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On Intuitionistic Fuzzy n-Controlled Metric Spaces with Application in Economics

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Abstract. In this paper, we introduce the concept of an intuitionistic fuzzy n-controlled metric space (IFnCMS) by using n non-comparable functions $\alpha_i : \Sigma \times \Sigma \to [1, \infty)$ $(1 \leq i \leq n)$ in the inequalities having the form $F(\varsigma_1^o, \varsigma_{n+1}^o, \iota_1^o + \iota_2^o + \cdots + \iota_n^o) \geq F\left(\varsigma_1^o, \varsigma_2^o, \frac{\iota_1^o}{\alpha_1(\varsigma_1^o, \varsigma_2^o)}\right) * F\left(\varsigma_2^o, \varsigma_3^o, \frac{\iota_2^o}{\alpha_2(\varsigma_2^o, \varsigma_3^o)}\right) * \cdots * F\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota_n^o}{\alpha_n(\varsigma_n^o, \varsigma_{n+1}^o)}\right) \forall\ \iota_n^o > 0$ and $N(\varsigma_1^o, \varsigma_{n+1}^o, \iota_1^o + \iota_2^o + \cdots + \iota_n^o) \leq N\left(\varsigma_1^o, \varsigma_2^o, \frac{\iota_1^o}{\alpha_1(\varsigma_1^o, \varsigma_2^o)}\right) \circ N\left(\varsigma_2^o, \varsigma_3^o, \frac{\iota_2^o}{\alpha_2(\varsigma_2^o, \varsigma_3^o)}\right) \circ \cdots \circ N\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota_n^o}{\alpha_n(\varsigma_n^o, \varsigma_{n+1}^o)}\right) \forall\ \iota_n^o > 0$. Further, we provide some non-trivial examples and prove several fixed point results by utilizing an intuitionistic fuzzy version of Banach contraction and generalized $\alpha - \phi-$intuitionistic fuzzy contractive mapping in the setting of IFnCMS. Furthermore, we present some of its consequences to illustrate the significance of our results. Ultimately, we apply fractional differential equations used in economics to support the main result.

1. Introduction

Intuitionistic fuzzy metric spaces (IFMSs) enhance classical metrics by integrating membership and non-membership degrees, resulting in a more complete representation of uncertainty. This framework has a wide range of applications in several fields. It improves data classification and feature selection in pattern recognition and machine learning by handling uncertainty more effectively. It improves risk assessment and decision-making models by accounting for complicated preferences and uncertainties in decision-making and multi-criteria analysis. It supports control systems and robots in the design of controllers and the navigation of unpredictable environments. It improves classification and segmentation accuracy in image processing by dealing with imperfect

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visual information. In healthcare, it aids in disease diagnosis and patient monitoring in the face of unknowns. It also improves forecasting and portfolio management in economic and financial modelling by addressing missing data. Finally, it assists linguistic and cognitive modelling by imitating human-like reasoning and allowing for more flexible representation of language factors.

Schweizer and Sklar [1] presented the notion of triangular norms. Zadeh [2] presented the notion of fuzzy sets (FSs). Several authors in [3]- [19] worked on fuzzy metric spaces (FMSs), its topological properties and fixed point results. Nădăban [20] introduced the concept of fuzzy b-metric spaces (fuzzy b-MSs) and discussed its some properties. Mehmood et al. [21] introduced the concept of extended fuzzy b-MSs. They used a function in the triangular inequality of fuzzy b-MSs instead of $b \geq 1$. Sezen [22] introduced the concept of controlled FMS by modifying the triangular inequality of fuzzy b-MSs. After that, Saleem et al. [23, 24] generalized the concept of controlled FMSs by introducing fuzzy double controlled and fuzzy triple controlled MSs. Zubair et al. [25] introduced the concept of fuzzy extended hexagonal b-MSs and proved several fixed point results. Hussain et al. [26] introduced the concept of pentagonal controlled FMSs as a generalization of fuzzy double controlled, fuzzy triple controlled and fuzzy extended hexagonal b-MSs. Recently, Furqan et al. [27] introduced the concept of fuzzy n-controlled metric spaces (FnCMSs) as a generalization of fuzzy pentagonal controlled MSs. Further, they proved several fixed point results for contraction mappings in the setting of FnCMSs.

In 2004, Park [28] presented the concept of IFMS as a generalization of FMSs. In FMSs only membership function have used, however, Park used membership and non-memebrship functions to introduce IFMSs. Further, he discussed several topological properties of IFMSs. After that, Alca et al. [29] extend the fuzzy version of Banach fixed point theorem in the setting of IFMSs and proved fixed point theorems. Konwar [30] introduced the concept of an intuitionistic fuzzy b-MSs (IFbMSs) as a generalization of IFMSs and proved several fixed point results. Kattan et al. [31] presented the concept of intuitionistic fuzzy rectangular extended b-MSs and extend Banach contraction principle in this setting. In 2022, Farheen et al. [32] presented the concept of intuitionistic fuzzy double controlled MSs (IFDCMSs) and proved several fixed point results in this new setting.

The main objectives of this study are as follows:

- To present the concept of IFnCMSs.
- To extend the Banach fixed point theorem in the context of IFnCMS.
- To present fixed point results via intuitionistic fuzzy Banach contraction principle and generalized $\alpha - \phi-$intuitionistic fuzzy contractive mappings.
- To find the existence and uniqueness of a solution of fractional differential equations by utilizing main result.

This paper contains five sections. In second section, we review some definitions from existing literature for the support of main study. In third section, we discuss IFnCMS, fixed point results via intuitionistic fuzzy Banach contraction principle and generalized $\alpha - \phi-$intuitionistic fuzzy contractive mappings with examples. In fourth part, we find the existence and uniqueness of fractional differential equations by applying main result. In fifth part, we provide conclusion.

2. Preliminaries

In this part, we go over the definition of CTN, CTCN, fuzzy b-MSs, extended FMSs, controlled FMSs, double controlled FMSs, triple controlled FMSs, fuzzy hexagonal extended b-MSs, pentagonal controlled FMSs, fuzzy n-controlled MSs, IFMSs, IFbMSs and IFDCMSs.

Definition 2.1. *[1] A mapping $\iota^o : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm (CTN) if its satisfied the below conditions:*

- **(i):** *Commutativity:* $\iota^o(\varsigma^o, \hbar^o) = \iota^o(\hbar^o, \varsigma^o)$;
- **(ii):** *Assosiativity:* $\iota^o(\varsigma^o, \iota^o(\hbar^o, z)) = \iota^o(\iota^o(\varsigma^o, \hbar^o), z)$;
- **(iii):** *Monotonicity* $\iota^o(\varsigma^o, \hbar^o) \leq \iota^o(w, z)$ *whenever* $\varsigma^o \leq w$ *and* $\hbar^o \leq z$;
- **(iv):** *The number 1 act as an identity element* $\iota^o(\varsigma^o, 1) = \varsigma^o$;
- **(v):** ι^o *is continuous.*

Definition 2.2. *[1] A mapping $\circ : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t- conorm (CTCN) if fulfill the below assertions:*

- **(i):** *Commutativity:* $\circ(\varsigma^o, \hbar^o) = \circ(\hbar^o, \varsigma^o)$;
- **(ii):** *Assosiativity:* $\circ(\varsigma^o, \circ(\hbar^o, z)) = \circ(\circ(\varsigma^o, \hbar^o), z)$;
- **(iii):** *Monotonicity* $(\varsigma^o, \hbar^o) \leq \circ(w, z)$ *whenever* $\varsigma^o \leq w$ *and* $\hbar^o \leq z$;
- **(iv):** $\circ(\varsigma^o, 1) = 1$;
- **(v):** $\circ$ *is continuous.*

Definition 2.3. *[8] A fuzzy b-MS is an order triple $(\Theta, F, *)$ where Θ is an arbitrary set,$*$ is CTN, and F is FS on $\Theta \times \Theta \times (0, \infty)$ verifies the assertions listed below for all $\varsigma^o, \hbar^o, z \in \Theta$ and $\iota^o, \varsigma^o > 0$:*

(F1) $F(\varsigma^o, \hbar^o, \iota^o) > 0$;
(F2) $F(\varsigma^o, \hbar^o, \iota^o) = 1$ *if and only if* $\varsigma^o = \hbar^o$;
(F3) $F(\varsigma^o, \hbar^o, \iota^o) = F(\hbar^o, \varsigma^o, \iota^o)$;
(F4) $F(\varsigma^o, z, b(\iota^o + \jmath)) \geq F(\varsigma^o, \hbar^o, \iota^o) * F(\hbar^o, z, \jmath)$; *where* $b \geq 1$;
(F5) $F(\varsigma^o, \iota^o, .) : (0, \infty) \to [0,1]$ *is left continuous.*
Then F is called a fuzzy b-metric on Θ.

Example 2.1. *Let (Θ, d, ς^o) be a b-MS. Define $F : \Theta \times \Theta \times (0, \infty) \to [0,1]$ by*

$$F(\varsigma^o, \hbar^o, \iota^o) = \frac{\iota^o}{\iota^o + d(\varsigma^o, \hbar^o)}.$$

Also consider that $\iota^o * \jmath = min\{\iota^o, \jmath\}$. Then $(\Theta, F, *)$ is fuzzy b-MS with constant $\jmath = \varsigma^o$.

Definition 2.4. *[22] A controlled FMS is a four tuple $(\Theta, M, *, \alpha)$ where Θ is an arbitrary set, $\alpha : \Theta \times \Theta \to [1, \infty)$ $*$ is CTN, and M is FS in $\Theta \times \Theta \times (0, \infty)$ that verifies the below axioms:*

(GV1) $\forall_{\varsigma^o, \hbar^o \in \Theta} \forall_{\iota^o > 0} \{M(\varsigma^o, \hbar^o, \iota^o) = 0\}$;

$(GV2)$ $\forall_{\varsigma^o,\hbar^o\in\Theta}$ $\{\forall_{\iota^o>0}\{M(\varsigma^o,\hbar^o,\iota^o)=1\}\Leftrightarrow\varsigma^o=\hbar^o\}$;
$(GV3)$ $\forall_{\varsigma^o,\hbar^o\in\Theta}$ $\forall_{\iota^o>0}$ $\{M(\varsigma^o,\hbar^o.\iota^o)=M(\hbar^o,\varsigma^o,\iota^o)\}$;
$(GV4)$ $\forall_{\varsigma^o,\hbar^o,z\in\Theta}$ $\forall_{\iota^o,\jmath>0}$ $\{M(\varsigma^o,z,\iota^o+\jmath)\geq M(\varsigma^o,\hbar^o,\frac{\iota^o}{\alpha(\varsigma^o,\hbar^o)})*M(\hbar^o,z,\frac{\jmath}{\alpha(\hbar^o,z)})\}$;
$(GV5)$ $M(\varsigma^o,\hbar^o,.):(0,\infty)\to[0,1]$ *is continuous, for all* $\varsigma^o,\hbar^o\in\Theta$.
Then M *is named a controlled fuzzy metric on* Θ.

