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SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY MILLER-ROSS-TYPE POISSON DISTRIBUTION SERIES

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Abstract. It is very well-known that the special functions and integral operators play a vital role in the research of applied and mathematical sciences. The main purpose of this paper is to introduce a new subclass of analytic functions involving Miller-Ross functions and obtain coefficient inequalities, distortion theorem, convex linear combination, partial sums, convolution, and neighborhood results for this class.

1. Introduction

The geometric characteristics of analytical functions are the subject of Geometric Function Theory, a significant area of complex analysis. Numerous mathematical disciplines, particularly pure and practical mathematics, heavily rely on this area of complex analysis. Certain geometric properties (such as convexity, starlikeness, or univalency) of some classes of analytic functions (in the unit disk) associated with some researchers have always drawn special functions have been studied by several researchers in the literature for some special classes of univalent functions. The distribution of random variables has garnered a lot of attention lately. In statistics and the concept of probability, especially in relation to distributions, probability density functions are

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fundamental. In real-world scenarios, there are many different types of distributions, such as the binomial, Poisson, and hypergeometric distributions. In the theory of geometric functions, simple distribution, along with Pascal, Poisson, logarithmic, binomial, beta negative binomial, has been partially studied from a theoretical point of view (see [1,8,22,25,43]) and two parameters of the Mittag Leffler-type probability distribution (see [9,20,39]). Now, let us review some well-known definitions and findings related to geometric function theory.

Let $\mathcal{A}$ indicate the class of all mapping $\aleph(\hbar)$ of the type

$$\aleph(\hbar) = \hbar + \sum_{\iota=2}^{\infty} \eta_\iota \hbar^\iota, \tag{1.1}$$

in the open unit disc $\mathbb{U} = \{\hbar \in \mathbb{C} : |\hbar| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent mapping and satisfy the following usual normalization condition $\aleph(0) = \aleph'(0) - 1 = 0$. We denote by S the subclass of $\mathcal{A}$ consisting of mappings $\aleph(\hbar)$ which are all univalent in $\mathbb{U}$. A function $\aleph \in \mathcal{A}$ is a starlike function of the order ξ, $0 \leq \xi < 1$, if it fulfills

$$\Re\left\{\frac{\hbar\aleph'(\hbar)}{\aleph(\hbar)}\right\} > \xi, \ \hbar \in \mathbb{U}. \tag{1.2}$$

We indicate this class with $S^*(\xi)$. A mapping $\aleph \in \mathcal{A}$ is a convex function of the order ξ, $0 \leq \xi < 1$, if it fulfills

$$\Re\left\{1 + \frac{\hbar\aleph''(\hbar)}{\aleph'(\hbar)}\right\} > \xi, \ \hbar \in \mathbb{U}. \tag{1.3}$$

We indicate this class with $K(\xi)$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$ respectively. For $\aleph \in \mathcal{A}$ given by (1.1) and $g(\hbar)$ given by

$$g(\hbar) = \hbar + \sum_{\iota=2}^{\infty} b_\iota \hbar^\iota \tag{1.4}$$

their convolution (or Hadamard product), signified by $(\aleph * g)$, is specified as

$$(\aleph * g)(\hbar) = \hbar + \sum_{\iota=2}^{\infty} \eta_\iota b_\iota \hbar^\iota = (g * \aleph)(\hbar), \ (\hbar \in \mathbb{U}). \tag{1.5}$$

Note that $\aleph * g \in \mathcal{A}$.
Let T indicates the class of functions analytic in $\mathbb{U}$ that are of the type

$$\aleph(\hbar) = \hbar - \sum_{\iota=2}^{\infty} \eta_\iota \hbar^\iota, \ \eta_\iota(\geq 0, \ \hbar \in \mathbb{U}) \tag{1.6}$$

and let $T^*(\xi) = T \cap S^*(\xi)$, $C(\xi) = T \cap K(\xi)$. Silverman [34] has examined the class $T^*(\xi)$ and its associated classes in great detail. These classes include numerous fascinating characteristics.

Miller and Ross proposed the following special function in their monograph (p. 314, [19]), which is now called the Miller-Ross function, defined as

$$E_{\nu,c}(\hbar) = \hbar^\nu e^{c\hbar} \gamma^*(\nu, c\hbar),$$

where γ^* is the incomplete gamma function. Using the properties of the incomplete gamma functions, the Miller-Ross function can easily be written as

$$E_{\nu,\, c}(\hbar) = \hbar^{\nu} \sum_{\iota=0}^{\infty} \frac{(c\hbar)^{\iota}}{\Gamma(\iota+\nu+1)}; \quad \hbar, c, \nu \in \mathbb{C}. \tag{1.7}$$

In this paper, we shall restrict our attention to the case of real-valued $c > 0$ and $\hbar \in \mathbb{U}$. It is clear that the Miller-Ross function $E_{\nu,\, c}(\hbar)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Miller-Ross function [41]:

$$\begin{aligned} E_{\nu,\, c}(\hbar) &= \hbar^{1-\nu}\Gamma(\nu+1)E_{\nu,\, c}(\hbar) \\ &= \hbar + \sum_{\iota=2}^{\infty} \frac{c^{\iota-1}\Gamma(\nu+1)}{\Gamma(\iota+\nu)}\hbar^{\iota} \end{aligned} \tag{1.8}$$

