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An Interior Point Algorithm for Quadratic Programming Based on a New Step-Length

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Abstract. Interior point methods have seen significant advancements in recent decades for solving linear, semi-definite and quadratic programming. Among these methods, the logarithmic barrier methods based on approximate functions have polynomial convergence and are known for their favorable numerical performance. In this work, a new minorant function for the barrier method is proposed for solving convex quadratic problems with inequality constraints. The proposed minorant function allows to compute the steplength easily and quickly, unlike the line search method, which is computationally intensive and time-consuming. Mathematical results concerning the convergence of the algorithm are established. The numerical comparisons with the inexact Wolfe line search technique show that the proposed method is promising and effective.

1. Introduction

Consider the following quadratic problem

$$\begin{cases} \min q(x) = & \frac{1}{2}x^tQx + c^tx \\ Ax \geq b, & x \in \mathbb{R}^n. \end{cases} \tag{P}$$

The convex quadratic programming problems are encountered in various fields, including social, economics, public planning and manufacturing [11,12,16,17]. Researchers have developed numerous algorithms that utilize interior point techniques to tackle this class of problems. We classify these methods into three basic classes: affine methods, potential reduction methods, and central trajectory methods. Karmarkar [13] proposed an efficient polynomial algorithm for linear programming. Vaidya and Tse [14] introduced an interior point algorithm based on Karmarkar's

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projective transformation. Chikhaoui et al. [6] proposed an algorithm that solves a quadratic function in its canonical form. On the other hand, the primal-dual path-following methods have become the most common among interior point methods; see for example the work of Achache [1], where he proposed a primal-dual path-following method for convex quadratic programming. Boudjellal et al. [3] have also introduced a primal-dual interior point algorithm based on kernel functions.

Alongside and unrelated to earlier works, different logarithmic barrier interior point methods using approximate functions (majorant and minorant ones) have been considered. They are primarily introduced by Crouzeix and Merikhi [7] for solving a semidefinite programming problem. Later, Menniche and Benterki proposed a barrier method for linear programming based on majorant functions. Inspired by previous works, Chaghoub and Benterki [5] introduced a penalty for convex quadratic programming. On the other hand, Leulmi et al. [8,9] have suggested a barrier method using a new minorant function for semidefinite programming, by providing a new minorant function for linear programming.

Based on the previous discussion, our focus in this paper is to minimize convex quadratic problems. We aim to elaborate an efficient minorant function for the logarithmic barrier method. Contrary to the line search method, the proposed function helps to compute the step length easily and without consuming a lot of time.

We have organized the remainder of this paper as follows: We present the problem and its associated perturbed problem in Section 2, providing the necessary theoretical results. Next, in Section 3 we introduce the new approximate function for solving the perturbed problem, as well as the algorithm description. The final section presents numerical results and draws some conclusions.

2. The statement of the problem and its theoretical study

2.1. **The perturbed associated problem.** Before defining the perturbed unconstrained problem associated to (P), we first require the following mild assumptions:

1. **Positive semidefinitness (PSD)**: Q is a $\mathbb{R}^{n\times n}$ symmetric and positive semidefinite matrix.
2. **Interior point condition (IPC)**: There exists $x_0 \in \mathbb{R}^n$ such that $Ax_0 > b$.
3. **The full rank condition (FR)**: A is a $(m \times n)$ full rank matrix ($\text{rank}(A) = m < n$) .
4. $c \in \mathbb{R}^n$, $b \in \mathbb{R}^p$, the set of optimal solutions of (P) is nonempty and bounded.

We define the perturbed unconstrained problem associated to (P) as follows:

$$\min_{x\in\mathbb{R}^n} q_\eta(x), \tag{P$_\eta$}$$

where the barrier function $q_\eta : \mathbb{R}^n \to (-\infty, +\infty]$ is

$$q_\eta(x) = \begin{cases} q(x) - \eta \sum_{i=1}^{m} \ln < e_i, Ax - b > & \text{if } Ax - b > 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where $< x, y >$ is the scalar product of $x, y \in \mathbb{R}^n$, $(e_1, e_2, ..., e_m)$ is the canonical basis in $\mathbb{R}^m$ and η is a strictly positive barrier scalar.

2.2. **Existence and uniqueness of solution.** To prove that (P_η) has a unique optimal solution, it is sufficient to show that the recession cone of q_η is reduced to zero. We need to use the following lemma provided in [10].