Example 2.2. *Assume* $\Theta=A\cup B$ *where* $A=[0,1]$ *and* $C=\mathbb{N}\backslash\{0,1\}$. *Define* $M:\Theta\times\Theta\times[0,\infty)\to[0,1]$ *as*

$$M(\iota^o,c,\iota^o)=\begin{cases}1, \text{ if } \iota^o=c,\\ \frac{\iota^o}{\iota^o+\frac{1}{c}} \text{ if } \iota^o\in A \text{ and } c\in C\\ \frac{\iota^o}{\iota^o+\frac{1}{\iota^o}} \text{ if } \iota^o\in A \text{ and } c\in C\\ \frac{\iota^o}{\iota^o+1} \text{ if otherwise}\end{cases}$$

with the CTN $*$ *such that* $\iota^o*\jmath=\iota^o.\jmath$ *Define* $\alpha:\Theta\times\Theta\to[1,\infty)$ *as*

$$\alpha(a,c)=\begin{cases}1 \text{ if } \iota^o,c\in A\\ \max\{\iota^o,c\} \text{ if otherwise.}\end{cases}$$

Then $(\Theta,M,*)$ *is a controlled FMS.*

Definition 2.5. *[23] Consider a nonempty set* Θ *and two non-comparable functions* $\alpha,\beta:\Theta\times\Theta\to[1,\infty)$. *Let* $*$ *be a CTN. A fuzzy double controlled metric on* Θ *is a FS and* M *defined on* $\Theta\times\Theta\times(0,\infty)$ *that fulfills the below assertions for any* $\varsigma^o,\hbar^o,z\in\Theta$ *:*
(M1) $\forall_{\varsigma^o,\hbar^o\in\Theta}$ $\{M(\varsigma^o,\hbar^o,\iota^o)>0\}$;
(M2) $\forall_{\varsigma^o,\hbar^o\in\Theta}$ $\{\forall_{\iota^o>0}\{M(\varsigma^o,\hbar^o,\iota^o)=1\}\Leftrightarrow\varsigma^o=\hbar^o\}$;
(M3) $\forall_{\varsigma^o,\hbar^o\in\Theta}$ $\forall_{\iota^o>0}$ $\{M(\varsigma^o,\hbar^o.\iota^o)=M(\hbar^o,\varsigma^o,\iota^o)\}$;
(M4) $\forall_{\varsigma^o,\hbar^o,z\in\Theta}$ $\forall_{\iota^o,\jmath>0}$ $\{M(\varsigma^o,z,\iota^o+\jmath)\geq M(\varsigma^o,\hbar^o,\frac{\iota^o}{\alpha(\varsigma^o,\hbar^o)})*M(\hbar^o,z,\frac{\jmath}{\beta(\hbar^o,z)})$.
(M5) $M(\varsigma^o,\hbar^o,.):(0,\infty)\to[0,1]$ *is continuous.*
Then $(\Theta,M,*)$ *is called a fuzzy double controlled MS.*

Definition 2.6. *[24] Suppose* $f,g,h:F\times F\to[1,\infty)$ *are three noncomparable functions,* $*$ *is a CTN, and* mq *is a FS on* $F\times F\times(0,\infty)$. *Then,* m_q *is said to be a fuzzy triple controlled metric if for any* $\varpi,\varsigma^o\in F$ *and all distinct* $\varepsilon,\eta\in F\setminus\{\varpi,\varsigma^o\}$, *that fulfills the below assertions:*

(m_q1) $m_q(\varpi,\varsigma^o,\iota^o)>0$;
(m_q2) $m_q(\varpi,\varsigma^o,\iota^o)=1$ *for all* $\iota^o>0$ *iff* $\varpi=\varsigma^o$;
(m_q3) $m_q(\varpi,\varsigma^o,\iota^o)=m_q(\varsigma^o,\varpi,\iota^o)$;
(m_q4) $m_q(\varpi,\varsigma^o,\iota^o+\varsigma^o+w)\geq m_q(\varpi,\varepsilon,\iota^o/f(\varpi,\varepsilon))*m_q(\varepsilon,\eta,\varsigma^o/g(\varepsilon,\eta))*m_q(\eta,\varsigma^o,w/h(\eta,\varsigma^o))$, *for all* $\iota^o,\varsigma^o,w>0$;
(m_q5) $m_q(\varpi,\varsigma^o,\cdot):(0,\infty)\to[0,1]$ *is continuous.*

Then, $(F,m_q,*)$ *is named a fuzzy triple controlled MS.*

Definition 2.7. *[25] Suppose Θ be a nonempty set. A 4-tuple $(\Theta, L, *)$ is a fuzzy extended hexagonal b-MS, if $*$ is a CTCN, L is a FS on $\Theta\times\Theta\times[0,\infty)$, and $Q:\Theta\times\Theta\to[1,\infty)$ verifies the below assertions for all $\varsigma^o, \hbar^o, e, f, g, k\in\Theta$, $\varsigma^o\neq e$, $e\neq f$, $f\neq g$, $g\neq k$, $k\neq\hbar^o$, and $\alpha,\beta,\sigma^o,\sigma_1^o, w\geq 0$;*

(F1) $L(\varsigma^o,\hbar^o,0)=0$;
(F2) $L(\varsigma^o,\hbar^o,\alpha)=1$ *implies* $\varsigma^o=\hbar^o$;
(F3) $L(\varsigma^o,\hbar^o,\alpha)=L(\hbar^o,\varsigma^o,\alpha)$;
(F4) $L(\varsigma^o,\hbar^o,Q(\varsigma^o,\hbar^o)(\alpha+\beta+\sigma^o+\sigma_1^o+w))\geq L(\varsigma^o,e,\alpha)*L(e,f,\beta)*L(f,g,\sigma^o)*L(g,k,\sigma_1^o)$ $*L(k,\hbar^o,w)$;
(F5) $L(\varsigma^o,\hbar^o,\cdot):[0,\infty)\to[0,1]$ *is left continuous.*

Definition 2.8. *[26] Suppose $\Theta\neq\phi$. A 4-tuple $(\Theta,H,*,Q)$ is a fuzzy controlled hexagonal MS if $*$ is a CTCN, H is a FS on $\Theta\times\Theta\times[0,\infty)$, and $Q:\Theta\times\Theta\to[1,\infty)$ that verifies the below axioms for all $\varsigma^o,\hbar^o,e,f,g,k\in\Theta$, $\varsigma^o\neq e$, $e\neq f$, $f\neq g$, $g\neq k$, $k\neq\hbar^o$, and $\alpha,\beta,\sigma^o,\sigma_1^o,w\geq 0$;*

(T1) $H(\varsigma^o,\hbar^o,0)=0$;
(T2) $H(\varsigma^o,\hbar^o,\alpha)=1$ *implies* $\varsigma^o=\hbar^o$;
(T3) $H(\varsigma^o,\hbar^o,\alpha)=H(\hbar^o,\varsigma^o,\alpha)$;
(T4) $H(\varsigma^o,\hbar^o,\alpha+\beta+\sigma^o+\sigma_1^o+w)\geq H(\varsigma^o,e,\alpha/Q(\varsigma^o,e))*H(e,f,\beta/Q(e,f))*H(f,g,\sigma^o/Q(f,g))$ $*H(g,k,\sigma_1^o/Q(g,k))*H(k,\hbar^o,w/Q(k,\hbar^o))$;
(T5) $H(\varsigma^o,\hbar^o,\cdot):[0,\infty)\to[0,1]$ *is left continuous and* $\lim_{\alpha\to\infty}H(\varsigma^o,\hbar^o,\alpha)=1$.

Example 2.3. *Let $\Theta=\{1,2,3,4,5,6\}$. Define $H:\Theta\times\Theta\times[0,\infty)\to[0,1]$ as*

$$H(\varsigma^o,\hbar^o,\alpha)=\left[e^{\frac{|\varsigma^o-\hbar^o|^2}{\alpha}}\right]^{-1}\quad \textit{for all } \alpha>0$$

with the CTN $*$ such that $\alpha_1*\alpha_2=\alpha_1\alpha_2$. Then, $(\Theta,H,*,Q)$ is a fuzzy controlled hexagonal MS with control functions $Q(\varsigma^o,\hbar^o)=1+\varsigma^o+\hbar^o$.

Definition 2.9. *[26] Let $\Theta\neq\phi$. A triplet $(\Theta,M,*)$ is a pentagonal controlled FMS if $*$ is a CTCN, M is a FS on $\Theta\times\Theta\times[0,\infty)$, and $Q,W,E,R,\Xi:\Theta\times\Theta\to[1,\infty)$ are five noncomparable functions that verifies the below axioms for all $\varsigma^o,\hbar^o,e,f,g,k\in\Theta$, $\varsigma^o\neq e$, $e\neq f$, $f\neq g$, $g\neq k$, $k\neq\hbar^o$, and $\alpha,\beta,\sigma^o,\sigma_1^o,w\geq 0$:*

(A1) $M(\varsigma^o,\hbar^o,0)=0$;
(A2) $M(\varsigma^o,\hbar^o,\alpha)=1$ *implies* $\varsigma^o=\hbar^o$;
(A3) $M(\varsigma^o,\hbar^o,\alpha)=M(\hbar^o,\varsigma^o,\alpha)$;
(A4) $M(\varsigma^o,\hbar^o,\alpha+\beta+\sigma^o+\sigma_1^o+w)\geq M(\varsigma^o,e,\alpha/Q(\varsigma^o,e))*M(e,f,\beta/W(e,f))*M(f,g,\sigma^o/E(f,g))$ $*M(g,k,\sigma_1^o/R(g,k))*M(k,\hbar^o,w/\Xi(k,\hbar^o))$;
(A5) $M(\varsigma^o,\hbar^o,\cdot):[0,\infty)\to[0,1]$ *is left continuous and* $\lim_{\alpha\to\infty}M(\varsigma^o,\hbar^o,\alpha)=1$.

Example 2.4. *Let $\Theta=\{1,2,3,4,5,6\}$. Define $M:\Theta\times\Theta\times[0,\infty)\to[0,1]$ as*

$$M(\varsigma^o,\hbar^o,\alpha)=\frac{\alpha}{\alpha+|\varsigma^o-\hbar^o|^6}\quad \textit{for all } \alpha>0,$$

with the CTN $*$ such that $\alpha_1 * \alpha_2 = \alpha_1\alpha_2$. Then, $(\Theta, M, *)$ is a pentagonal controlled FMS with non-comparable control functions $Q(\varsigma^o, \hbar^o) = 1 + \varsigma^o + \hbar^o, W(\varsigma^o, \hbar^o) = 1 + \varsigma^{o^2} + \hbar^{o^2}, E(\varsigma^o, \hbar^o) = 1 + \frac{\varsigma^o}{\hbar^o}, R(\varsigma^o, \hbar^o) = 1 + \frac{\hbar^o}{\varsigma^o}, T(\varsigma^o, \hbar^o) = 1 + \varsigma^{o^2} + \hbar^o$.

Definition 2.10. *[27] Assume $\Sigma \neq \phi$ and $\alpha_i : \Sigma \times \Sigma \to [1, \infty)$ where $i \in \mathbb{N}$ be the non-comparable functions n. A FS M_n^c on $\Sigma \times \Sigma \times (0, \infty)$, together with a CTN $*$, is said to be a fuzzy n-controlled metric, if M_n^c fulfills:*
(M1) $M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) > 0$;
(M2) $M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) = 1$ for all $\iota^o > 0$, if and only if $\varsigma_1^o = \varsigma_2^o$;
(M3) $M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) = M_n^c(\varsigma_2^o, \varsigma_1^o, \iota^o)$;
(M4)$M_n^c(\varsigma_1^o, \varsigma_{n+1}^o, \iota_1^o + \iota_2^o + \cdots + \iota_n^o) \geq M_n^c\left(\varsigma_1^o, \varsigma_2^o, \frac{\iota_1^o}{\alpha_1(\varsigma_1^o, \varsigma_2^o)}\right) * M_n^c\left(\varsigma_2^o, \varsigma_3^o, \frac{\iota_2^o}{\alpha_2(\varsigma_2^o, \varsigma_3^o)}\right)$
$* \cdots * M_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota_n^o}{\alpha_n(\varsigma_n^o, \varsigma_{n+1}^o)}\right)$ for all $\iota_n^o > 0$;
(M5) $M_n^c(\varsigma_1^o, \varsigma_2^o, \cdot) : (0, \infty) \to [0, 1]$ is continuous;
for all distinct $\varsigma_1^o, \varsigma_2^o, \varsigma_3^o, \ldots, \varsigma_{n+1}^o \in \Sigma$. The quadruple $(\Sigma, M_n^c, \alpha_n, *)$ is called a FnCMS.

Remark 2.1. *(i) If we take the following $\varsigma_1^o, \varsigma_2^o, \varsigma_3^o, \varsigma_4^o, \varsigma_5^o, \varsigma_6^o$ distinct elements, then a (FnCMS) reduces to a fuzzy pentagonal controlled MS with controlled functions α_i, $1 \leq i \leq 5$.*
(ii) If we take the following $\varsigma_1^o, \varsigma_2^o, \varsigma_3^o, \varsigma_4^o$, distinct elements, then a FnCMS reduces to a fuzzy triple controlled MS with controlled functions α_i, $1 \leq i \leq 3$.
(iii) If we take the following $\varsigma_1^o, \varsigma_2^o, \varsigma_3^o$, distinct elements, then a (FnCMS) reduces to the definition with controlled functions α_i, $1 \leq i \leq 2$.

Remark 2.2. *Pentagonal, hexagonal, triple controlled, double controlled, b-rectangular, b-extended and controlled rectangular, and several other FMSs are also not Hausdorff.*

Represent Φ, the collection of all functions $\phi : [0, \infty) \to [0, \infty)$ which are nondecreasing and having the assertions:
(i) $\phi(\iota^o) < \iota^o$,
(ii) $\lim_{n\to\infty} \phi^n(\iota^o) = 0$,
for all $\iota^o > 0$, where ϕ^n represents the n-th iteration of ϕ.