For $c, \nu \in \mathbb{C}$, we can write the following

$$\begin{aligned} E_{\nu,\, c}(1) - 1 &= \sum_{\iota=2}^{\infty} \frac{c^{\iota-1}\Gamma(\nu+1)}{\Gamma(\iota+\nu)}, \\ E'_{\nu,\, c}(1) - 1 &= \sum_{\iota=2}^{\infty} \frac{\iota\, c^{\iota-1}\Gamma(\nu+1)}{\Gamma(\iota+\nu)}, \\ E''_{\nu,\, c}(1) &= \sum_{\iota=2}^{\infty} \frac{\iota(\iota-1)c^{\iota-1}\Gamma(\nu+1)}{\Gamma(\iota+\nu)}. \end{aligned}$$

In recent years, a large literature has evolved on the use of distribution series such as Poisson, Pascal, Borel, etc., in geometric function theory. Many researchers have examined some important features in the field of geometric function theory, such as coefficient estimates, inclusion relations, and conditions of being in some known classes, using different probability distributions, see for example [10-15].

We now recall that a discrete random variable X whose probability mass function is given by

$$P[X = i] = \frac{e^{-\zeta}\zeta^{i}}{i!}, \quad i = 0, 1, 2 \cdots, \quad \zeta > 0$$

is said to have a Poisson distribution with parameter ζ.

Recently, Porwal and Dixit [26] introduced Mittag-Leffler-type Poisson distribution and obtained moments, moment generating function. Bajpai [2] introduced Mittag-Leffler-type Poisson distribution series. Lately, Srivastava et al. [40] introduced the Poisson distribution, a two-parameter Mittag-Leffler-type Poisson distribution. Motivated by results on connections between various subclasses of analytic univalent functions using special functions and distribution series [11, 12, 15, 30, 31, 33, 39, 42, 45] we obtain coefficient inequalities, distortion theorem, convex linear combination, partial sums, convolution, and neighborhood property for the Miller -Ross-type Poisson distribution series to be in classes. First, we recall the definition of the Miller-Ross-type distribution.

The probability mass function of the Miller-Ross-type Poisson distribution is given by

$$P_{\nu,c}(\zeta,\iota)=\frac{\zeta^{\nu}(c\zeta)^{\iota}}{E_{\nu,c}(\zeta)\Gamma(\iota+\nu+1)},\quad \iota=0,1,2,\cdots, \tag{1.9}$$

where $\nu>-1, c>0$ and $E_{\nu,\,c}(\hbar)$ is Miller-Ross function given in (1.7).
The Miller-Ross-type Poisson distribution series is defined by

$$\mathbb{F}^{\zeta}_{\nu,c}(\hbar)=\hbar+\sum_{\iota=2}^{\infty}\frac{\zeta^{\nu}(c\zeta)^{\iota-1}}{\Gamma(\iota+\nu)E_{\nu,c}(\zeta)}\hbar^{\iota},\quad \hbar\in\mathbb{U}. \tag{1.10}$$

(see [18], see also [22]). Furthermore, using the convolution (or Hadamard product), we define

$$\begin{aligned}\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)&=\mathbb{F}^{\zeta}_{\nu,c}(\hbar)*\aleph(\hbar)\\&=\hbar+\sum_{\iota=2}^{\infty}\frac{\zeta^{\nu}(c\zeta)^{\iota-1}}{\Gamma(\iota+\nu)E_{\nu,c}(\zeta)}\eta_{\iota}\hbar^{\iota}\\&=\hbar+\sum_{\iota=2}^{\infty}\Phi^{\nu}_{c}(\iota,\zeta)\eta_{\iota}\hbar^{\iota},\end{aligned} \tag{1.11}$$

where

$$\Phi^{\nu}_{c}(\iota,\zeta)=\frac{\zeta^{\nu}(c\zeta)^{\iota-1}}{\Gamma(\iota+\nu)E_{\nu,c}(\zeta)}. \tag{1.12}$$

Inspired by the work of [13,17,21], we introduce the new subclass involving Miller -Ross -type Poisson distribution series $\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)$, as below:

Definition 1.1. *For $0\le\wp<1, 0\le\ell<1$, we say $\aleph(\hbar)\in A$ is in the class $\varphi^{\zeta}_{\nu,c}(\wp,\ell)$ if it satisfies the condition*

$$\Re\left(\frac{\hbar\left(\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)\right)'+\wp\hbar^{2}\left(\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)\right)''}{\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)}\right)>\ell,\ (\hbar\in\mathbb{U}). \tag{1.13}$$

Also we denote by $T\varphi^{\zeta}_{\nu,c}(\wp,\ell)=\varphi^{\zeta}_{\nu,c}(\wp,\ell)\cap T$.

2. Coefficient Inequalities

In this section, we obtain a sufficient condition for a function $\aleph$ given by (1.1) to be in $\varphi^{\zeta}_{\nu,c}(\wp,\ell)$.