Lemma 2.1. *Assume that Assumption 4 hold, if* $\left(q_\eta\right)_\infty(d) \leq 0$ *and* $Ad > 0$ *then* $d = 0$.

Consider the cone of recession C_{q_η} of q_η that is defined as follows

$$C_{q_\eta} = \{d \in \mathbb{R}^n : (q)_\infty(d) \leq 0, d \geq 0\}, \tag{2.1}$$

amounts to zero, i.e.,

$$(q)_\infty(d) \leq 0 \Rightarrow d = 0$$

where $(q)_\infty(d)$ is the asymptotic function of q_η, which is defined by

$$\left(q_\eta\right)_\infty(d) = \lim_{\alpha \to +\infty} \frac{q_\eta(x_0 + \alpha d) - q_\eta(x_0)}{\alpha} \leq 0,$$

Thus, we can infer that

$$\{d \in \mathbb{R}^n : (q_\eta)_\infty(d) \leq 0\} = \{0\},$$

hence $C_{q_\eta} = \{0\}$. This means that the strictly convex problem (P_η) admits a unique optimal solution x_η^* for each η.

2.3. **The convergence of the perturbed problem.** The solution of the problem (P) reduces to the solution of the series of problems (P_η). The sequence of the solutions x_η of (P_η) should converge to the solution of (P) when η tends to 0. The next lemma addresses this issue.

Lemma 2.2. *(see [5]) Let* $\eta > 0$. *If* x_η *is an optimal solution of the problem* (P_η) *such that* $\lim_{\eta \to 0} x_\eta = x^*$, *then* x^* *is an optimal solution of the problem* (P).

3. The perturbed problem and the corresponding algorithm

We interest now by finding the solution of perturbed problem (P_η).

3.1. **The Newton descent direction.** The necessary and sufficient optimality conditions of the convex problem (P_η) assure that x_η is an optimal solution of (P_η) if and only if it is a solution of the following nonlinear system:

$$\nabla q_\eta(x_\eta) = 0.$$

To address this system, we propose a logarithmic interior point method based on Newton's approach. This method involves generating a sequence of interior points $(x_\eta)_k = x_k$, which converges to the optimal solution of (P_η). Newton's iteration is given by $x_{k+1} = x_k + d_k$, where d_k represents the descent direction obtained by solving the following linear system.

$$\nabla^2 q_\eta(x_\eta) d_k = -\nabla q_\eta(x_\eta). \tag{3.1}$$

Unfortunately, this strategy does not ensure that each iterate x_{k+1} produced by the algorithm is feasible, meaning we cannot guarantee that $A(x_k + d_k) > b$. To overcome this drawback, we incorporate a step length α_k at each iteration, which assures the feasibility of the new point $x_{k+1} = x_k + \alpha_k d_k$.

3.2. **Computation of the step-length.** There are two main techniques to determine the step-length α_k:

(1) **Line search methods**: such as Fibonacci, Armijo, Goldstein or Wolfe methods. These techniques are time-consuming and very sensitive. They aim to minimize the following unidimensional functions

$$\phi(\alpha) = \min_{\alpha>0} q_\eta(x_\eta + \alpha d).$$

(2) **Minorant function:** This technique was originally proposed by Leulmi et al. [8] for positive semidefinite programming problems; it relies on approximating the function

$$G(\alpha) = \frac{1}{\eta}(q_\eta(x_\eta + \alpha d) - q_\eta(x_\eta)), \tag{3.2}$$

by another function whose minimum is easily computed. This function permits to determine the step-length at each iteration quickly with a smaller number of instructions, in contrast to line search techniques.

Now, let $\widehat{\alpha} = \min\limits_{i \in I_-}\{\frac{-1}{y_i}\}$, with $y_i = \frac{<e_i,Ad>}{<e_i,Ax-b>}$, $I_- = \{i : y_i < 0\}$ and $i = 1, \dots, m$. In the following result we establish that for $\alpha \in [0, \widehat{\alpha}[$, the function function G can be written in the following way

$$G(\alpha) = \frac{1}{\eta}\left(\frac{1}{2}\alpha^2 d^t Qd - \alpha d^t Qd\right) + \alpha\left(\sum_{i=1}^m y_i - \|y\|^2\right) - \sum_{i=1}^m \ln(1 + \alpha y_i),$$

where and $\|y\|$ denotes the Euclidean norm of $y \in \mathbb{R}^n$.

