk_{(\hbar_1,\hbar_2)}$. Now, $\complement_{\sigma}^{T^+}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{T^+}(\mathcal{F}(\varkappa))} = \complement_{\Bbbk}^{T^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\varkappa)} \geq \hbar_1, \complement_{\sigma}^{T^+}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{T^+}(\mathcal{F}(\partial))} = \complement_{\Bbbk}^{T^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\partial)} \geq \hbar_1$. Thus, $\complement_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega \daleth_{\sigma}^{T^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \geq \complement_{\Bbbk}^{T^+}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}((\varkappa\Box_1\partial))} \geq \hbar_1$. Now, $\complement_{\sigma}^{I^+}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{I^+}(\mathcal{F}(\varkappa))} = \complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{I^+}(\varkappa)} \geq \hbar_1, \complement_{\sigma}^{I^+}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{I^+}(\mathcal{F}(\partial))} = \complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{I^+}(\partial)} \geq \hbar_1$. Thus, $\complement_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega \daleth_{\sigma}^{I^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \geq \complement_{\Bbbk}^{I^+}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{I^+}((\varkappa\Box_1\partial))} \geq \hbar_1$. Now, $\complement_{\sigma}^{F^+}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{F^+}(\mathcal{F}(\varkappa))} = \complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\varkappa)} \leq \hbar_2, \complement_{\sigma}^{F^+}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{F^+}(\mathcal{F}(\partial))} = \complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\partial)} \leq \hbar_2$. Thus, $\complement_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega \daleth_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \leq \complement_{\Bbbk}^{F^+}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}((\varkappa\Box_1\partial))} \leq \hbar_2$,for all $\mathcal{F}(\varkappa), \mathcal{F}(\partial) \in \mathfrak{S}_2$. Similarly other operations, $\mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$ is an SBS of CBNSBS σ of $\mathfrak{S}_2$.

Theorem 3.8. *If $\mathcal{F}: \mathfrak{S}_1 \to \mathfrak{S}_2$ is any homomorphism, then $\Bbbk_{(\hbar_1,\hbar_2)}$ is an SBS of CBNSBS $\Bbbk$ of $\mathfrak{S}_1$.*

Proof. The mapping $\mathcal{F}: \mathfrak{S}_1 \to \mathfrak{S}_2$ be any homomorphism. We have $\mathcal{F}((\varkappa\Box_1\partial)) = \mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial), \mathcal{F}((\varkappa\Box_2\partial)) = \mathcal{F}(\varkappa)\varnothing_2\mathcal{F}(\partial)$ and $\mathcal{F}((\varkappa\Box_3\partial)) = \mathcal{F}(\varkappa)\varnothing_3\mathcal{F}(\partial)$ for all $\varkappa, \partial \in \mathfrak{S}_1$. Let $\sigma = \mathcal{F}(\Bbbk)$, σ is a CBNSBS of $\mathfrak{S}_2$. By Theorem 3.6, $\Bbbk$ is a CBNSBS of $\mathfrak{S}_1$. Let $\mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$ be an SBS of σ. Suppose that $\mathcal{F}(\varkappa), \mathcal{F}(\partial) \in \mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$. Now, $\mathcal{F}((\varkappa\Box_1\partial)), \mathcal{F}((\varkappa\Box_2\partial))$ and $\mathcal{F}((\varkappa\Box_3\partial)) \in \mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$. Now, $\complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}(\varkappa)} = \complement_{\sigma}^{T^-}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{T^-}(\mathcal{F}(\varkappa))} \leq \hbar_1, \complement_{\Bbbk}^{T^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}(\partial)} = \complement_{\sigma}^{T^-}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{T^-}(\mathcal{F}(\partial))} \leq \hbar_1$. Thus, $\complement_{\Bbbk}^{T^-}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}((\varkappa\Box_1\partial))} \leq \max\{\complement_{\Bbbk}^{T^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}(\varkappa)}, \complement_{\Bbbk}^{T^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{T^-}(\partial)}\} \leq \hbar_1$. Now, $\complement_{\Bbbk}^{I^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa)} = \complement_{\sigma}^{I^-}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{I^-}(\mathcal{F}(\varkappa))} \leq \hbar_1, \complement_{\Bbbk}^{I^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial)} = \complement_{\sigma}^{I^-}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{I^-}(\mathcal{F}(\partial))} \leq \hbar_1$.

Thus, $\complement_{\Bbbk}^{I^-}((\varkappa\Box_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}((\varkappa\Box_1\partial))} \leq \frac{\complement_{\Bbbk}^{I^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\varkappa)} + \complement_{\Bbbk}^{I^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{I^-}(\partial)}}{2} \leq \hbar_1$.

Now, $\complement_{\Bbbk}^{F^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa)} = \complement_{\sigma}^{F^-}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{F^-}(\mathcal{F}(\varkappa))} \geq \hbar_2, \complement_{\Bbbk}^{F^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial)} = \complement_{\sigma}^{F^-}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{F^-}(\mathcal{F}(\partial))} \geq \hbar_2$.