Example 2.5. *Assume $\Sigma = [0, \infty)$, define $\alpha : \Sigma \times \Sigma \times [0, \infty)$ by*

$$\alpha(\varsigma_1^o, \varsigma_2^o, \iota^o) = \begin{cases} \frac{e^{\iota^o \varsigma_2^o}}{\varsigma_1^o}, & \text{if } \varsigma_1^o \geq \varsigma_2^o, \varsigma_1^o \neq 0 \\ 0, & \text{if } \varsigma_1^o < \varsigma_2^o \end{cases}$$

and $\Xi\varsigma^o = 5\varsigma^o$. Then clearly Ξ is α-admissible.

Definition 2.11. *[28] An IFMS is a five tuple $(\Theta, M, N, *, \circ)$ where Θ is an arbitrary set,$*$ is CTN, $\circ$ is CTCN and M, N are FSs on $\Theta \times \Theta \times (0, \infty)$ verifies the below assertions:*

(IF1) $\forall_{\varsigma^o, \hbar^o \in \Theta} \{M(\varsigma^o, \hbar^o, \iota^o) + N(\varsigma^o, \hbar^o, \iota^o) \leq 1\}$;
(IF2) $\forall_{\varsigma^o, \hbar^o \in \Theta} \{M(\varsigma^o, \hbar^o, \iota^o) > 0\}$;
(IF3) $\forall_{\varsigma^o, \hbar^o \in \Theta} \{\forall_{\iota^o > 0}\{M(\varsigma^o, \hbar^o, \iota^o) = 1\} \Leftrightarrow \varsigma^o = \hbar^o\}$;

(IF4) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \forall_{\iota^o>0}\ \{M(\varsigma^o,\hbar^o.\iota^o)=M(\hbar^o,\varsigma^o,\iota^o)\}$;

(IF5) $\forall_{\varsigma^o,\hbar^o,z\in\Theta}\ \forall_{\iota^o,\jmath>0}\ \{M(\varsigma^o,z,\iota^o+\jmath)\geq M(\varsigma^o,\hbar^o,\iota^o)*M(\hbar^o,z,\jmath)\}$;

(IF6) $M(\varsigma^o,\hbar^o,.):[0,\infty)\to[0,1]$ *is continuous for all* $\varsigma^o,\hbar^o\in\Theta$;

(IF7) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{N(\varsigma^o,\hbar^o,\iota^o)>0\}$;

(IF8) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{\forall_{\iota^o>0}\{N(\varsigma^o,\hbar^o,\iota^o)=0\}\Leftrightarrow\varsigma^o=\hbar^o\}$;

(IF9) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \forall_{\iota^o>0}\ \{N(\varsigma^o,\hbar^o.\iota^o)=N(\hbar^o,\varsigma^o,\iota^o)\}$;

(IF10) $\forall_{\varsigma^o,\hbar^o,z\in\Theta}\ \forall_{\iota^o,\jmath>0}\ \{N(\varsigma^o,z,\iota^o+\jmath)\leq N(\varsigma^o,\hbar^o,\iota^o)\circ N(\hbar^o,z,\jmath)\}$;

(IF11) $N(\varsigma^o,\hbar^o,.):[0,\infty)\to[0,1]$ *is continuous for all* $\varsigma^o,\hbar^o\in\Theta$.

The pair (M,N) is referred to as an IFM on Θ. The functions $M(\varsigma^o,\hbar^o,\iota^o)$ and $N(\varsigma^o,\hbar^o,\iota^o)$ represent the degree of closeness and the level of non-nearness between ς^o and $\hbar^o$ with regard to ι^o, respectively.

Definition 2.12. *[30] An is a six tuple $(\Theta,M,N,*,\circ,b)$ where Θ is an arbitrary non-empty set, $*$ is a CTN, $\circ$ is a CTCN and M, N are FSs on $\Theta\times\Theta\times(0,\infty)$ if $\forall\ \varsigma^o,\iota^o>0$ and a given number $b\geq1$ fulfills the below assertions:*

(IF1) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{M(\varsigma^o,\hbar^o,\iota^o)+N(\varsigma^o,\hbar^o,\iota^o)\leq1\}$;

(IF2) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{M(\varsigma^o,\hbar^o,\iota^o)>0\}$;

(IF3) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{\forall_{\iota^o>0}\{M(\varsigma^o,\hbar^o,\iota^o)=1\}\Leftrightarrow\varsigma^o=\hbar^o\}$;

(IF4) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \forall_{\iota^o>0}\ \{M(\varsigma^o,\hbar^o.\iota^o)=M(\hbar^o,\varsigma^o,\iota^o)\}$;

(IF5) $\forall_{\varsigma^o,\hbar^o,z\in\Theta}\ \forall_{\iota^o,\jmath>0}\ \{M(\varsigma^o,z,\iota^o+\jmath)\geq M(\varsigma^o,\hbar^o,\frac{\iota^o}{b})*M(\hbar^o,z,\frac{\jmath}{b})\}$;

(IF6) $M(\varsigma^o,\hbar^o,.):[0,\infty)\to[0,1]$ *is continuous for all* $\varsigma^o,\hbar^o\in\Theta$;

(IF7) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{N(\varsigma^o,\hbar^o,\iota^o)>0\}$;

(IF8) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{\forall_{\iota^o>0}\{N(\varsigma^o,\hbar^o,\iota^o)=0\}\Leftrightarrow\varsigma^o=y\}$;

(IF9) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \forall_{\iota^o>0}\ \{N(\varsigma^o,\hbar^o.\iota^o)=N(\hbar^o,\varsigma^o,\iota^o)\}$;

(IF10) $\forall_{\varsigma^o,\hbar^o,z\in\Theta}\ \forall_{\iota^o,\jmath>0}\ \{N(\varsigma^o,z,\iota^o+\jmath)\leq N(\varsigma^o,\hbar^o,\frac{\iota^o}{b})\circ N(\hbar^o,z,\frac{\jmath}{b})\}$;

(IF11) $N(\varsigma^o,\hbar^o,.):[0,\infty)\to[0,1]$ *is continuous for all* $\varsigma^o,\hbar^o\in\Theta$.

Then (M,N) is called an intuitionistic fuzzy b-metric on Θ.

Example 2.6. *[30] Consider that $M(\varsigma^o,\hbar^o,\iota^o)=e^{\frac{-|\varsigma^o-\hbar^o|^p}{\iota^o}}$ and $N(\varsigma^o,\hbar^o,\iota^o)=1-e^{\frac{-|\varsigma^o-\hbar^o|^p}{\iota^o}}$ where $p>1$ is a real number. Then $(\Theta,M,N,*,\circ,b)$ is become an IFbMS with $b=2^{p-1}$.*

Definition 2.13. *[32] An IFDCMS is a six tuple $(\Theta,M,N,*,\circ,b)$ where Θ is an arbitrary non-empty set, $*$ is a CTN, $\circ$ is a CTCN and M, N are FSs on $\Theta\times\Theta\times(0,\infty)$ if $\forall\ \varsigma^o,\iota^o>0$ and $\alpha_1,\alpha_2:\Theta\times\Theta\to[1,\infty)$ be two non-comparable functions satisfies the following conditions:*

(IF1) $\forall_{\varsigma^o,\hbar^o\in\Theta}\{M(\varsigma^o,\hbar^o,\iota^o)+N(\varsigma^o,\hbar^o,\iota^o)\leq1\}$;

(IF2) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{M(\varsigma^o,\hbar^o,\iota^o)>0\}$;

(IF3) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \{\forall_{\iota^o>0}\{M(\varsigma^o,\hbar^o,\iota^o)=1\}\Leftrightarrow\varsigma^o=\hbar^o\}$;

(IF4) $\forall_{\varsigma^o,\hbar^o\in\Theta}\ \forall_{\iota^o>0}\{M(\varsigma^o,\hbar^o.\iota^o)=M(\hbar^o,\varsigma^o,\iota^o)\}$;

(IF5) $\forall\varsigma^o,\hbar^o,z\in\Theta\forall_{\iota^o,\jmath>0}\left\{M(\varsigma^o,z,\iota^o+\jmath)\geq M\left(\varsigma^o,\hbar^o,\frac{\iota^o}{\alpha_1(\varsigma^o,\hbar^o)}\right)*M\left(\hbar^o,z,\frac{\jmath}{\alpha_2(\hbar^o,z)}\right)\right\}$

(IF6) $M(\varsigma^o,\hbar^o,.):[0,\infty)\to[0,1]$ *is continuous for all* $\varsigma^o,\hbar^o\in\Theta$;

(IF7) $\forall_{\varsigma^o,\hbar^o\in\Theta}\{N(\varsigma^o,\hbar^o,\iota^o)>0\}$;

(IF8) $\forall_{\varsigma^o,\hbar^o\in\Theta}\{\forall_{\iota^o>0}\{N(\varsigma^o,\hbar^o,\iota^o)=0\}\Leftrightarrow\varsigma^o=\hbar^o\}$;

(IF9) $\forall_{\varsigma^o,\hbar^o\in\Theta}\forall_{\iota^o>0}\{N(\varsigma^o,\hbar^o.\iota^o)=N(\hbar^o,\varsigma^o,\iota^o)\}$;

(IF10) $\forall\varsigma^o,\hbar^o,z\in\Theta\forall_{\iota^o,\jmath>0}\left\{N(\varsigma^o,z,\iota^o+\jmath)\leq N\left(\varsigma^o,\hbar^o,\frac{\iota^o}{\alpha_1(\varsigma^o,\hbar^o)}\right)\circ N\left(\hbar^o,z,\frac{\jmath}{\alpha_2(\hbar^o,z)}\right)\right\}$

(IF11) $N(\varsigma^o,\hbar^o,.):[0,\infty)\to[0,1]$ *is continuous for all* $\varsigma^o,\hbar^o\in\Theta$.

Then (M,N) *is called an intuitionistic fuzzy double controlled metric on* Θ.

3. Main Results

In this section, we introduce the concept of IFnCMS as a generalization of IFDCMSs [32], FbMSs [30], FnCMS [27] and several other generalized spaces. Further, we prove some generalized fixed point results in the setting of IFnCMSs.

Definition 3.1. *Suppose* $\Sigma\neq\phi$ *and* $\alpha_i:\Sigma\times\Sigma\to[1,\infty)$ $(1\leq i\leq n)$ *be n non-comparable functions. Let* $*$ *and* $\circ$ *show the CTN and CTCN, respectively. The six tuple* $(\Sigma,F,N,\alpha_n,*,\circ)$ *is called an IFnCMS when the below axioms are hold for all distinct* $\varsigma_1^o,\varsigma_2^o,\ldots,\varsigma_{n+1}^o\in\Sigma$ *and* $t>0$:

(IF1) $F(\varsigma_1^o,\varsigma_2^o,\iota^o)>0$;

(IF2) $F(\varsigma_1^o,\varsigma_2^o,\iota^o)=1\ \forall\ \iota^o>0$ *, if and only if* $\varsigma_1^o=\varsigma_2^o$;

(IF3) $F(\varsigma_1^o,\varsigma_2^o,\iota^o)=F(\varsigma_2^o,\varsigma_1^o,\iota^o)$;

(IF4) $F(\varsigma_1^o,\varsigma_{n+1}^o,\iota_1^o+\iota_2^o+\cdots+\iota_n^o)\geq F(\varsigma_1^o,\varsigma_2^o,\frac{\iota_1^o}{\alpha_1(\varsigma_1^o,\varsigma_2^o)})*F(\varsigma_2^o,\varsigma_3^o,\frac{\iota_2^o}{\alpha_2(\varsigma_2^o,\varsigma_3^o)})$
$*\cdots*F(\varsigma_n^o,\varsigma_{n+1}^o,\frac{\iota_n^o}{\alpha_n(\varsigma_n^o,\varsigma_{n+1}^o)})\ \forall\ \iota_n^o>0$;

(IF5 $F(\varsigma_1^o,\varsigma_2^o,.):(0,\infty)\to[0,1]$ *is continuous;*

(IF6) $N(\varsigma_1^o,\varsigma_2^o,\iota^o)>0$;

(IF7) $N(\varsigma_1^o,\varsigma_2^o,\iota^o)=1\ \forall\ \iota^o>0$ *, if and only if* $\varsigma_1^o=\varsigma_2^o$;

(IF8) $N(\varsigma_1^o,\varsigma_2^o,\iota^o)=N(\varsigma_2^o,\varsigma_1^o,\iota^o)$;

(IF9) $N(\varsigma_1^o,\varsigma_{n+1}^o,\iota_1^o+\iota_2^o+\cdots+\iota_n^o)\leq N(\varsigma_1^o,\varsigma_2^o,\frac{\iota_1^o}{\alpha_1(\varsigma_1^o,\varsigma_2^o)})\circ N(\varsigma_2^o,\varsigma_3^o,\frac{\iota_2^o}{\alpha_2(\varsigma_2^o,\varsigma_3^o)})$
$\circ\cdots\circ N(\varsigma_n^o,\varsigma_{n+1}^o,\frac{\iota_n^o}{\alpha_n(\varsigma_n^o,\varsigma_{n+1}^o)})\ \forall\ \iota_n^o>0$;

(IF10 $N(\varsigma_1^o,\varsigma_2^o,.):(0,\infty)\to[0,1]$ *is continuous;*

(IF11) $F(\varsigma_1^o,\varsigma_2^o,\iota^o)+N(\varsigma_1^o,\varsigma_2^o,\iota^o)\leq1$.