Proposition 3.1. *For all $\alpha \in [0, \widehat{\alpha}[$, the function G given in equation* (3.2) *can be written as*

$$G(\alpha) = \frac{1}{\eta}\left(\frac{1}{2}\alpha^2 d^T Qd - \alpha d^T Qd\right) - \sum_{i=1}^m \ln(1 + \alpha y_i) + \alpha\left(\sum_{i=1}^m y_i - \|y\|^2\right), \tag{3.3}$$

where $\widehat{\alpha} = \min\limits_{i \in I_-}\{\frac{-1}{y_i}\}$, with $y_i = \frac{<e_i,Ad>}{<e_i,Ax-b>}$, $I_- = \{i : y_i < 0\}$ and $i = 1, \dots, m$.

Proof. From Equation (3.2), we have

$$\begin{aligned} G(\alpha) &= \frac{1}{\eta}(q_\eta(x + \alpha d) - q_\eta(x)) \\ &= \frac{1}{\eta}c^T(x + \alpha d) + \frac{1}{2\eta}(x + \alpha d)^T Q(x + \alpha d) - \frac{1}{2\eta}x^T Qx - \frac{1}{\eta}c^T x - \sum_{i=1}^m \ln\left(1 + \alpha\frac{< e_i, Ad >}{< e_i, Ax - b >}\right) \\ &= \frac{1}{2\eta}\alpha^2 d^T Qd + \frac{1}{2\eta}\alpha x^T Qd + \frac{1}{2\eta}\alpha d^T Qx + \frac{1}{\eta}\alpha c^T d - \sum_{i=1}^m \ln\left(1 + \alpha\frac{< e_i, Ad >}{< e_i, Ax - b >}\right), \end{aligned}$$

therefore, since Q is symmetric, it results that

$$G(\alpha) = \frac{1}{2\eta}\alpha^2 d^T Q d + \frac{1}{\eta}\alpha d^T Q x + \frac{1}{\eta}\alpha c^T d - \sum_{i=1}^{m} \ln\left(1 + \alpha \frac{< e_i, Ad >}{< e_i, Ax - b >}\right).$$

On the other hand, from Assumption (4), and since

$$\nabla q_\eta(x) = Qx - \eta \sum_{i=1}^{m} \frac{A^T e_i}{< e_i, Ax - b >} + c,$$

and

$$\nabla^2 q_\eta(x) = Q + \eta \sum_{i=1}^{m} \frac{A^T e_i (A^T e_i)^T}{(< e_i, Ax - b >)^2},$$

then, from relation (3.1) we obtain

$$d^T \nabla^2 q_\eta(x) d = -d^T \nabla q_\eta(x).$$

Therefore,

$$d^T Q x + d^T c = -d^T Q d - \eta \sum_{i=1}^{m} \frac{< e_i, Ad >^2}{< e_i, Ax - b >^2} + \eta d^T \sum_{i=1}^{m} \frac{A^T e_i}{< e_i, Ax - b >}$$

which implies that,

$$\begin{aligned} G(\alpha) &= \frac{1}{\eta}\left(\frac{1}{2}\alpha^2 d^T Q d - \alpha d^T Q d - \alpha\eta \sum_{i=1}^{m} \frac{< e_i, Ad >^2}{< e_i, Ax - b >^2} + \alpha\eta d^T \sum_{i=1}^{m} \frac{A^T e_i}{< e_i, Ax - b >} - \sum_{i=1}^{m} \ln\left(1 + \alpha \frac{< e_i, Ad >}{< e_i, Ax - b >}\right)\right) \\ &= \frac{1}{\eta}\left(\frac{1}{2}\alpha^2 d^T Q d - \alpha d^T Q d\right) + \alpha\left(\sum_{i=1}^{m} y_i - \|y\|^2\right) - \sum_{i=1}^{m} \ln(1 + \alpha y_i). \end{aligned}$$

and the proof is complete. □

3.3. **The new minorant function.** Before providing our new minorant function , which is the main point of the paper, let's talk about the following useful inequalities, which are related to a statistical series $\{y_1, y_2, \ldots, y_n\}$ of n real numbers. Wolkowicz et al. [15] proved that