Theorem 2.1. *A function $\aleph\in A$ belongs to the class $\varphi^{\zeta}_{\nu,c}(\wp,\ell)$ if*

$$\sum_{\iota=2}^{\infty}[\iota+\wp\iota(\iota-1)-\ell]\Phi^{\nu}_{c}(\iota,\zeta)|\eta_{\iota}|\le 1-\ell. \tag{2.1}$$

Proof. Since $0\le\ell<1$ and $\wp\ge 0$, now if we put

$$\varrho(\hbar)=\frac{\hbar\left(\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)\right)'+\wp\hbar^{2}\left(\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)\right)''}{\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)},\ (\hbar\in\mathbb{U})$$

then it is sufficient to prove that $|\varrho(\hbar)-1|<1-\ell, (\hbar\in\mathbb{U})$.
Indeed if $\aleph(\hbar)=\hbar,\ (\hbar\in\mathbb{U})$, then we have $\varrho(\hbar)=\hbar,\ (\hbar\in\mathbb{U})$.
This implies the desired equality (2.1) holds.

If $\aleph(\hbar) \neq \hbar$ $(|\hbar| = r < 1)$, then there exist a coefficient $\Phi_c^{\nu}(\iota, \zeta)\eta_\iota \neq 0$, for some $\iota \geq 2$. It follows that $\sum\limits_{\iota=2}^{\infty} \Phi_c^{\nu}(\iota, \zeta)|\eta_\iota| > 0$. Further, note that

$$\sum_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - \ell]\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota| > (1-\ell)\sum_{\iota=2}^{\infty}\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota|$$

$$\Rightarrow \sum_{\iota=2}^{\infty}\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota| < 1.$$

By (2.1), we obtain

$$\begin{aligned}
|\varrho(\hbar) - 1| &= \left| \frac{\sum\limits_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - 1]\Phi_c^{\nu}(\iota, \zeta)\eta_\iota\hbar^{\iota-1}}{1 + \sum\limits_{\iota=2}^{\infty}\Phi_c^{\nu}(\iota, \zeta)\eta_\iota\hbar^{\iota-1}} \right| \\
&< \frac{\sum\limits_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - 1]\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota|}{1 - \sum\limits_{\iota=2}^{\infty}\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota|} \\
&\leq \frac{\sum\limits_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - \ell]\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota| - (1-\ell)\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota|}{1 - \sum\limits_{\iota=2}^{\infty}\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota|} \\
&\leq \frac{(1-\ell) - (1-\ell)\sum\limits_{\iota=2}^{\infty}\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota|}{1 - \sum\limits_{\iota=2}^{\infty}\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota|} \\
&= 1 - \ell, \ (\hbar \in \mathbb{U}).
\end{aligned}$$

Hence we obtain

$$\Re\left(\frac{\hbar\left(\mathbb{K}_{\nu,c}^{\zeta}\aleph(\hbar)\right)' + \wp\hbar^2\left(\mathbb{K}_{\nu,c}^{\zeta}\aleph(\hbar)\right)''}{\mathbb{K}_{\nu,c}^{\zeta}\aleph(\hbar)}\right) = \Re(\varrho(\hbar)) > 1 - (1-\ell) = \ell, (\hbar \in \mathbb{U}).$$

Then $\aleph \in \varphi_{\nu,c}^{\zeta}(\wp, \ell)$. This completes the proof. □

In the next theorem, we prove that the condition (2.1) is also necessary for a function $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp, \ell)$.

Theorem 2.2. *Let $\aleph$ be given by (1.6). Then the function $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp, \ell)$ if and only if*

$$\sum_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - \ell]\Phi_c^{\nu}(\iota, \zeta)|\eta_\iota| \leq 1 - \ell. \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove that $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp, \ell)$ satisfies the coefficient inequality (2.1). If $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp, \ell)$ then the function

$$\varrho(\hbar) = \frac{\hbar\left(\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)\right)' + \wp\hbar^2\left(\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)\right)''}{\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar)}, \ (\hbar \in \mathbb{U})$$

satisfies $\Re(\varrho(\hbar)) > \ell, \ (\hbar \in \mathbb{U})$. This implies that

$$\mathbb{K}^{\zeta}_{\nu,c}\aleph(\hbar) = \hbar - \sum_{\iota=2}^{\infty}\Phi^{\nu}_{c}(\iota,\zeta)|\eta_\iota|\hbar^\iota \neq 0, \ (\hbar \in \mathbb{U}\setminus\{0\}).$$

Noting that $\frac{\mathbb{K}^{\zeta}_{\nu,c}\aleph(r)}{r}$ is the real continuous function in the open interval $(0,1)$ with $\aleph(0) = 1$, we have

$$\frac{\mathbb{K}^{\zeta}_{\nu,c}\aleph(r)}{r} = 1 - \sum_{\iota=2}^{\infty}\Phi^{\nu}_{c}(\iota,\zeta)|\eta_\iota|r^{\iota-1} > 0, (0 < r < 1). \tag{2.3}$$

Now

$$\ell < \varrho(r) = \frac{1 - \sum\limits_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1)]\Phi^{\nu}_{c}(\iota,\zeta)|\eta_\iota|r^{\iota-1}}{1 - \sum\limits_{\iota=2}^{\infty}\Phi^{\nu}_{c}(\iota,\zeta)|\eta_\iota|r^{\iota-1}}$$

and consequently by (2.3), we get

$$\sum_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - \ell]\Phi^{\nu}_{c}(\iota,\zeta)|\eta_\iota|r^{\iota-1} \leq 1 - \ell.$$

Letting $r \to 1$, we get

$$\sum_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - \ell]\Phi^{\nu}_{c}(\iota,\zeta)|\eta_\iota| \leq 1 - \ell.$$

This proves the converse part. □

Remark 2.1. *If a function $\aleph$ of the form (1.6) belongs to the class $T\varphi^{\zeta}_{\nu,c}(\wp,\ell)$ then*

$$|\eta_\iota| \leq \frac{1-\ell}{[\iota + \wp\iota(\iota-1) - \ell]\Phi^{\nu}_{c}(\iota,\zeta)}, \iota \geq 2.$$

The equality holds for the functions

$$\aleph_\iota(\hbar) = \hbar - \frac{1-\ell}{[\iota + \wp\iota(\iota-1) - \ell]\Phi^{\nu}_{c}(\iota,\zeta)}\hbar^\iota, (\hbar \in \mathbb{U}, \iota \geq 2). \tag{2.4}$$

3. Distortion Theorem

In the section, distortion bounds for functions belonging to the class $T\varphi^{\zeta}_{\nu,c}(\wp,\ell)$.