Thus, $\complement_{\Bbbk}^{F^-}((\varkappa\square_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}((\varkappa\square_1\partial))} = \complement_{\sigma}^{F^-}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega \daleth_{\sigma}^{F^-}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))}$
$\geq \min\{\complement_{\Bbbk}^{F^-}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\varkappa)}, \complement_{\Bbbk}^{F^-}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{F^-}(\partial)}\} \geq \hbar_2$ for all $\varkappa, \partial \in \mathfrak{S}_1$.
Similarly other operations, $\Bbbk_{(\hbar_1,\hbar_2)}$ is an SBS of CBNSBS $\Bbbk$ of $\mathfrak{S}_1$.

The mapping $\mathcal{F} : \mathfrak{S}_1 \to \mathfrak{S}_2$ be any homomorphism. We have $\mathcal{F}((\varkappa\square_1\partial)) = \mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial), \mathcal{F}((\varkappa\square_2\partial)) = \mathcal{F}(\varkappa)\varnothing_2\mathcal{F}(\partial)$ and $\mathcal{F}((\varkappa\square_3\partial)) = \mathcal{F}(\varkappa)\varnothing_3\mathcal{F}(\partial)$ for all $\varkappa, \partial \in \mathfrak{S}_1$. Let $\sigma = \mathcal{F}(\Bbbk)$, σ be a CBNSBS of $\mathfrak{S}_2$. By Theorem 3.6, $\Bbbk$ is a CBNSBS of $\mathfrak{S}_1$. Let $\mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$ be an SBS of σ. Suppose that $\mathcal{F}(\varkappa), \mathcal{F}(\partial) \in \mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$. Now, $\mathcal{F}((\varkappa\square_1\partial)), \mathcal{F}((\varkappa\square_2\partial))$ and $\mathcal{F}((\varkappa\square_3\partial)) \in \mathcal{F}(\Bbbk_{(\hbar_1,\hbar_2)})$.
Now, $\complement_{\Bbbk}^{T^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\varkappa)} = \complement_{\sigma}^{T^+}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{T^+}(\mathcal{F}(\varkappa))} \geq \hbar_1, \complement_{\Bbbk}^{T^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\partial)} = \complement_{\sigma}^{T^+}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{T^+}(\mathcal{F}(\partial))} \geq \hbar_1$.
Thus, $\complement_{\Bbbk}^{T^+}((\varkappa\square_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}((\varkappa\square_1\partial))} \geq \min\{\complement_{\Bbbk}^{T^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\varkappa)}, \complement_{\Bbbk}^{T^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{T^+}(\partial)}\} \geq \hbar_1$. Now, $\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{I^+}(\varkappa)} = \complement_{\sigma}^{I^+}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{I^+}(\mathcal{F}(\varkappa))} \geq \hbar_1, \complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{I^+}(\partial)} = \complement_{\sigma}^{I^+}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{I^+}(\mathcal{F}(\partial))} \geq \hbar_1$.
Thus, $\complement_{\Bbbk}^{I^+}((\varkappa\square_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{I^+}((\varkappa\square_1\partial))} \geq \frac{\complement_{\Bbbk}^{I^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{I^+}(\varkappa)} + \complement_{\Bbbk}^{I^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{I^+}(\partial)}}{2} \geq \hbar_1$. Now, $\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\varkappa)} = \complement_{\sigma}^{F^+}(\mathcal{F}(\varkappa))\cdot e^{i\omega \daleth_{\sigma}^{F^+}(\mathcal{F}(\varkappa))} \leq \hbar_2, \complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\partial)} = \complement_{\sigma}^{F^+}(\mathcal{F}(\partial))\cdot e^{i\omega \daleth_{\sigma}^{F^+}(\mathcal{F}(\partial))} \leq \hbar_2$.
Thus, $\complement_{\Bbbk}^{F^+}((\varkappa\square_1\partial))\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}((\varkappa\square_1\partial))} = \complement_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))\cdot e^{i\omega \daleth_{\sigma}^{F^+}((\mathcal{F}(\varkappa)\varnothing_1\mathcal{F}(\partial)))} \leq \max\{\complement_{\Bbbk}^{F^+}(\varkappa)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\varkappa)}, \complement_{\Bbbk}^{F^+}(\partial)\cdot e^{i\omega \daleth_{\Bbbk}^{F^+}(\partial)}\} \leq \hbar_2$,for all $\varkappa, \partial \in \mathfrak{S}_1$.
Similarly other operations, $\Bbbk_{(\hbar_1,\hbar_2)}$ is an SBS of CBNSBS $\Bbbk$ of $\mathfrak{S}_1$.

4. Conclusion and future direction

This study introduces a new class of CBNSBSs. The complex bipolar neutrosophic subbisemiring, which takes an innovative approach to the notion of three grades, expresses three grades in terms of a complex number. A two-dimensional parameter with three grades is called a complex form of three grades. It was classified as a bipolar neutrosophic SBS carrying. CBNSBS and CBNNSBS level sets were established. Our goal is to apply the set to the bisemiring. Additionally, a study of the characteristics of different conversions is done. We are trying to use cubic soft sets and soft sets to handle novel fuzzy structures. Consequently, we need to consider utilizing soft set CBNSBS and CBNNSBS in the future.

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