$$\begin{aligned} \bar{y} - \sigma_y \sqrt{n-1} &\leq \min_i y_i \leq \bar{y} - \frac{\sigma_y}{\sqrt{n-1}}, \\ \bar{y} + \frac{\sigma_y}{\sqrt{n-1}} &\leq \max_i y_i \leq \bar{y} + \sigma_y \sqrt{n-1}. \end{aligned} \tag{3.4}$$

where $\bar{y}$ and σ_y are reactively the mean and standard deviation

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i, \quad \sigma_y^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - \bar{y}^2 = \frac{1}{n}\sum_{i=1}^{n} (y_i - \bar{y})^2.$$

Also, Crouzeix and Merikhi [7] proposed following useful inequalities related to the maximum and the minimum of a statistical series with $y_i > 0$ for $i = 1, \ldots, n$

$$n \ln\left(\bar{y} - \sigma_z \sqrt{n-1}\right) \leq A \leq \sum_{i=1}^{n} \ln(y_i) \leq B \leq n \ln(\bar{y}), \tag{3.5}$$

where

$$A = (n-1)\ln\left(\bar{y} + \frac{\sigma_y}{\sqrt{n-1}}\right) + \ln\left(\bar{y} - \sigma_y\sqrt{n-1}\right)$$

and

$$B = \ln\left(\bar{y} + \sigma_y\sqrt{n-1}\right) + (n-1)\ln\left(\bar{y} - \frac{\sigma_y}{\sqrt{n-1}}\right),$$

Now, in order to compute the step-length α_k, we have proposed a simpler minorant function , noted $\widetilde{G}$, which is defined as

$$\widetilde{G}(\alpha) = (n\bar{y} - \|y\|^2)\alpha - \frac{1}{\eta}\widehat{\alpha}d^tQd - \tau\ln(1+\phi\alpha), \quad \forall\alpha > 0, \ 0 < \tau < 1.$$

where $\phi = \frac{\|y\|^2}{(n-1)\beta+\gamma}$, $\beta = \bar{y} - \frac{\sigma_y}{\sqrt{n-1}}$ and $\gamma = \bar{y} + \sigma_y\sqrt{n-1}$.

Lemma 3.1. *Consider the function G defined in* (3.3)*; then for all $\alpha > 0$, the proposed minorant function $\widetilde{G}$ is strictly convex and we have:*

$$\widetilde{G}(\alpha) \leq G(\alpha).$$

Proof. Let's define the following function

$$\begin{aligned} P(\alpha) =& \alpha\left(\sum_{i=1}^m y_i - \|y\|^2\right) - \sum_{i=1}^m \ln(1+\alpha y_i) - (n\bar{y} - \|y\|^2)\alpha + \tau\ln(1+\phi\alpha) \\ =& -\sum_{i=1}^m \ln(1+\alpha y_i) + \tau\ln(1+\phi\alpha) \end{aligned}$$

For proving $\widetilde{G}(\alpha) \leq G(\alpha)$ it is sufficient to show that $P(\alpha) \geq 0$.

We have $P'(0) = P(0) = 0$ where

$$P'(\alpha) = -\sum_{i=1}^m \frac{y_i}{1+\alpha y_i} + \tau\frac{\phi}{1+\alpha\phi},$$

and

$$P''(\alpha) = \sum_{i=1}^m \frac{y_i^2}{(1+\alpha y_i)^2} - \frac{\|y\|^2}{(1+\alpha\phi)^2}.$$

Since $y_i \leq \|y\|$, $n\bar{y} \leq \|y\|$ and $\phi \geq y_i$ it results that

$$\frac{1}{(1+\alpha y_i)^2} \geq \frac{1}{(1+\alpha\phi)^2}.$$

Hence, for all $\alpha > 0$, $P''(\alpha) \geq 0$. Furthermore, Since

$$\frac{1}{2\eta}\alpha^2 d^tQd \geq 0 \ \text{ and } \ -\frac{1}{\eta}\alpha d^tQd \geq -\frac{1}{\eta}\widehat{\alpha}d^tQd, \forall\alpha \in]0, \widehat{\alpha}[,$$

then,

$$\alpha\left(\sum_{i=1}^m y_i - \|y\|^2\right) - \sum_{i=1}^m \ln(1+\alpha y_i) \geq (n\bar{y} - \|y\|^2)\alpha - \tau\ln(1+\phi\alpha),$$