Theorem 3.1. *Let $\aleph$ be in the class $T\varphi^{\zeta}_{\nu,c}(\wp,\ell)$ and $|\hbar| = r < 1$. Then*

$$r - \frac{1-\ell}{[2\wp - \ell + 2]\Phi^{\nu}_{c}(2,\zeta)}r^2 \leq |\aleph(\hbar)| \leq r + \frac{1-\ell}{[2\wp - \ell + 2]\Phi^{\nu}_{c}(2,\zeta)}r^2 \tag{3.1}$$

and

$$1 - \frac{2(1-\ell)}{[2\wp - \ell + 2]\Phi^{\nu}_{c}(2,\zeta)}r \leq |\aleph'(\hbar)| \leq 1 + \frac{2(1-\ell)}{[2\wp - \ell + 2]\Phi^{\nu}_{c}(2,\zeta)}r. \tag{3.2}$$

The result is sharp with the extremal function $\aleph_2(\hbar)$ is given by (2.4).

Proof. Since $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$, we apply Theorem 2.2 to obtain

$$[2\wp-\ell+2]\Phi_c^{\nu}(2,\zeta)\sum_{\iota=2}^{\infty}|\eta_\iota| \le \sum_{\iota=2}^{\infty}[\iota+\wp\iota(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)|\eta_\iota| \le 1-\ell.$$

$$\text{Thus } |\aleph(\hbar)| \le |\hbar| + |\hbar|^2\sum_{\iota=2}^{\infty}|\eta_\iota| \le r + \frac{1-\ell}{[2\wp-\ell+2]\Phi_c^{\nu}(2,\zeta)}\, r^2.$$

$$\text{Also we have, } |\aleph(\hbar)| \le |\hbar| - |\hbar|^2\sum_{\iota=2}^{\infty}|\eta_\iota| \le r - \frac{1-\ell}{[2\wp-\ell+2]\Phi_c^{\nu}(2,\zeta)}\, r^2,$$

and (3.1) follows. In similar manner for $\aleph'$, the inequalities

$$|\aleph'(\hbar)| \le 1 + \sum_{\iota=2}^{\infty}\iota|\eta_\iota||\hbar|^{\iota-1} \le 1 + |\hbar|\sum_{\iota=2}^{\infty}\iota|\eta_\iota|$$

and

$$\sum_{\iota=2}^{\infty}\iota|\eta_\iota| \le \frac{2(1-\ell)}{[2\wp-\ell+2]\Phi_c^{\nu}(2,\zeta)}$$

are satisfied, which leads to (3.2). This completes the proof. □

4. Radii of Close-to-Convexity and Starlikeness

In this section, the radii of close-to-convex and starlikeness of this class $T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$ will be obtained.

Theorem 4.1. *Let $\aleph$ be given by* (1.6) *is in $T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$. Then $\aleph$ is close-to-convex of order δ $(0 \le \delta < 1)$ in the disc $|\hbar| < t_1$, where*

$$t_1 = \inf_{\iota\ge 2}\left[\frac{(1-\delta)[\iota+\iota\wp(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)}{\iota(1-\ell)}\right]^{\frac{1}{\iota-1}}. \tag{4.1}$$

The result is sharp with the extremal function $\aleph(\hbar)$ is given by (2.4).

Proof. If $\aleph \in T$ and $\aleph$ is close-to-convex of order δ then we get

$$|\aleph'(\hbar)-1| \le 1-\delta. \tag{4.2}$$

For the left hand side of (4.2), we obtain

$$|\aleph'(\hbar)-1| \le \sum_{\iota=2}^{\infty}\iota\eta_\iota|\hbar|^{\iota-1} < 1-\delta$$

$$\Rightarrow \sum_{\iota=2}^{\infty}\frac{\iota}{1-\delta}\eta_\iota|\hbar|^{\iota-1} \le 1.$$

We know that $\aleph(\hbar) \in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$ if and only if

$$\sum_{\iota=2}^{\infty}\frac{[\iota+\iota\wp(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)}{(1-\ell)}\eta_\iota \le 1.$$

Thus (4.2) holds true if

$$\frac{\iota}{1-\delta}|\hbar|^{\iota-1} \leq \frac{[\iota+\iota\wp(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)}{(1-\ell)}$$

or equivalently

$$|\hbar| \leq \left[\frac{(1-\delta)[\iota+\iota\wp(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)}{\iota(1-\ell)}\right]^{\frac{1}{\iota-1}}$$

and hence, the proof is complete. □

Theorem 4.2. *Let* $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$. *Then* $\aleph$ *is starlike of order* δ, $(0 \leq \delta < 1)$ *in the disc* $|\hbar| < t_2$, *where*

$$t_2 = \inf_{\iota\geq 2}\left[\frac{(1-\delta)[\iota+\iota\wp(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)}{(\iota-\delta)(1-\ell)}\right]^{\frac{1}{\iota-1}}. \tag{4.3}$$

The result is sharp with the extremal function $\aleph(\hbar)$ *is given by* (2.4).