Then, (Σ,F,N) *is called intuitionsitics n-controlled fuzzy metric. The function* $F(\varsigma_1^o,\varsigma_2^o,\iota^o)$ *and* $N(\varsigma_1^o,\varsigma_2^o,\iota^o)$ *represents the degree of nearness and the degree of non-nearness between* ς_1^o *and* ς_2^o *with respect to* ι^o*, respectively.*

We have the below remarks from the preceding definition:

Remark 3.1. *(i) If we take* $n=5$*, i.e,* $\varsigma_1^o,\varsigma_2^o,\ldots,\varsigma_6^o$*, six distinct elements, then IFnCMS reduces to intuitionsitic fuzzy pentagonal controlled MS with controlled functions* $\alpha_i, 1\leq i\leq5$.
(ii) If we take $n=2$*, i.e,* $\varsigma_1^o,\varsigma_2^o,\varsigma_3^o$*, three distinct elements, then IFnCMS reduces to IFDCMS [32] with controlled functions* $\alpha_i, 1\leq i\leq2$.
(iii) Every FnCMS [27] is an IFnCMS of the form $(\Sigma,F,1-F,\alpha_n,*,\circ)$ *such that* $*$ *and* $\circ$ *are associated as*

$\varsigma^o \circ \hbar^o = 1 - ((1-\varsigma^o) * (1-\hbar^o))$ for all $\varsigma^o, \hbar^o \in \Theta$.
(iv) If we take $\varsigma_3^o = \varsigma_4^o = \cdots = \varsigma_{n+1}^o, \iota_1^{o'} = \iota_2^o + \iota^o 3 + \cdots + \iota_n^o$, then IFnCMS reduces to IFMS [28].

From the above remark, we can deduce intuitionistic fuzzy rectangular extended b-MS [31], IFbMS [30], fuzzy pentagonal controlled MS [26], fuzzy hexagonal extended b-MS [25], fuzzy triple controlled MS [24], fuzzy double controlled MS [23], fuzzy controlled MS [22], fuzzy extended b-MS [21], fuzzy b-MS [20] and FMS [19].

Example 3.1. *Let $\Theta = \{1,2,3,\cdots,7\}$ and $a_i : \Theta \times \Theta \to [1,\infty)(1 \le i \le 6)$ be defined as $\alpha_1 = 1 + \varsigma_1^o + \varsigma_2^o, \alpha_2 = 1 + \varsigma_2^o + \varsigma_3^o, \alpha_3 = 1 + \varsigma_3^o + \varsigma_4^o, \alpha_4 = 1 + \varsigma_4^o + \varsigma_5^o, \alpha_5 = 1 + \varsigma_5^o + \varsigma_6^o, \alpha_6 = 1 + \varsigma_6^o + \varsigma_7^o$. Define $F, N : \Theta \times \Theta \times (0,\infty) \to [0,1]$ by*

$$F(\varsigma_1^o, \varsigma_2^o, \iota^o) = \frac{\min\{\varsigma_1^o, \varsigma_2^o\} + \iota^o}{\mu\varsigma^o\{\varsigma_1^o, \varsigma_2^o\} + \iota^o}$$

and

$$N(\varsigma_1^o, \varsigma_2^o, \iota^o) = 1 - \frac{\min\{\varsigma_1^o, \varsigma_2^o\} + \iota^o}{\max\{\varsigma_1^o, \varsigma_2^o\} + \iota^o}.$$

Define CTN "$*$" by $\iota_1^o * \iota_2^o = \iota_1^o \iota_2^o$ and CTCN "$\circ$" by $\iota_1^o \circ \iota_2^o = \max\{\iota_1^o, \iota_2^o\}$. Then $(\Theta, F, N, \alpha_6, *, \circ)$ is an IFnCMS.

Proof. Here, we prove (IF4) and (IF8) as other all are obvious.
(IF4): Let $\varsigma_1^o = 1$ and $\varsigma_7^o = 7$, then left hand side

$$\begin{aligned} F(\varsigma_1^o, \varsigma_7^o, \iota_1^o + \iota_2^o + \cdots + \iota_6^o) &= \frac{\min\{\varsigma_1^o, \varsigma_7^o\} + \iota_1^o + \iota_2^o + \cdots + \iota_6^o}{\max\{\varsigma_1^o, \varsigma_7^o\} + \iota_1^o + \iota_2^o + \cdots + \iota_6^o} \\ &= \frac{\min\{1,7\} + \iota_1^o + \iota_2^o + \cdots + \iota_6^o}{\max\{1,7\} + \iota_1^o + \iota_2^o + \cdots + \iota_6^o} \\ &= \frac{1 + \iota_1^o + \iota_2^o + \cdots + \iota_6^o}{7 + \iota_1^o + \iota_1^o + \cdots + \iota_6^o}. \end{aligned}$$

Now, right hand side

$$F\left(1,2,\frac{\iota_1^o}{\alpha_1(1,2)}\right) = \frac{\min\{1,2\} + \frac{\iota_1^o}{\alpha_1(1,2)}}{\max\{1,2\} + \frac{\iota_1^o}{\alpha_1(1,2)}} = \frac{1 + \frac{\iota_1^o}{3}}{2 + \frac{\iota_1^o}{3}} = \frac{3 + \iota_1^o}{6 + \iota_1^o}$$

$$F\left(2,3,\frac{\iota_2^o}{\alpha_2(2,3)}\right) = \frac{\min\{2,3\} + \frac{\iota_2^o}{\alpha_2(2,3)}}{\max\{2,3\} + \frac{\iota_2^o}{\alpha_2(2,3)}} = \frac{2 + \frac{\iota_2^o}{5}}{3 + \frac{\iota_2^o}{5}} = \frac{10 + \iota_2^o}{15 + \iota_2^o}$$

$$F\left(3,4,\frac{\iota_3^o}{\alpha_3(3,4)}\right) = \frac{\min\{3,4\} + \frac{\iota_3^o}{\alpha_3(3,4)}}{\max\{3,4\} + \frac{\iota_3^o}{\alpha_3(3,4)}} = \frac{3 + \frac{\iota_3^o}{7}}{4 + \frac{\iota_3^o}{7}} = \frac{21 + \iota_3^o}{28 + \iota_3^o}$$

$$F\left(4,5,\frac{\iota_4^o}{\alpha_4(4,5)}\right) = \frac{\min\{4,5\} + \frac{\iota_4^o}{\alpha_4(4,5)}}{\max\{4,5\} + \frac{\iota_4^o}{\alpha_4(4,5)}} = \frac{4 + \frac{\iota_4^o}{9}}{5 + \frac{\iota_4^o}{9}} = \frac{36 + \iota_4^o}{45 + \iota_4^o}$$

$$F\left(5,6,\frac{\iota_5^o}{\alpha_5(5,6)}\right)=\frac{\min\{5,6\}+\frac{\iota_5^o}{\alpha_5(5,6)}}{\max\{5,6\}+\frac{\iota_5^o}{\alpha_5(5,6)}}=\frac{5+\frac{\iota_5^o}{11}}{6+\frac{\iota_5^o}{11}}=\frac{55+\iota_5^o}{66+\iota_5^o}$$

$$F\left(6,7,\frac{\iota_6^o}{\alpha_6(6,7)}\right)=\frac{\min\{6,7\}+\frac{\iota_6^o}{\alpha_6(6,7)}}{\max\{6,7\}+\frac{\iota_6^o}{\alpha_6(6,7)}}=\frac{6+\frac{\iota_6^o}{13}}{7+\frac{\iota_6^o}{13}}=\frac{78+\iota_6^o}{91+\iota_6^o},$$

which implies

$$F(\varsigma_1^o,\varsigma_7^o,\iota_1^o+\iota_+^o\cdots+\iota_6^o)\geq F\left(1,2,\frac{\iota_1^o}{\alpha_1(1,2)}\right)*F\left(2,3,\frac{\iota_2^o}{\alpha_2(2,3)}\right)*F\left(3,4,\frac{\iota_3^o}{\alpha_3(3,4)}\right)$$
$$*F\left(4,5,\frac{\iota_4^o}{\alpha_4(4,5)}\right)*F\left(5,6,\frac{\iota_5^o}{\alpha_5(5,6)}\right)*F\left(6,7,\frac{\iota_6^o}{\alpha_6(6,7)}\right). \tag{3.1}$$

Since $\iota_1^o*\iota_2^o=\iota_1^o\iota_2^o$, we can write

$$\frac{1+\iota_1^o+\iota_1^o+\cdots+\iota_6^o}{7+\iota_1^o+\iota_1^o+\cdots+\iota_6^o}\geq\frac{3+\iota_1^o}{6+\iota_1^o}\cdot\frac{10+\iota_2^o}{15+\iota_2^o}\cdot\frac{21+\iota_3^o}{28+\iota_3^o}$$
$$\cdot\frac{36+\iota_4^o}{45+\iota_4^o}\cdot\frac{55+\iota_5^o}{66+\iota_5^o}\cdot\frac{78+\iota_6^o}{91+\iota_6^o},$$

which satisfies for all $\iota_1^o,\iota_2^o,\ldots\iota_6^o>0$. Other cases can be obtain similarly.
Now, we can get

$$\begin{aligned}N(\varsigma_1^o,\varsigma_7^o,\iota_1^o+\iota_2^o+\cdots+\iota_7^o)&=1-\frac{\min\{\varsigma_1^o,\varsigma_7^o\}+\iota_1^o+\iota_2^o+\cdots+\iota_7^o}{\max\{\varsigma_1^o,\varsigma_7^o\}+\iota_1^o+\iota_2^o+\cdots+\iota_7^o}\\&=1-\frac{\min\{1,7\}+\iota_1^o+\iota_2^o+\cdots+\iota_7^o}{\max\{1,7\}+\iota_1^o+\iota_2^o+\cdots+\iota_7^o}\\&=1-\frac{1+\iota_1^o+\iota_2^o+\cdots+\iota_7^o}{7+\iota_1^o+\iota_1^o+\cdots+\iota_7^o}\\&=1-F(\varsigma_1^o,\varsigma_7^o,\iota_1^o+\iota_2^o+\cdot\varsigma^o+\iota_7^o).\end{aligned}$$

Similarly, we can get

$$\begin{aligned}N\left(1,2,\frac{\iota_1^o}{\alpha_1(1,2)}\right)&=1-F\left(1,2,\frac{\iota_1^o}{\alpha_1(1,2)}\right)&&=1-\frac{3+\iota_1^o}{6+\iota_1^o},,\\N\left(2,3,\frac{\iota_2^o}{\alpha_2(2,3)}\right)&=1-F\left(2,3,\frac{\iota_2^o}{\alpha_2(2,3)}\right)&&=1-\frac{10+\iota_2^o}{15+\iota_2^o},\\N\left(3,4,\frac{\iota_3^o}{\alpha_3(3,4)}\right)&=1-F\left(3,4,\frac{\iota_3^o}{\alpha_3(3,4)}\right)&&=1-\frac{21+\iota_3^o}{28+\iota_3^o},\\N\left(4,5,\frac{\alpha_4}{\alpha_4(4,5)}\right)&=1-F\left(4,5,\frac{\iota_4^o}{\alpha_4(5,6)}\right)&&=1-\frac{36+\iota_4^o}{45+\iota_4^o},\\N\left(5,6,\frac{\iota_5^o}{\alpha_5(5,6)}\right)&=1-F\left(5,6,\frac{\iota_5^o}{\alpha_5(5,6)}\right)&&=1-\frac{55+\iota_5^o}{66+\iota_5^o}\end{aligned}$$

$$N\left(6,7,\frac{\iota_6^o}{\alpha_6(6,7)}\right)=1-F\left(6,7,\frac{\iota_6^o}{\alpha_6(6,7)}\right) \qquad =1-\frac{78+\iota_6^o}{91+\iota_6^o},$$

$$N(1,7,\iota_1^o+\iota_+^o\cdots+\iota_7^o)\leq N\left(1,2,\frac{\iota_1^o}{\alpha_1(1,2)}\right)\circ N\left(2,3,\frac{\iota_2^o}{\alpha_2(2,3)}\right)\circ N\left(3,4,\frac{\iota_3^o}{\alpha_3(3,4)}\right)$$
$$\circ N\left(4,5,\frac{\iota_4^o}{\alpha_4(4,5)}\right)\circ N\left(5,6,\frac{\iota_5^o}{\alpha_5(5,6)}\right)\circ N\left(6,7,\frac{\iota_6^o}{\alpha_6(6,7)}\right).$$