therefore,

$$\underbrace{\frac{1}{2\eta}\alpha^2 d^t Qd + \alpha\left(\sum_{i=1}^{m} y_i - \|y\|^2\right) - \sum_{i=1}^{m}\ln(1+\alpha y_i) - \frac{1}{\eta}\alpha d^t Qd}_{G(\alpha)} \geq \underbrace{(n\bar{y} - \|y\|^2)\alpha - \tau\ln(1+\phi\alpha) - \frac{1}{\eta}\widehat{\alpha} d^t Qd.}_{\widetilde{G}(\alpha)}$$

and the proof is complete. □

Not that, the minimum of the convex function $\widetilde{G}$ is obtained by solving the equation $\widetilde{G}'(\alpha) = 0$ on $]0, \widehat{\alpha}[$, where $\widehat{\alpha} = \sup\{\alpha : 1 + \alpha y_i > 0, \text{ for } i = 1, ..., m\}$ and $y_i = \frac{<e_i, Ad>}{<e_i, Ax-b>}$, $i \in \{1, ..., m\}$. By straightforward calculations, the optimal step-length is

$$\alpha^* = \frac{\tau}{n\bar{y} - \|y\|^2} - \frac{1}{\phi}. \tag{3.6}$$

where $\phi = \frac{\|y\|^2}{(n-1)\beta+\gamma}$ and $0 < \tau < 1$.

The following Lemma shows that the generated interior points $\{x_k\}_{k\in\mathbb{N}}$ decrease iteratively the value of the function q_η.

Theorem 3.1. *The function q_η decreases substantially from iteration k to iteration $k+1$. In other words, if x_k and x_{k+1} represent feasible solutions obtained at iterations k and $k+1$, respectively, then*

$$q_\eta(x_{k+1}) < q_\eta(x_k).$$

Proof. According to Taylor's development, we have

$$q_\eta(x_{k+1}) - q_\eta(x_k) = \nabla q_\eta(x_k)\alpha_k d_k + o\left(\|\alpha_k d_k\|\right).$$

Furthermore, since

$$\nabla^2 q_\eta(x_\eta) d_k = -\nabla q_\eta(x_\eta),$$

it results that,

$$q_\eta(x_{k+1}) - q_\eta(x_k) \simeq -\alpha_k \nabla^2 q_\eta(x_k) d_k. \tag{3.7}$$

In the other hand, by convexity of q_η, we have

$$\nabla^2 q_\eta(x_k) d_k \geq 0.$$

then,

$$-\alpha_k \nabla^2 q_\eta(x_k) d_k \leq 0. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$q_\eta(x_{k+1}) - q_\eta(x_k) < 0.$$

which completes the proof. □

3.4. **The proposed algorithm.** The main step of the algorithm to obtain an optimal solution $\bar{x}$ of the problem (P) are described in Algorithm 1 below.

Algorithm 1: The proposed algorithm

Input: Choose the parameters $\eta > 0$, $0 < \tau < 1$, $0 < \sigma < 1$ and a parameter ε to stop the algorithm.

// **Initialization**

- Select a feasible point x_0 of the quadratic problem (P) and set $k = 0$.

while $\left\|\nabla q_\eta(x_k)\right\| > \varepsilon$ **do**

– Compute the the descent direction d_k by solving the following linear system

$$\nabla^2 q_\eta(x_\eta) d_k = -\nabla q_\eta(x_\eta).$$

– Determine optimal step-length α_k by setting

$$\alpha_k = \frac{\tau}{n\bar{y} - \|y\|^2} - \frac{1}{\phi}.$$

– Set $x_{k+1} = x_k + \alpha_k d_k$.

– Put $\eta = \sigma\eta$, and set $k = k + 1$.