Proof. We have $\aleph \in T$ and $\aleph$ is starlike of order δ, we have

$$\left|\frac{\hbar\aleph'(\hbar)}{\aleph(\hbar)}-1\right| < 1-\delta. \tag{4.4}$$

For the left hand side of (4.4), we have

$$\left|\frac{\hbar\aleph'(\hbar)}{\aleph(\hbar)}-1\right| \leq \frac{\sum\limits_{\iota=2}^{\infty}(\iota-1)\eta_\iota|\hbar|^{\iota-1}}{1-\sum\limits_{\iota=2}^{\infty}\eta_\iota|\hbar|^{\iota-1}}$$

$(1-\delta)$ is bigger than the right-hand side of the left relation if

$$\sum_{\iota=2}^{\infty}\frac{\iota-\delta}{1-\delta}\eta_\iota|\hbar|^{\iota-1} < 1.$$

We know that $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$ if and only if

$$\sum_{\iota=2}^{\infty}\frac{[\iota+\iota\wp(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)}{(1-\ell)}\eta_\iota \leq 1.$$

Thus (4.4) is true if

$$\frac{\iota-\delta}{1-\delta}|\hbar|^{\iota-1} \leq \frac{[\iota+\iota\wp(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)}{(1-\ell)}$$

or equivalently

$$|\hbar| \leq \left[\frac{(1-\delta)[\iota+\iota\wp(\iota-1)-\ell]\Phi_c^{\nu}(\iota,\zeta)}{(\iota-\delta)(1-\ell)}\right]^{\frac{1}{\iota-1}}.$$

It yields the star likeness of the family. □

5. Convex Linear Combinations

Theorem 5.1. *Let* $\aleph_1(\hbar) = \hbar$ *and*

$$\aleph_\iota(\hbar) = \hbar - \frac{1-\ell}{[\iota + \wp\iota(\iota-1) - \ell]\Phi_c^{\nu}(\iota,\zeta)}\hbar^{\iota}, \quad (\hbar \in \mathbb{U}, \iota \geq 2). \tag{5.1}$$

Then $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$ *if and only if* $\aleph$ *can be expressed in the form*

$$\aleph(\hbar) = \sum_{\iota=1}^{\infty} \mu_\iota \aleph_\iota(\hbar), \mu_\iota \geq 0 \tag{5.2}$$

and $\sum\limits_{\iota=1}^{\infty} \mu_\iota = 1$.

Proof. If a function $\aleph$ is of the form $\aleph(\hbar) = \sum\limits_{\iota=1}^{\infty} \mu_\iota \aleph_\iota(\hbar), \mu_\iota \geq 0$ and $\sum\limits_{\iota=1}^{\infty} \mu_\iota = 1$ then

$$\begin{aligned}
&\sum_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - \ell]\Phi_c^{\nu}(\iota,\zeta)|\eta_\iota| \\
=&\sum_{\iota=2}^{\infty}[\iota + \wp\iota(\iota-1) - \ell]\Phi_c^{\nu}(\iota,\zeta)\frac{(1-\ell)\mu_\iota}{[\iota + \wp\iota(\iota-1) - \ell]\Phi_c^{\nu}(\iota,\zeta)} \\
=&\sum_{\iota=2}^{\infty}(1-\ell)\mu_\iota = (1-\mu_1)(1-\ell) \\
\leq&(1-\ell)
\end{aligned}$$

which provides (2.2), hence $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$, by Theorem 2.2.

Conversely, if $\aleph$ is in the class $\aleph \in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$, then we may set

$$\mu_\iota = \frac{[\iota + \wp\iota(\iota-1) - \ell]\Phi_c^{\nu}(\iota,\zeta)}{1-\ell}|\eta_\iota|, \iota \geq 2,$$

and $\mu_1 = 1 - \sum\limits_{\iota=2}^{\infty} \mu_\iota$.

Then the function $\aleph$ is of the form (5.2), and this completes the proof. □

6. Partial Sums

For various interesting developments concerning partial sums of analytic univalent functions, the reader may be referred to the works of Brickman et al. [3], Caglar and Orhan [4], Lin and Owa [18], Deniz and Orhan [6,7], Kazimoglu et al. [16], Shiel-Small [32]. Recently, some researchers have studied on partial sums of special functions (see [5,16,28,44]). For a function $\aleph \in A$ given by (1.1), Silverman [35] investigated the partial sums $\aleph$ defined by

$$\aleph_1(\hbar) = \hbar \text{ and } \aleph_j(\hbar) = \hbar + \sum_{\iota=2}^{j} \eta_\iota \hbar^{\iota}. \tag{6.1}$$

In [35], Silverman examined sharp lower bounds on the real part of the quotients between the normalized convex or starlike functions and their sequences of partial sums. Also, Srivastava et al. [37], Silvia [36] and Owa et al. [24] have investigated interesting results on the partial sums. In this section, we consider partial sums of functions in the class $\varphi^{\zeta}_{\nu,c}(\wp,\ell)$ and obtain sharp lower bounds for the ratios of real part of $\aleph$ to $\aleph_j$ and $\aleph'$ to $\aleph'_j$.