Since $a\circ b=\max\{a,b\}$, we can write

$$1-\frac{1+\iota_1^o+\iota_2^o+\cdots+\iota_7^o}{7+\iota_1^o+\iota_2^o+\cdots+\iota_7^o}\leq \max\{(1-\frac{3+\iota_1^o}{6+\iota_1^o}),(1-\frac{10+\iota_2^o}{15+\iota_2^o}),(1-\frac{21+\iota_3^o}{28+\iota_3^o}),$$
$$(1-\frac{36+\iota_4^o}{45+\iota_4^o}),(1-\frac{55+\iota_5^o}{66+\iota_5^o}),(1-\frac{78+\iota_6^o}{91+\iota_6^o})\},$$

which satisfies for all $\iota_1^o,\iota_2^o,\ldots\iota_7^o>0$. Similarly, we can prove other cases.
Hence,$(\Sigma,F,N,\alpha_n,*,\circ)$ is an IFnCMS. On the same lines, we can prove for higher value of n. Below Figure 1, shows the graphical behaviour of this example.

□

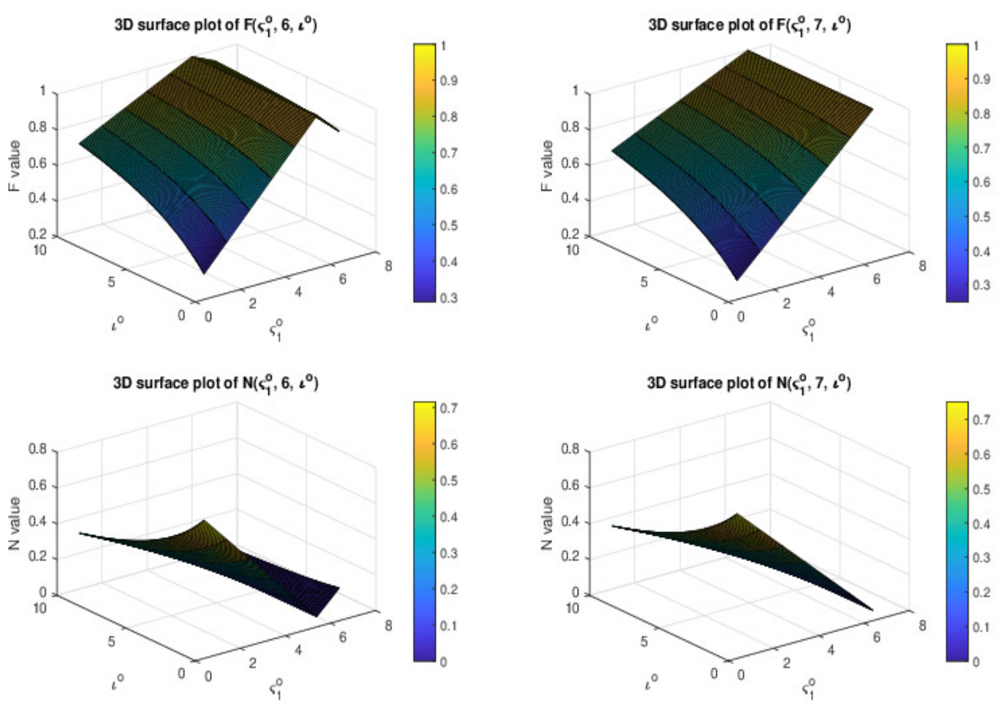

Figure 1. Depicts the graphical behaviour of F and N in the above example.

Now, we introduce generalized intuitionistic fuzzy contraction mappings and prove several fixed point results by using such contractions. Also, we provide an example to show the validity of main result.

Theorem 3.1. *Suppose $\alpha_n : \Sigma \times \Sigma \to [1, \infty)$ and $(\Sigma, M_n^c, N_n^c, \alpha_n, *, \circ)$ is a complete IFnCMS equipped with*

$$\lim_{\iota^o \to \infty} M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) = 1 \text{ and } \lim_{\iota^o \to \infty} N_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) = 0. \tag{3.-15}$$

Also, assume Ξ be a self-mapping on Σ verifying:

$$M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, K\iota^o) \geq M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) \text{ and } N_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, K\iota^o) \leq N_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) \tag{3.-14}$$

for all $\varsigma_1^o, \varsigma_2^o \in \Sigma$. Then the mapping Ξ has a unique fixed point in Σ.

Proof:

Suppose $\varsigma_0^o \in \Sigma$ and the sequence $\Xi^n \varsigma_0^o = \varsigma_{n+1}^o$. We have

$$\begin{aligned} M_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) &= M_n^c(\Sigma\varsigma_{n-1}^o, \Sigma\varsigma_n^o, \iota^o) \\ &\geq M_n^c(\Sigma\varsigma_{n-1}^o, \Sigma\varsigma_n^o, \frac{\iota^o}{K}) \\ &\vdots \\ &\geq M_n^c(\varsigma_0^o, \varsigma_1^o, \frac{\iota^o}{K^n}) \end{aligned}$$

and

$$\begin{aligned} N_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) &= N_n^c(\Sigma\varsigma_{n-1}^o, \Sigma\varsigma_n^o, \iota^o) \\ &\leq N_n^c(\Sigma\varsigma_{n-1}^o, \Sigma\varsigma_n^o, \frac{\iota^o}{K}) \\ &\vdots \\ &\leq N_n^c(\varsigma_0^o, \varsigma_1^o, \frac{\iota^o}{K^n}). \end{aligned}$$

We get

$$M_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) \geq M_n^c(\varsigma_0^o, \varsigma_1^o, \frac{\iota^o}{K^n}), \tag{3.-13}$$

$$N_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) \leq N_n^c(\varsigma_0^o, \varsigma_1^o, \frac{\iota^o}{K^n}). \tag{3.-12}$$

Assume the sequence $\{\varsigma_n^o\}$ in Σ, then

Case 1. When $p = 2q+1$ (odd), we take $\iota^o = \frac{(2q+1)\iota^o}{2q+1} = \frac{\iota^o}{2q+1} + \frac{\iota^o}{2q+1} + \ldots + \frac{\iota^o}{2q+1}$, we have

$$\begin{aligned} &M_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \\ &\geq M_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)}) * M_n^c(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}) \\ &\quad * M_n^c(\varsigma_{n+2}^o, \varsigma_{n+3}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}) * \cdots * M_n^c(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_n(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o)}) \end{aligned}$$

and

$$\begin{aligned} & N_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \\ & \leq N_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)}) \circ M_n^c(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}) \\ & \quad \circ N_n^c(\varsigma_{n+2}^o, \varsigma_{n+3}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}) \circ \cdots \circ N_n^c(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_n(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o)}). \end{aligned}$$

Using (3.-2) and (3.-1), we have

$$\begin{aligned} & M_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \\ & \geq M_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)K^n}\right) * M_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)K^{n+1}}\right) \\ & * M_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)K^{n+2}}\right) * \cdots * M_n^c\left((\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o), \frac{\frac{\iota^o}{2q+1}}{\alpha_n(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o)K^{n+2q}}\right) \end{aligned}$$

and

$$\begin{aligned} & N_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \\ & \leq N_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)K^n}\right) \circ N_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)K^{n+1}}\right) \\ & \circ N_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)K^{n+2}}\right) \circ \cdots \circ N_n^c\left((\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o), \frac{\frac{\iota^o}{2q+1}}{\alpha_n(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o)K^{n+2q}}\right). \end{aligned}$$

Applying limit $n \to \infty$, we have

$$\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \geq 1 * 1 * 1 \cdots * 1 = 1$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \leq 0 \circ 0 \circ 0 \cdots \circ 0 = 0.$$

Case 2.

$$\text{When p = 2q (even), we take } \iota^o = \frac{(2q)\iota^o}{2q} = \frac{\iota^o}{2q} + \frac{\iota^o}{2q} + \cdots + \frac{\iota^o}{2q},$$

we have

$$\begin{aligned} M_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o) & \geq M_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\frac{\iota^o}{2q}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)}\right) * M_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\frac{\iota^o}{2q}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) \\ & * M_n^c\left(\varsigma_{n+2}^o, \varsigma_{n+3}^o, \frac{\frac{\iota^o}{2q}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) * \cdots * M_n^c\left(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o, \frac{\frac{\iota^o}{2q}}{\alpha_n(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o)}\right) \end{aligned}$$

and

$$N_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o) \leq N_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\frac{\iota^o}{2q}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)}\right) \circ N_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\frac{\iota^o}{2q}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right)$$
$$\circ N_n^c\left(\varsigma_{n+2}^o, \varsigma_{n+3}^o, \frac{\frac{\iota^o}{2q}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) \circ \cdots \circ N_n^c\left(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o, \frac{\frac{\iota^o}{2q}}{\alpha_n(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o)}\right).$$

Using (3.-2) and (3.-1), we have

$$\begin{aligned}
&M_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o)\\
&\geq M_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)K^n}\right) * M_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)K^{n+1}}\right)\\
&* M_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)K^{n+2}}\right) * \ldots * M_n^c\left(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o, \frac{\frac{\iota^o}{2q}}{\alpha_n(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o)K^{2q-1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&N_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o)\\
&\leq N_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)K^n}\right) \circ N_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)K^{n+1}}\right)\\
&\circ N_n^c\left(\varsigma_0^o, \varsigma_1^o, \frac{\frac{\iota^o}{2q}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)K^{n+2}}\right) \circ \ldots \circ N_n^c\left(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o, \frac{\frac{\iota^o}{2q}}{\alpha_n(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o)K^{2q-1}}\right)
\end{aligned}$$

applying limit $n \to \infty$, we have

$$\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o) \geq 1 * 1 * 1 * \cdots * 1 = 1$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o) \leq 0 \circ 0 \circ 0 \circ \cdots \circ 0 = 0.$$

Thus in both cases, we have

$$\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma_{n+p}^o, \iota^o) = 1$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma_{n+p}^o, \iota^o) = 0.$$

Showing $\{\varsigma_n^o\}$ is Cauchy in Σ and converges in Σ, so

$$\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma^o, \iota^o) = 1.$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma^o, \iota^o) = 0.$$

Further, to examine that ς^o is the fixed point of Ξ. Again, two cases occurred as previous.

Case 1. When $n = 2q+1$ (odd), we take $\iota^o = \frac{(2q+1)\iota^o}{2q+1} = \frac{\iota^o}{2q+1} + \frac{\iota^o}{2q+1} + \ldots + \frac{\iota^o}{2q+1}$, we have

$$\begin{aligned}
&M_n^c(\varsigma^o, \Xi\varsigma^o, \iota^o)\\
&\geq M_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{(2q+1)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) * M_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota^o}{(2q+1)\alpha_2(\varsigma_n^o, \varsigma_{n+1}^o)}\right)\\
&\quad * M_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\iota^o}{(2q+1)\alpha_3(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) * \cdots\\
&\quad * M_n^c\left(\varsigma_{2q+1}^o, \Xi\varsigma^o, \frac{\iota^o}{(2q+1)\alpha_n(\varsigma_{2q+1}^o, \Xi\varsigma^o)}\right)\\
&\geq M_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{2q+1)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) * M_n^c\left(\Xi\varsigma_n^o, \Xi\varsigma_{n+1}^o, \frac{\iota^o}{(2q+1)\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right)\\
&\quad * M_n^c\left(\Xi\varsigma_{n+1}^o, \Xi\varsigma_{n+2}^o, \frac{\iota^o}{(2q+1)\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) * \ldots\\
&\quad * M_n^c\left(\Xi\varsigma_{2q}^o, \Xi\varsigma^o, \frac{\iota^o}{(2q+1)\alpha_n(\varsigma_{2q+1}^o, \Xi\varsigma^o)}\right)\\
&\geq M_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{(2q+1)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) * M_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota^o}{(2q+1)\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)K}\right)\\
&\quad * M_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\iota^o}{(2q+1)\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)K}\right) * \cdots\\
&\quad * M_n^c\left(\varsigma_{2q}^o, \varsigma^o, \frac{\iota^o}{(2q+1)\alpha_n(\varsigma_{2q+1}^o, \Xi\varsigma^o)K}\right)\\
&\to 1 * 1 * 1 = 1,
\end{aligned}$$

as $n \to \infty$.