Theorem 6.1. *Let be a function $\aleph$ of the form (1.1) belong to the class $\varphi^{\zeta}_{\nu,c}(\wp,\ell)$ and satisfy (2.1). Then*

$$\Re\left(\frac{\aleph(\hbar)}{\aleph_j(\hbar)}\right) \geq \frac{d_{j+1}-1+\ell}{d_{j+1}}, (\hbar \in \mathbb{U}), \tag{6.2}$$

where

$$d_\iota \geq \begin{cases} 1-\ell, & \text{if } \iota = 2,3,\cdots,j; \\ d_{j+1}, & \text{if } \iota = j+1, j+2, j+3,\cdots. \end{cases} \tag{6.3}$$

The result (6.2) *is sharp with the function given by*

$$\aleph(\hbar) = \hbar + \frac{1-\ell}{d_{j+1}}\hbar^{j+1}. \tag{6.4}$$

Proof. Define the function $\wp(\hbar)$ by

$$\begin{aligned}\frac{1+\wp(\hbar)}{1-\wp(\hbar)} &= \frac{d_{j+1}}{1-\ell}\left\{\frac{\aleph(\hbar)}{\aleph_j(\hbar)} - \left(\frac{d_{j+1}-1+\ell}{d_{j+1}}\right)\right\} \\ &= \left[\frac{1+\sum\limits_{\iota=2}^{j}\eta_\iota\hbar^{\iota-1} + \frac{d_{j+1}}{1-\ell}\sum\limits_{\iota=j+1}^{\infty}\eta_\iota\hbar^{\iota-1}}{1+\sum\limits_{\iota=2}^{j}\eta_\iota\hbar^{\iota-1}}\right].\end{aligned} \tag{6.5}$$

It suffices to show $|\wp(\hbar)| \leq 1$. Now, from (6.5) we can write

$$\wp(\hbar) = \frac{\frac{d_{j+1}}{1-\ell}\sum\limits_{\iota=j+1}^{\infty}\eta_\iota\hbar^{\iota-1}}{2+2\sum\limits_{\iota=2}^{j}\eta_\iota\hbar^{\iota-1} + \frac{d_{j+1}}{1-\ell}\sum\limits_{\iota=j+1}^{\infty}\eta_\iota\hbar^{\iota-1}}$$

$$\Rightarrow |\wp(\hbar)| \leq \frac{\frac{d_{j+1}}{1-\ell}\sum\limits_{\iota=j+1}^{\infty}|\eta_\iota|}{2-2\sum\limits_{\iota=2}^{j}|\eta_\iota| - \frac{d_{j+1}}{1-\ell}\sum\limits_{\iota=j+1}^{\infty}|\eta_\iota|}.$$

Now $|\wp(\hbar)| \leq 1$ if and only if

$$2\frac{d_{j+1}}{1-\ell}\sum_{\iota=j+1}^{\infty}|\eta_\iota| \leq 2-2\sum_{\iota=2}^{j}|\eta_\iota|$$

$$\Rightarrow \sum_{\iota=2}^{j}|\eta_\iota| + \sum_{\iota=j+1}^{\infty}\frac{d_{j+1}}{1-\ell}|\eta_\iota| \leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{\iota=2}^{j}|\eta_\iota| + \sum_{\iota=j+1}^{\infty}\frac{d_{j+1}}{1-\ell}|\eta_\iota| \leq \sum_{\iota=2}^{\infty}\frac{d_\iota}{1-\ell}|\eta_\iota|$$

which is equivalent to

$$\sum_{\iota=2}^{j}\left(\frac{d_j-1+\ell}{1-\ell}\right)|\eta_\iota|-\sum_{\iota=j+1}^{\infty}\frac{d_{j+1}}{1-\ell}|\eta_\iota|\geq 0. \tag{6.6}$$

To see that the function given by (6.4) gives the sharp result, we observe that for $\hbar=re^{\frac{i\pi}{n}}$,

$$\begin{aligned}\frac{\aleph(\hbar)}{\aleph_j(\hbar)}&=1+\frac{1-\ell}{d_{j+1}}\hbar^j\to 1-\frac{1-\ell}{d_{j+1}}\\&=\frac{d_{j+1}-1+\ell}{d_{j+1}},\ \text{when } r\to 1^-.\end{aligned}$$

□

Theorem 6.2. *Let be a function $\aleph$ of the form (1.1) belong to the class $\varphi^{\zeta}_{\nu,c}(\wp,\ell)$ and satisfy (2.1). Then*

$$\Re\left(\frac{\aleph_j(\hbar)}{\aleph(\hbar)}\right)\geq\frac{d_{j+1}}{d_{j+1}+1-\ell},\ (\hbar\in\mathbb{U}), \tag{6.7}$$

where $d_{j+1}\geq 1-\ell$ and

$$d_\iota\geq\begin{cases}1-\ell, & \text{if } \iota=2,3,\cdots,j;\\ d_{j+1}, & \text{if } \iota=j+1,j+2,j+3,\cdots.\end{cases} \tag{6.8}$$

The result (6.7) *is sharp with the function given by* (6.4).