Also, we have

$$\begin{aligned}
&N_n^c(\varsigma^o, \Xi\varsigma^o, \iota^o)\\
&\leq N_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{(2q+1)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) \circ N_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota^o}{(2q+1)\alpha_2(\varsigma_n^o, \varsigma_{n+1}^o)}\right)\\
&\quad \circ N_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\iota^o}{(2q+1)\alpha_3(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) \circ \cdots\\
&\quad \circ N_n^c\left(\varsigma_{2q+1}^o, \Xi\varsigma^o, \frac{\iota^o}{(2q+1)\alpha_n(\varsigma_{2q+1}^o, \Xi\varsigma^o)}\right)\\
&\leq N_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{2q+1)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) \circ N_n^c\left(\Xi\varsigma_n^o, \Xi\varsigma_{n+1}^o, \frac{\iota^o}{(2q+1)\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right)\\
&\quad \circ N_n^c\left(\Xi\varsigma_{n+1}^o, \Xi\varsigma_{n+2}^o, \frac{\iota^o}{(2q+1)\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) \circ \cdots
\end{aligned}$$

$$\circ N_n^c\left(\Xi\varsigma_{2q}^o, \Xi\varsigma^o, \frac{\iota^o}{(2q+1)\alpha_n(\varsigma_{2q+1}^o, \Xi\varsigma^o)}\right)$$

$$\leq N_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{(2q+1)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) \circ N_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota^o}{(2q+1)\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)K}\right)$$

$$\circ N_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\iota^o}{(2q+1)\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)K}\right) \circ \cdots$$

$$\circ N_n^c\left(\varsigma_{2q}^o, \varsigma^o, \frac{\iota^o}{(2q+1)\alpha_n(\varsigma_{2q+1}^o, \Xi\varsigma^o)K}\right)$$

$$\to 0 \circ 0 \circ 0 = 0,$$

as $n \to \infty$.

Case 2.

When n = 2q (even), we take $\iota^o = \frac{(2q)\iota^o}{2q} = \frac{\iota^o}{2q} + \frac{\iota^o}{2q} + \cdots + \frac{\iota^o}{2q}$,

we have

$$M_n^c(\varsigma^o, \Xi\varsigma^o, \iota^o)$$

$$\geq M_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{(2q)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) * M_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota^o}{(2q)\alpha_2(\varsigma_n^o, \varsigma_{n+1}^o)}\right)$$

$$* M_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\iota^o}{(2q)\alpha_3(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) * \cdots$$

$$* M_n^c\left(\varsigma_{2q}^o, \Xi\varsigma^o, \frac{\iota^o}{(2q)\alpha_n(\varsigma_{2q}^o, \Xi\varsigma^o)}\right)$$

$$\geq M_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{2q)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) * M_n^c\left(\Xi\varsigma_n^o, \Xi\varsigma_{n+1}^o, \frac{\iota^o}{(2q)\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right)$$

$$* M_n^c\left(\Xi\varsigma_{n+1}^o, \Xi\varsigma_{n+2}^o, \frac{\iota^o}{(2q)\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) * \cdots$$

$$* M_n^c\left(\Xi\varsigma_{2q}^o, \Xi\varsigma^o, \frac{\iota^o}{(2q)\alpha_n(\varsigma_{2q+1}^o, \Xi\varsigma^o)}\right)$$

$$\geq M_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{(2q)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) * M_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota^o}{(2q)\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)K}\right)$$

$$* M_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\iota^o}{(2q)\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)K}\right) * \cdots$$

$$* M_n^c\left(s_{2q}, \varsigma^o, \frac{\iota^o}{(2q)\alpha_n(\varsigma_{2q}^o, \Xi\varsigma^o)K}\right)$$

$$\to 1 * 1 * 1 = 1,$$

as $n \to \infty$.

$$
\begin{aligned}
& N_n^c(\varsigma^o, \Xi\varsigma^o, \iota^o) \\
& \leq N_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{(2q)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) \circ N_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota^o}{(2q)\alpha_2(\varsigma_n^o, \varsigma_{n+1}^o)}\right) \\
& \quad \circ N_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\iota^o}{(2q)\alpha_3(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) \circ \cdots \\
& \quad \circ M_n^c\left(\varsigma_{2q}^o, \Xi\varsigma^o, \frac{\iota^o}{(2q)\alpha_n(\varsigma_{2q}^o, \Xi\varsigma^o)}\right) \\
& \leq N_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{2q)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) \circ N_n^c\left(\Xi\varsigma_n^o, \Xi\varsigma_{n+1}^o, \frac{\iota^o}{(2q)\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) \\
& \quad \circ N_n^c\left(\Xi\varsigma_{n+1}^o, \Xi\varsigma_{n+2}^o, \frac{\iota^o}{(2q)\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) \circ \cdots \\
& \quad \circ N_n^c\left(\Xi\varsigma_{2q}^o, \Xi\varsigma^o, \frac{\iota^o}{(2q)\alpha_n(\varsigma_{2q+1}^o, \Xi\varsigma^o)}\right) \\
& \leq N_n^c\left(\varsigma^o, \varsigma_n^o, \frac{\iota^o}{(2q)\alpha_1(\varsigma^o, \varsigma_n^o)}\right) \circ N_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota^o}{(2q)\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)K}\right) \\
& \quad \circ N_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\iota^o}{(2q)\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)K}\right) \circ \cdots \\
& \quad \circ N_n^c\left(\varsigma_{2q}^o, \varsigma^o, \frac{\iota^o}{(2q)\alpha_n(\varsigma_{2q}^o, \Xi\varsigma^o)K}\right) \\
& \to 0 \circ 0 \circ 0 = 0,
\end{aligned}
$$

as $n \to \infty$. Hence in either cases, ς^o is the fixed point of Ξ.

Uniqueness. Assume $\Xi\varsigma^{o'} = \varsigma^{o'}$ for any other $\varsigma^{o'} \in \Sigma$, then

$$M_n^c(\varsigma^o, \varsigma^{o'}, \iota^o) = M_n^c(\Xi\varsigma^o, \Xi\varsigma^{o'}, \iota^o) \geq M_n^c\left(\varsigma^o, \varsigma^{o'}, \frac{\iota^o}{K}\right),$$

$$N_n^c(\varsigma^o, \varsigma^{o'}, \iota^o) = N_n^c(\Xi\varsigma^o, \Xi\varsigma^{o'}, \iota^o) \leq N_n^c\left(\varsigma^o, \varsigma^{o'}, \frac{\iota^o}{K}\right),$$

which is a contradiction. Which shows the uniqueness of ς^o.

Definition 3.2. *Let $(\Sigma, M_n^c, N_n^c, \alpha_n, *, \circ)$ be an IFnCMS. Then the mapping $\Xi : \Sigma \to \Sigma$ is said to be a generalized $\alpha - \phi$-intuitionistic fuzzy contractive mapping if for two functions $\phi \in \Phi$ and $\alpha : \Sigma \times \Sigma \times (0, \infty) \to [0, \infty)$, we obtain*

$$\alpha(\varsigma_1^o, \varsigma_2^o, \iota^o)\left(\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right) \leq \varphi\left(\frac{1}{M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)} - 1\right) \tag{3.-11}$$

and

$$\alpha(\varsigma_1^o, \varsigma_2^o, \iota^o) N_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o) \leq \varphi(N_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)) \tag{3.-10}$$

for all $\varsigma_1^o, \varsigma_2^o \in \Sigma$ *and* $\iota^o > 0$.

Theorem 3.2. *Let* $(\Sigma, M_n^c, \alpha_n, *, \circ)$ *be a complete IFnCMS with* $\alpha_n : \Sigma \times \Sigma \to [1, \infty)$ *be n non-comparable functions and* $\Xi : \Sigma \to \Sigma$ *be generalized* $\alpha - \phi-$*intuitionistic fuzzy contractive mapping fulfilling:*
(i) Ξ *is* α*-admissible,*
(ii) *for all* $\iota^o > 0$, *there exists* $\varsigma_0^o \in \Sigma$ *fulfilling* $\alpha(\varsigma_0^o, \Xi\varsigma_0^o, \iota^o) \geq 1$,
(iii) Ξ *is continuous.*
Then, Ξ *has a fixed point.*

Proof. Suppose $\varsigma_0^o \in \Sigma_0$ is an arbitrary element and let for all $n \in N$, the sequence ς_n^o in Σ by the formula is given by $\varsigma_n^o = \Xi\varsigma_{n-1}^o$. Let for all $n \in \mathbb{N}$, $\varsigma_n^o \neq \varsigma_{n-1}^o$. Therefore, Ξ is α-admissible and

$\alpha(\varsigma_0^o, \varsigma_1^o, \iota^o) = \alpha(\varsigma_0^o, \Xi\varsigma_0^o, \iota^o) \geq 1$, so for any $\iota^o > 0$, we deduce $\alpha(\varsigma_1^o, \varsigma_2^o, \iota^o) = \alpha(\Xi\varsigma_0^o, \Xi\varsigma_1^o, \iota^o) \geq 1$. Ultimately, $\alpha(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o) \geq 1$. Now using 3.-11 and 3.-10, we can write

$$\begin{aligned}
\frac{1}{M_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o)} - 1 &= \frac{1}{M_n^c(\Xi\varsigma_{n-1}^o, \Xi\varsigma_n^o, \iota^o)} - 1 \\
&\leq \alpha(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o)\left(\frac{1}{M_n^c(\Xi\varsigma_{n-1}^o, \Xi\varsigma_n^o, \iota^o)} - 1\right) \\
&\leq \varphi\left(\frac{1}{M_n^c(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o)} - 1\right)
\end{aligned} \tag{3.-9}$$

and

$$\begin{aligned}
N_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) &= N_n^c(\Xi\varsigma_{n-1}^o, \Xi\varsigma_n^o, \iota^o) \\
&\leq \alpha(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o)\left(N_n^c(\Xi\varsigma_{n-1}^o, \Xi\varsigma_n^o, \iota^o)\right) \\
&\leq \varphi\left(N_n^c(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o)\right).
\end{aligned} \tag{3.-8}$$

Now, as $\varphi(\iota^o) < \iota^o$, we have

$$\frac{1}{M_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o)} - 1 \leq \varphi\left(\frac{1}{M_n^c(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o)} - 1\right) < \frac{1}{M_n^c(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o)} - 1 \tag{3.-7}$$

and

$$N_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) \leq \varphi\left(N_n^c(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o)\right) < N_n^c(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o). \tag{3.-6}$$

Hence, $F_c^n(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) > M_n^c(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o))$ and $N_c^n(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) < N_n^c(\varsigma_{n-1}^o, \varsigma_n^o, \iota^o))$. So, the sequence $\{M_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o)\}$ is strictly increasing and the sequence $\{N_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o)\}$ is strictly decreasing in $[0, 1]$, for all $\iota^o > 0$. Let for all $\iota^o > 0$, $S(\iota^o) = \lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o)$ and $L(\iota^o) = \lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o)$. We claim that $S(\iota^o) = 1$ and $L(\iota^o) = 0$. On the contrary, assume $S^*(\iota_0^o) < 1$ and $L^*(\iota_0^o) > 0$, for some $\iota_0^o > 0$. Taking the limit on both sides of 3.-7 and 3.-6, we have

$$\frac{1}{S^*(t_0)} - 1 \leq \varphi\left(\frac{1}{S^*(\iota_0^o)} - 1\right) < \frac{1}{S^*(\iota_0^o)} - 1$$

and

$$L^*(\iota_0^o) \leq \varphi\left(L^*(\iota_0^o)\right) < L^*(\iota_0^o),$$

which are the contradictions. Thus, we have

$$\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) = 1, \quad \iota^o > 0 \tag{3.-5}$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma_{n+1}^o, \iota^o) = 0, \quad \iota^o > 0. \tag{3.-4}$$

To prove the Cauchyness of $\{\varsigma_n^o\}$, consider the cases as follows:

Case 1. When $p = 2q+1$ (odd), we take $\iota^o = \frac{(2q+1)\iota^o}{2q+1} = \frac{\iota^o}{2q+1} + \frac{\iota^o}{2q+1} + \ldots + \frac{\iota^o}{2q+1}$, we have

$$\begin{aligned} M_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \geq{} & M_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)}\right) * M_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) \\ & * M_n^c\left(\varsigma_{n+2}^o, \varsigma_{n+3}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) * \cdots * M_n^c\left(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_n(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o)}\right) \end{aligned}$$

and

$$\begin{aligned} N_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \leq{} & N_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)}\right) \circ N_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) \\ & \circ N_n^c\left(\varsigma_{n+2}^o, \varsigma_{n+3}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) \circ \cdots \circ N_n^c\left(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o, \frac{\frac{\iota^o}{2q+1}}{\alpha_n(\varsigma_{n+2q}^o, \varsigma_{n+2q+1}^o)}\right). \end{aligned}$$

Now, applying limit $n \to \infty$, we have

$$\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \geq 1 * 1 * 1 = 1$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma_{n+2q+1}^o, \iota^o) \leq 0 \circ 0 \circ 0 = 0.$$

Case 2.

$$\text{When p = 2q (even), we take } \iota^o = \frac{(2q)\iota^o}{2q} = \frac{\iota^o}{2q} + \frac{\iota^o}{2q} + \cdots + \frac{\iota^o}{2q},$$

we have

$$\begin{aligned} M_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o) \geq{} & M_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\frac{\iota^o}{2q}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)}\right) * M_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\frac{\iota^o}{2q}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right) \\ & * M_n^c\left(\varsigma_{n+2}^o, \varsigma_{n+3}^o, \frac{\frac{\iota^o}{2q}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) * \cdots * M_n^c\left(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o, \frac{\frac{\iota^o}{2q}}{\alpha_n(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o)}\right) \end{aligned}$$

and

$$N_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o) \leq N_n^c\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\frac{\iota^o}{2q}}{\alpha_1(\varsigma_n^o, \varsigma_{n+1}^o)}\right) \circ N_n^c\left(\varsigma_{n+1}^o, \varsigma_{n+2}^o, \frac{\frac{\iota^o}{2q}}{\alpha_2(\varsigma_{n+1}^o, \varsigma_{n+2}^o)}\right)$$
$$\circ N_n^c\left(\varsigma_{n+2}^o, \varsigma_{n+3}^o, \frac{\frac{\iota^o}{2q}}{\alpha_3(\varsigma_{n+2}^o, \varsigma_{n+3}^o)}\right) \circ \cdots \circ N_n^c\left(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o, \frac{\frac{\iota^o}{2q}}{\alpha_n(\varsigma_{n+2q-1}^o, \varsigma_{n+2q}^o)}\right).$$

By taking the limit $n \to \infty$, we get

$$\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o) \geq 1 * 1 * 1 * 1 = 1$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma_{n+2q}^o, \iota^o) \leq 0 \circ 0 \circ 0 \circ 0 = 0.$$

Hence in either cases, $\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma_{n+p}^o, \iota^o) = 1$ and $\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma_{n+p}^o, \iota^o) = 0$, showing Cauchyness of $\{\varsigma_n^o\}$ and convergence to $s \in \Sigma$, so

$$\lim_{n\to\infty} M_n^c(\varsigma_n^o, \varsigma^o, \iota^o) = 1$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_n^o, \varsigma^o, \iota^o) = 0.$$

Now as Ξ is continuous, we get $\Xi\varsigma_n^o \to \Xi\varsigma^o$ for all $\iota^o > 0$. Now, we have

$$\lim_{n\to\infty} M_n^c(\varsigma_{n+1}^o, \Xi\varsigma^o, \iota^o) = \lim_{n\to\infty} M_n^c(\Xi\varsigma_n^o, \Xi\varsigma^o, \iota^o) = 1$$

and

$$\lim_{n\to\infty} N_n^c(\varsigma_{n+1}^o, \Xi\varsigma^o, \iota^o) = \lim_{n\to\infty} N_n^c(\Xi\varsigma_n^o, \Xi\varsigma^o, \iota^o) = 0,$$

for all $\iota^o > 0$, that is $\varsigma_n^o \to \Xi\varsigma^o$. The uniqueness of the limit implies that $\Xi\varsigma^o = \varsigma^o$, i.e., ς^o is the fixed point of Ξ. □

Theorem 3.3. *Let $\alpha_n : \Sigma \times \Sigma \to [1, \infty)$ and $(\Sigma, M_n^c, N_n^c, \alpha_n, *, \circ)$ be a complete IFnCMS with*

$$\lim_{\iota^o\to\infty} M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) = 1 \text{ and } \lim_{\iota^o\to\infty} N_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) = 0. \tag{3.-3}$$

Also, let Ξ be a self-mapping on Σ satisfying:

$$\left[\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right] \leq K\left[\frac{1}{M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)} - 1\right] \tag{3.-2}$$

and

$$N_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o) \leq K N_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) \tag{3.-1}$$

for all $\varsigma_1^o, \varsigma_2^o \in \Sigma$. Then the mapping Ξ has a unique fixed point in Σ.

Proof. It is easy to show on the lines of Theorems 3.1 and 3.2. □

Example 3.2. *Suppose*

$$\Sigma_1 = \left\{\frac{p}{q} : p = 0, 1, 3, 9, \ldots, \text{ and } q = 1, 4, \ldots, 3k+1, \ldots\right\},$$

$$\Sigma_2 = \left\{\frac{p}{q} : p = 1, 3, 9, \ldots, \text{ and } q = 2, 5, \ldots, 3k+2, \ldots\right\},$$

$$\Sigma_3 = \{2k : k \in \mathbb{N}\},$$

and $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. *Let* $\iota_1^o * \iota_1^o = \iota_1^o \iota_2^o$ *and* $\iota_1^o \circ \iota_1^o = \max\{\iota_1^o, \iota_1^o\}$ *for all* $\iota_1^o, \iota_2^o \in [0,1]$ *and*

$$M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) = \frac{\iota^o}{\iota^o + |\varsigma_1^o - \varsigma_2^o|^n} \text{ and } N_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) = \frac{|\varsigma_1^o - \varsigma_2^o|^n}{\iota^o + |\varsigma_1^o - \varsigma_2^o|^n}$$

for all $\varsigma_1^o, \varsigma_2^o \in \Sigma$ *with* $\alpha_n : \Sigma \times \Sigma \to [1, \infty)$ *and* $\iota^o > 0$. *Define* $\Xi : \Sigma \to \Sigma$ *by*

$$\Xi\varsigma^o = \begin{cases} \frac{3\varsigma^o}{11}, & \varsigma^o \in \Sigma_1, \\ \frac{\varsigma^o}{8}, & \varsigma^o \in \Sigma_2, \\ 2\varsigma^o, & \varsigma^o \in \Sigma_3. \end{cases}$$

If $\varsigma_1^o, \varsigma_2^o \in \Sigma_1$, *then*

$$\begin{aligned}\left(\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right) &= \frac{\left|\frac{3\varsigma_1^o}{11} - \frac{3\varsigma_2^o}{11}\right|^n}{\iota^o} \\ &\leq \left(\frac{3}{11}\right)^n \frac{\left|\varsigma_1^o - \varsigma_2^o\right|^n}{\iota^o} = K\left(\frac{1}{M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)} - 1\right)\end{aligned}$$

and

$$\begin{aligned}N_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o) &= \frac{\left|\frac{3\varsigma_1^o}{11} - \frac{3\varsigma_2^o}{11}\right|^n}{\iota^o + \left|\frac{3\varsigma_1^o}{11} - \frac{3\varsigma_2^o}{11}\right|^n} \\ &= \frac{\left|\varsigma_1^o - \varsigma_2^o\right|^n}{\left(\frac{11}{3}\right)^n \iota^o + \left|\varsigma_1^o - \varsigma_2^o\right|^n} \\ &\leq K\frac{\left|\varsigma_1^o - \varsigma_2^o\right|^n}{\iota^o + \left|\varsigma_1^o - \varsigma_2^o\right|^n} = KN_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o).\end{aligned}$$

If $\varsigma_1^o, \varsigma_2^o \in \Sigma_2$, *then*

$$\begin{aligned}\left(\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right) &= \frac{\left|\frac{\varsigma_1^o}{8} - \frac{\varsigma_2^o}{8}\right|^n}{\iota^o} = \left(\frac{1}{8}\right)^n \frac{\left|\varsigma_1^o - \varsigma_2^o\right|^n}{\iota^o} \\ &\leq K\left(\frac{1}{M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)} - 1\right)\end{aligned}$$

and

$$\begin{aligned} N_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o) &= \frac{\left|\frac{\varsigma_1^o}{8} - \frac{\varsigma_2^o}{8}\right|^n}{\iota^o + \left|\frac{\varsigma_1^o}{8} - \frac{\varsigma_2^o}{8}\right|^n} \\ &= \frac{\left|\varsigma_1^o - \varsigma_2^o\right|^n}{(8)^n\iota^o + \left|\varsigma_1^o - \varsigma_2^o\right|^n} \\ &\leq K\frac{\left|\varsigma_1^o - \varsigma_2^o\right|^n}{t + \left|\varsigma_1^o - \varsigma_2^o\right|^n} = KN_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o). \end{aligned}$$

If $\varsigma_1^o, \varsigma_2^o \in \Sigma_3$, then (3.-2) and (3.-1) similarly holds. Now, if $\varsigma_1^o \in \Sigma_1$ and $\varsigma_2^o \in \Sigma_2$, then

$$\left(\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right) = \frac{\left|\frac{3\varsigma_1^o}{11} - \frac{\varsigma_2^o}{8}\right|^n}{\iota^o} = \left(\frac{3}{11}\right)^n \frac{\left|\varsigma_1^o - \frac{11}{24}\varsigma_2^o\right|^n}{\iota^o}$$

So, if $\varsigma_1^o > \frac{11}{24}\varsigma_2^o$, then

$$\begin{aligned} \left(\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right) &= \left(\frac{3}{11}\right)^n \frac{\left|\varsigma_1^o - \frac{11}{24}\varsigma_2^o\right|^n}{\iota^o} \leq \left(\frac{3}{11}\right)^n \frac{\left|\varsigma_1^o - \varsigma_2^o\right|^n}{\iota^o} \\ &\leq K\left(\frac{1}{M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)} - 1\right), \end{aligned}$$

and if $\varsigma_1^o < \frac{11}{24}\varsigma_2^o$, then

$$\begin{aligned} \left(\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right) &= \left(\frac{3}{11}\right)^n \frac{\left|\frac{11}{24}\varsigma_2^o - \varsigma_1^o\right|^n}{\iota^o} \leq \left(\frac{3}{11}\right)^n \frac{\left|\varsigma_2^o - \varsigma_1^o\right|^n}{\iota^o} \\ &\leq K\left(\frac{1}{M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)} - 1\right). \end{aligned}$$

We see that

$$\left(\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right) \leq K\left(\frac{1}{M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)} - 1\right)$$

and

$$\begin{aligned} N_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o) &= \frac{\left|\frac{3\varsigma_1^o}{11} - \frac{\varsigma_2^o}{8}\right|^n}{\iota^o + \left|\frac{3\varsigma_1^o}{11} - \frac{\varsigma_2^o}{8}\right|^n} \\ &= \frac{\left|\varsigma_1^o - \frac{11\varsigma_2^o}{24}\right|^n}{\left(\frac{11}{3}\right)^n \iota^o + \left|\varsigma_1^o - \frac{11\varsigma_2^o}{24}\right|^n} \\ &\leq K\frac{\left|\varsigma_1^o - \varsigma_2^o\right|^n}{\iota^o + \left|\varsigma_1^o - \varsigma_2^o\right|^n} = KN_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o) \end{aligned}$$

for all $\varsigma_1^o, \varsigma_2^o \in \Sigma_1 \cup \Sigma_2$*. So, we get*

$$\left(\frac{1}{M_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o)} - 1\right) \leq K\left(\frac{1}{M_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)} - 1\right)$$

and

$$N_n^c(\Xi\varsigma_1^o, \Xi\varsigma_2^o, \iota^o) \leq K N_n^c(\varsigma_1^o, \varsigma_2^o, \iota^o)$$

for all $\varsigma_1^o, \varsigma_2^o \in \Sigma$*. That is, all conditions of Theorem 3.3 fulfilled. Hence,* 0 *is a unique fixed point of* Ξ*.*

4. Application to Fractional Differential Equations

In this section, we provide an application to fractional differential equations. We apply main result to find the existence and uniqueness of the fractional differential equation.