Proof. We write by

$$\begin{aligned}\frac{1+\wp(\hbar)}{1-\wp(\hbar)}&=\frac{d_{j+1}+1-\ell}{1-\ell}\left\{\frac{\aleph_j(\hbar)}{\aleph(\hbar)}-\left(\frac{d_{j+1}}{d_{j+1}+1-\ell}\right)\right\}\\&=\left[\frac{1+\sum\limits_{\iota=2}^{j}\eta_\iota\hbar^{\iota-1}-\frac{d_{j+1}}{1-\ell}\sum\limits_{\iota=j+1}^{\infty}\eta_\iota\hbar^{\iota-1}}{1+\sum\limits_{\iota=2}^{\infty}\eta_\iota\hbar^{\iota-1}}\right],\end{aligned} \tag{6.9}$$

where

$$|\wp(\hbar)|\leq\frac{\frac{d_{j+1}+1-\ell}{1-\ell}\sum\limits_{\iota=j+1}^{\infty}|\eta_\iota|}{2-2\sum\limits_{\iota=2}^{j}|\eta_\iota|-\frac{d_{j+1}+1-\ell}{1-\ell}\sum\limits_{\iota=j+1}^{\infty}|\eta_\iota|}\leq 1.$$

This last inequality is equivalent to

$$\sum_{\iota=2}^{j}|\eta_\iota|+\sum_{\iota=j+1}^{\infty}\frac{d_{j+1}}{1-\ell}|\eta_\iota|\leq 1.$$

We are making use of (2.1) to get (6.6). Finally, equality holds in (6.7) for the extremal function $\aleph(\hbar)$ given by (6.4).

□

We next turn to ratios involving derivatives.

Theorem 6.3. *Let be a function $\aleph$ of the form (1.1) belong to the class $\varphi^{\zeta}_{v,c}(\wp,\ell)$ and satisfy (2.1). Then*

$$\Re\left(\frac{\aleph'(\hbar)}{\aleph'_j(\hbar)}\right) \geq \frac{d_{j+1}-(j+1)(1-\ell)}{d_{j+1}}, (\hbar \in \mathbb{U}),$$

$$\Re\left(\frac{\aleph'_j(\hbar)}{\aleph'(\hbar)}\right) \geq \frac{d_{j+1}}{d_{j+1}-(j+1)(1-\ell)}, (\hbar \in \mathbb{U}), \tag{6.10}$$

where $d_{j+1} \geq (j+1)(1-\ell)$ and

$$d_\iota \geq \begin{cases} \iota(1-\ell), & \text{if } \iota = 2,3,\cdots,j; \\ \iota\frac{d_{j+1}}{j+1}, & \text{if } \iota = j+1, j+2, j+3, \cdots. \end{cases} \tag{6.11}$$

The result is sharp with the function given by (6.4).

Proof. We write by

$$\frac{1+\wp(\hbar)}{1-\wp(\hbar)} = \frac{d_{j+1}}{(j+1)(1-\ell)}\left\{\frac{\aleph'(\hbar)}{\aleph'_j(\hbar)} - \left(\frac{d_{j+1}-(j+1)(1-\ell)}{d_{j+1}}\right)\right\},$$

where

$$\wp(\hbar) = \frac{\frac{d_{j+1}}{(j+1)(1-\ell)}\sum_{\iota=j+1}^{\infty}\iota\eta_\iota\hbar^{\iota-1}}{2+2\sum_{\iota=2}^{j}\iota\eta_\iota\hbar^{\iota-1}+\frac{d_{j+1}}{(j+1)(1-\ell)}\sum_{\iota=j+1}^{\infty}\iota\eta_\iota\hbar^{\iota-1}}.$$

Now $|\wp(\hbar)| \leq 1$ if and only if

$$\sum_{\iota=2}^{j}\iota|\eta_\iota| + \frac{d_{j+1}}{(j+1)(1-\ell)}\sum_{\iota=j+1}^{\infty}\iota|\eta_\iota| \leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{\iota=2}^{j}\iota|\eta_\iota| + \frac{d_{j+1}}{(j+1)(1-\ell)}\sum_{\iota=j+1}^{\infty}\iota|\eta_\iota| \leq \sum_{\iota=2}^{\infty}\frac{d_\iota}{1-\ell}|\eta_\iota|$$

which is equivalent to

$$\sum_{\iota=2}^{j}\left(\frac{d_\iota-(1-\ell)\iota}{1-\ell}\right)|\eta_\iota| + \sum_{\iota=j+1}^{\infty}\frac{(j+1)d_\iota-\iota d_{j+1}}{(j+1)(1-\ell)}|\eta_\iota| \geq 0.$$

To prove the result (6.10), define the function $\wp(\hbar)$

$$\frac{1+\wp(\hbar)}{1-\wp(\hbar)} = \frac{(j+1)(1-\ell)+d_{j+1}}{(j+1)(1-\ell)}\left\{\frac{\aleph'_j(\hbar)}{\aleph'(\hbar)} - \left(\frac{d_{j+1}}{d_{j+1}+(j+1)(1-\ell)}\right)\right\}$$

where

$$\wp(\hbar)=\frac{-\left(\frac{d_{j+1}+1}{(j+1)(1-\ell)}\right)\sum\limits_{\iota=j+1}^{\infty}\iota\eta_\iota\hbar^{\iota-1}}{2+2\sum\limits_{\iota=2}^{j}\iota\eta_\iota\hbar^{\iota-1}+\frac{1-d_{j+1}}{(j+1)(1-\ell)}\sum\limits_{\iota=j+1}^{\infty}\iota\eta_\iota\hbar^{\iota-1}}.$$

Now $|\wp(\hbar)|\leq 1$ if and only if

$$\sum_{\iota=2}^{j}\iota|\eta_\iota|+\sum_{\iota=j+1}^{\infty}\frac{d_{j+1}}{(j+1)(1-\ell)}\iota|\eta_\iota|\leq 1. \tag{6.12}$$

It suffices to show that the left hand side of (6.12) is bounded previously by the condition

$$\sum_{\iota=2}^{\infty}\frac{d_\iota}{1-\ell}|\eta_\iota|,$$

which is equivalent to

$$\sum_{\iota=2}^{\infty}\left(\frac{d_\iota}{1-\ell}-\iota\right)|\eta_\iota|+\sum_{\iota=j+1}^{\infty}\left(\frac{d_\iota}{1-\ell}-\frac{d_{j+1}}{(j+1)(1-\ell)}\right)\iota|\eta_\iota|\geq 0$$

□

7. Convolution Properties

In this section, we will prove that the class $T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$ is closed under convolution.

Theorem 7.1. *Let $g(\hbar)$ of the form*

$$g(\hbar)=\hbar-\sum_{\iota=2}^{\infty}b_\iota\hbar^\iota$$

be analytic in U. If $\aleph\in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$ then the function $\aleph*g$ is in the class $T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$. Here the symbol $*$ denoted to the Hadmard product (or convolution).

Proof. Since $\aleph\in T\varphi_{\nu,c}^{\zeta}(\wp,\ell)$, we have

$$\sum_{\iota=2}^{\infty}[\iota+\wp\iota(\iota-1)-\ell]\Phi_c^\nu(\iota,\zeta)|\eta_\iota|\leq 1-\ell.$$

By utilizing the last inequality and the fact that

$$\aleph(\hbar)*g(\hbar)=\hbar-\sum_{\iota=2}^{\infty}\eta_\iota b_\iota\hbar^\iota.$$

We obtain

$$\begin{aligned}&\sum_{\iota=2}^{\infty}[\iota+\wp\iota(\iota-1)-\ell]\Phi_c^\nu(\iota,\zeta)|\eta_\iota||b_\iota|\\ \leq&\sum_{\iota=2}^{\infty}[\iota+\wp\iota(\iota-1)-\ell]\Phi_c^\nu(\iota,\zeta)|\eta_\iota|\\ \leq&1-\ell\end{aligned}$$

and hence, in view of Theorem 2.1, the result follows. □

8. Neighborhood Property

Following [14,29], we defined the $\alpha-$neighbourhood of the function $\aleph(\hbar) \in T$ by

$$N_\alpha(\aleph) = \left\{ g \in T : g(\hbar) = \hbar - \sum_{\iota=2}^{\infty} b_\iota \hbar^\iota \text{ and } \sum_{\iota=2}^{\infty} \iota|\eta_\iota - b_\iota| \leq \alpha \right\}, \ \alpha \geq 0. \tag{8.1}$$

Definition 8.1. *A function $\aleph \in A$ is said to in the class $T\varphi_{\nu,c}^{\zeta}(\wp, \ell)$ if there exists a function $h \in T\varphi_{\nu,c}^{\zeta}(\wp, \ell)$ such that*

$$\left| \frac{\aleph(\hbar)}{h(\hbar)} - 1 \right| < 1 - \gamma, \quad (\hbar \in \mathbb{U}, \ 0 \leq \gamma < 1). \tag{8.2}$$

Theorem 8.1. *If $h \in T\varphi_{\nu,c}^{\zeta}(\wp, \ell)$ and*

$$\gamma = 1 - \frac{\alpha(2\wp - \ell + 2)\Phi_c^\nu(2, \zeta)}{2(2\wp - \ell + 2)\Phi_c^\nu(2, \zeta) - (1 + \ell)}$$

then $N_\alpha(h) \subseteq T\phi_c^{\delta,\gamma}(\wp, \ell)$.

Proof. Let $\aleph \in N_\alpha(h)$. We then find from that

$$\sum_{\iota=2}^{\infty} \iota|\eta_\iota - b_\iota| \leq \alpha,$$

which is easily implies the coefficient inequality

$$\sum_{\iota=2}^{\infty} |\eta_\iota - b_\iota| \leq \frac{\alpha}{\iota}, \quad (n \in N).$$

Since $h \in T\varphi_{\nu,c}^{\zeta}(\wp, \ell)$, we have from equation (2.1) that

$$\sum_{\iota=2}^{\infty} |\eta_\iota| \leq \frac{1 - \ell}{(2\wp - \ell + 2)\Phi_c^\nu(2, \zeta)}$$

and

$$\begin{aligned} \left| \frac{\aleph(\hbar)}{h(\hbar)} - 1 \right| &< \frac{\sum_{\iota=2}^{\infty} \iota|\eta_\iota - b_\iota|}{1 - \sum_{\iota=2}^{\infty} b_\iota} \\ &\leq \frac{\alpha}{2} \, \frac{(2\wp - \ell + 2)\Phi_c^\nu(2, \zeta)}{(2\wp - \ell + 2)\Phi_c^\nu(2, \zeta) - (1 + \ell)} \\ &= 1 - \gamma. \end{aligned}$$

This completes the proof of the theorem. □

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