Scientific research has significantly improved through the use of fractional calculus. It works with the variable derivative, which increases accuracy and facilitates the development of mathematical problem models. However, because it works with derivatives of integer order, a regular derivative was not as useful in this context. Typically, fractional derivatives and integrals are connected to Liouville. However, derivatives of fractional order have already been examined by mathematicians. Leibnitz's research focused on fractional calculus. Euler also contributed to it later on. The contributions of Liouville, Reimann, Abel, Litnikov, Hadamard, Weyl, and a number of other mathematicians from the past and present have made great progress in the study of fractional calculus, which is today a basic problem in mathematics. This section's goal is to show that the following fractional differential equation, which involves the Caputo fractional derivative, has a unique solution:

$$D_0^{\sigma^o} + \varsigma^o(\varepsilon) + g(\varepsilon, \varsigma^o(\varepsilon)) = 0, \quad 0 < \varepsilon < 1, \tag{4.1}$$

where $1 < \sigma^o \leq 2$, and the boundary conditions $\varsigma^o(0) + \varsigma^{o'}(0) = 0$ and $\varsigma^o(1) + \varsigma^{o'}(1) = 0$ are imposed and $g : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Describe a complete IFnCMS $(\Sigma, M_n^c, N_n^c, \alpha_n, *, \circ)$ on $\Sigma = C([0,1], \mathbb{R})$ by:

$$M_n^c(\varsigma^o, \varpi, \iota^o) = \exp\left(-\frac{\sup_{\varepsilon\in[0,1]} |\varsigma^o(\varepsilon) - \varpi(\varepsilon)|^n}{\iota^o}\right)$$

$$N_n^c(\varsigma^o, \varpi, \iota^o) = 1 - \exp\left(-\frac{\sup_{\varepsilon\in[0,1]} |\varsigma^o(\varepsilon) - \varpi(\varepsilon)|^n}{\iota^o}\right)$$

for all $\varsigma^o, \varpi \in \Sigma$, $\iota^o > 0$, where $*$ and $\circ$ operations between ι_1^o and ι_2^o are defined as $\iota_1^o * \iota_2^o = \iota_1^o\iota_2^o$ and $\iota_1^o \circ \iota_2^o = \min\{\iota_1^o, \iota_2^o\}$. It's noteworthy that $\varsigma^o \in \Sigma$ solves (4.1) whenever $\varsigma^o \in \Sigma$ is the solution of:

$$\varsigma^o(\varepsilon) = \frac{1}{\Gamma(\sigma^o)}\int_0^1 (1-ð)^{\sigma^o-1}(1-\varepsilon)g(ð, \varsigma^o(ð))\,dð + \frac{1}{\Gamma(\sigma^o-1)}\int_0^1 (1-ð)^{\sigma^o-2}(1-\varepsilon)g(ð, \varsigma^o(ð))\,dð$$
$$+ \frac{1}{\Gamma(\sigma^o)}\int_\varepsilon^1 (\varepsilon-ð)^{\sigma^o-1}g(ð, \varsigma^o(ð))\,dð.$$

Theorem 4.1. *Let the mapping $H:\Sigma\to\Sigma$ defined as:*

$$H\varsigma^o(\varepsilon) =\frac{1}{\Gamma(\sigma^o)}\int_0^1 (1-ð)^{\sigma^o-1}(1-\varepsilon)g(ð,\varsigma^o(ð))\,dð \quad +\frac{1}{\Gamma(\sigma^o-1)}\int_0^1 (1-ð)^{\sigma^o-2}(1-\varepsilon)g(ð,\varsigma^o(ð))\,dð + \frac{1}{\Gamma(\sigma^o)}\int_0^\varepsilon (\varepsilon-ð)^{\sigma^o-1}g(ð,\varsigma^o(ð))\,dð.$$

Suppose the following conditions hold:

(i) $\forall\, \varsigma^o,\varpi\in\Sigma$, $g:[0,1]\times[0,\infty)\to[0,\infty)$ *verifies that*

$$|g(ð,\varsigma^o(ð))-g(ð,\varpi(ð))|\le K^{\frac{1}{n}}|\varsigma^o(ð)-\varpi(ð)|,$$

(ii)

$$\sup_{\varepsilon\in(0,1)}\left(\frac{1-\varepsilon}{\Gamma(\sigma^o+1)}+\frac{1-\varepsilon}{\Gamma(\sigma^o)}+\frac{\varepsilon^{\sigma^o}}{\Gamma(\sigma^o+1)}\right)^n=\eta<1,$$

Then equation (4.1) has a unique solution.

Proof. *Let $\varsigma^o,\varpi\in\Sigma$ and consider*

$$\begin{aligned}|H\varsigma^o(\varepsilon)-H\varpi(\varepsilon)|^n =&\Bigg|\frac{1}{\Gamma(\sigma^o)}\int_0^1 (1-ð)^{\sigma^o-1}(1-\varepsilon)(g(ð,\varsigma^o(ð))-g(ð,\varpi(ð)))\,dð\\
&+\frac{1}{\Gamma(\sigma^o-1)}\int_0^1 (1-ð)^{\sigma^o-2}(1-\varepsilon)(g(ð,\varsigma^o(ð))-g(ð,\varpi(ð)))\,dð\\
&+\frac{1}{\Gamma(\sigma^o)}\int_0^\varepsilon (\varepsilon-ð)^{\sigma^o-1}(g(ð,\varsigma^o(ð))-g(ð,\varpi(ð)))\,dð\Bigg|^n\\
\le&\Bigg(\frac{1}{\Gamma(\sigma^o)}\int_0^1 (1-ð)^{\sigma^o-1}(1-\varepsilon)|g(ð,\varsigma^o(ð))-g(ð,\varpi(ð))|\,dð\\
&+\frac{1}{\Gamma(\sigma^o-1)}\int_0^1 (1-ð)^{\sigma^o-2}(1-\varepsilon)|g(ð,\varsigma^o(ð))-g(ð,\varpi(ð))|\,dð\\
&+\frac{1}{\Gamma(\sigma^o)}\int_0^\varepsilon (\varepsilon-ð)^{\sigma^o-1}|g(ð,\varsigma^o(ð))-g(ð,\varpi(ð))|\,dð\Bigg)^n\\
\le&\Bigg(\frac{1}{\Gamma(\sigma^o)}\int_0^1 (1-ð)^{\sigma^o-1}(1-\varepsilon)K^{\frac{1}{n}}|\varsigma^o(ð)-\varpi(ð)|\,dð\\
&+\frac{1}{\Gamma(\sigma^o-1)}\int_0^1 (1-ð)^{\sigma^o-2}(1-\varepsilon)K^{\frac{1}{n}}|\varsigma^o(ð)-\varpi(ð)|\,dð\\
&+\frac{1}{\Gamma(\sigma^o)}\int_0^\varepsilon (\varepsilon-ð)^{\sigma^o-1}K^{\frac{1}{n}}|\varsigma^o(ð)-\varpi(ð)|\,dð\Bigg)^n\\
=&K\left|\varsigma^o(ð)-\varpi(ð)\right|^n\left(\frac{1}{\Gamma(\sigma^o)}\int_0^1 (1-ð)^{\sigma^o-1}(1-\varepsilon)\right)dð\\
&+\frac{1}{\Gamma(\sigma^o-1)}\int_0^1 (1-ð)^{\sigma^o-2}(1-\varepsilon)dð\\
&+\frac{1}{\Gamma(\sigma^o)}\int_0^\varepsilon (\varepsilon-ð)^{\sigma^o-1}dð\Bigg)^n\end{aligned}$$

$$= K\left|\varsigma^o(\eth) - \varpi(\eth)\right|^n \left(\frac{1-\varepsilon}{\Gamma(\sigma^o+1)} + \frac{1-\varepsilon}{\Gamma(\sigma^o)} + \frac{\varepsilon^{\sigma^o}}{\Gamma(\sigma^o+1)}\right)^n$$

$$\leq K\left|\varsigma^o(\eth) - \varpi(\eth)\right|^n \sup_{\varepsilon\in[0,1]} \left(\frac{1-\varepsilon}{\Gamma(\sigma^o+1)} + \frac{1-\varepsilon}{\Gamma(\sigma^o)} + \frac{\varepsilon^{\sigma^o}}{\Gamma(\sigma^o+1)}\right)^n$$

$$= \eta.K|\varsigma^o(\eth) - \varpi(\eth)|^n$$

$$\leq K|\varsigma^o(\eth) - \varpi(\eth)|^n.$$

So, we have

$$|H\varsigma^o(\varepsilon) - H\varpi(\varepsilon)|^n \leq K|\varsigma^o(\eth) - \varpi(\eth)|^n,$$

i.e.,

$$-\frac{\sup_{\varepsilon\in[0,1]} |H\varsigma^o(\varepsilon) - H\varpi(\varepsilon)|^n}{K\iota^o} \geq -\frac{\sup_{\varepsilon\in[0,1]} |\varsigma^o(\eth) - \varpi(\eth)|^n}{\iota^o},$$

$$\exp\left(-\frac{\sup_{\varepsilon\in[0,1]} |H\varsigma^o(\varepsilon) - H\varpi(\varepsilon)|^n}{K\iota^o}\right) \geq \exp\left(-\frac{\sup_{\varepsilon\in[0,1]} |\varsigma^o(\eth) - \varpi(\eth)|^n}{\iota^o}\right)$$

and

$$1-\exp\left(-\frac{\sup_{\varepsilon\in[0,1]} |H\varsigma^o(\varepsilon) - H\varpi(\varepsilon)|^n}{K\iota^o}\right) \leq 1-\exp\left(-\frac{\sup_{\varepsilon\in[0,1]} |\varsigma^o(\eth) - \varpi(\eth)|^n}{\iota^o}\right).$$

Thus, we have

$$F_c^n(H\varsigma^o(\varepsilon), H\varpi(\varepsilon), K\iota^o) \geq F_c^n(\varsigma^o(\varepsilon), \varpi(\varepsilon), \iota^o)$$

and

$$N_c^n(H\varsigma^o(\varepsilon), H\varpi(\varepsilon), K\iota^o) \leq N_c^n(\varsigma^o(\varepsilon), \varpi(\varepsilon), \iota^o).$$

All conditions of Theorem 3.1, that is, an equation (4.1) has a unique solution.

5. Conclusion

In this study, we extended the concept of FnCMSs and IFMSs by using n non-comparable functions $\alpha_i : \Sigma\times\Sigma \to [1,\infty)$ $(1 \leq i \leq n)$ in the inequalities having the form $F(\varsigma_1^o, \varsigma_{n+1}^o, \iota_1^o + \iota_2^o + \cdots + \iota_n^o) \geq F\left(\varsigma_1^o, \varsigma_2^o, \frac{\iota_1^o}{\alpha_1(\varsigma_1^o,\varsigma_2^o)}\right) * F\left(\varsigma_2^o, \varsigma_3^o, \frac{\iota_2^o}{\alpha_2(\varsigma_2^o,\varsigma_3^o)}\right) * \cdots * F\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota_n^o}{\alpha_n(\varsigma_n^o,\varsigma_{n+1}^o)}\right)$ $\forall\ \iota_n^o > 0$ and $N(\varsigma_1^o, \varsigma_{n+1}^o, \iota_1^o + \iota_2^o + \cdots + \iota_n^o) \leq N\left(\varsigma_1^o, \varsigma_2^o, \frac{\iota_1^o}{\alpha_1(\varsigma_1^o,\varsigma_2^o)}\right) \circ N\left(\varsigma_2^o, \varsigma_3^o, \frac{\iota_2^o}{\alpha_2(\varsigma_2^o,\varsigma_3^o)}\right) \circ \cdots \circ N\left(\varsigma_n^o, \varsigma_{n+1}^o, \frac{\iota_n^o}{\alpha_n(\varsigma_n^o,\varsigma_{n+1}^o)}\right)$ $\forall\ \iota_n^o > 0$ and presented the notion of IFnCMSs. Further, we presented several fixed point results including an intuitionistic fuzzy version of the Banach contraction principle and generalized $\alpha-\phi-$intuitionistic fuzzy contractive mappings. Furthermore, we provided non-trivial examples and an application to non-linear fractional differential equations to show the validity of the main results.

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