Output: $\bar{x} = x_k$ is the prescribed solution

4. Numerical experiments

To evaluate our algorithm's efficiency based on our minorant function, we conducted a comparative numerical analysis on different test problems taken from the literature [1,2,4] and implemented in MATLAB R2013a on I5, 8350 (3.6 GHz) with 8 Go RAM between our function and Wolfe line search method. For this purpose, we consider the following quadratic problem:

$$\begin{cases} \min q(x) = & \frac{1}{2} x^t Q x + c^t x \\ Ax \geq b, & x \in \mathbb{R}^n \end{cases}$$

where $q(x) = \frac{1}{2} x^t Q x + c^t x$. The test problems are described below with different dimensions n, with accuracy $\varepsilon = 10^{-5}$. The obtained numerical results are recorded in Tables 1 - 5, where "iter" denotes the number of iterations necessary to obtain an optimal solution, "CPU(s)" is the calculation time to reach the solution in seconds (s), "nMF" denotes the strategy of new minorant function introduced in this paper, and "AGls" for the classical Armijo-Goldstein line search technique.

Problem 4.1. *Consider the following quadratic problem with $n = 2m$, where the matrix Q is defined by:*

$$\begin{aligned} Q[i,j] &= \begin{cases} 2j-1 & \text{if } i > j \\ 2i-1 & \text{if } i < j \\ (i+1)\,i-1 & \text{if } i = j,\ i,j = 1,..,n. \end{cases} \\ A[i,j] &= \begin{cases} 1 & \text{if } i = j \text{ or } j = m+i,\ i = 1,..,m \text{ and } j = 1,..,n \\ 0 & \text{otherwise.} \end{cases} \\ c[i] &= -1,\ c[m+i] = 0 \text{ and } b[i] = 2,\ \forall i = 1,..,m. \end{aligned}$$

The obtained results are reported in Table 1 below.

$\dim(m\times n)$	nMF		AGls	
	iter	CPU(s)	iter	CPU(s)
200×400	7	4.15129	26	39.25833
300×600	11	50.03129	35	156.00981
600×1200	20	71.66481	48	333.23551
1000×2000	29	111.21982	51	651.91698
1500×3000	38	275.17251	78	2657.10689

TABLE 1. Numerical results for Problem 4.1.

Problem 4.2. *Consider the following quadratic problem with $n = 2m$, where the matrix Q is defined by:*

$$
\begin{aligned}
Q[i,j] &= \left\{ \tfrac{1}{j+i} \quad \textit{for} \quad i,j = 1,..,n. \right. \\
A[i,j] &= \begin{cases} 1 & \textit{if } i=j \textit{ or } j = m+i,\ i = 1,..,m \textit{ and } j = 1,..,n \\ 0 & \textit{otherwise.} \end{cases} \\
c[j] &= 2j \textit{ and } b[i] = i^2,\ \forall i = 1,..,m.
\end{aligned}
$$

The obtained results are reported in Table 2 below.

$\dim(m\times n)$	nMF		AGls	
	iter	CPU(s)	iter	CPU(s)
200×400	7	4.122233	16	12.65894
300×600	8	12.022361	27	99.96582
600×1200	26	41.205912	55	213.00235
1000×2000	32	155.524377	68	1999.02589
1500×3000	51	312.338133	124	5002.021583

TABLE 2. Numerical results for problem 4.2.

Problem 4.3. *Consider the following quadratic problem with $n = 2m$, where the matrix Q is defined by:*

$$
\begin{aligned}
Q[i,j] &= \begin{cases} Q[1,1] = 1 \\ Q[i,i] = i^2+1 \\ Q[i,i-1] = Q[i-1,i] = i, \quad i = 2,..,n. \end{cases} \\
A[i,j] &= \begin{cases} 1 & \textit{if } i=j \textit{ or } j = i+m,\ i = 1,..,m \textit{ and } j = 1,..,n \\ 0 & \textit{otherwise.} \end{cases} \\
c[j] &= j \textit{ and } b[i] = \frac{i+1}{2},\ \forall i = 1,..,m.
\end{aligned}
$$

The obtained results are reported in Table 3 below.

dim($m \times n$)	nMF		AGls	
	iter	CPU(s)	iter	CPU(s)
200×400	18	7.77882	38	30.12365
300×600	26	51.18824	45	100.26581
600×1200	31	61.23586	66	332.98765
1000×2000	45	101.54756	70	2541.98756
1500×3000	77	910.54142	101	5002.16528

TABLE 3. Numerical results for Problem 4.3.

Problem 4.4. *Consider the following quadratic problem with $n = m + 2$, where the matrix Q is defined by:*

$$Q[i,j] = \begin{cases} 2 & \text{if } i = j = 1 \text{ or } i = j = m \\ 4 & \text{if } i = j \text{ and } i \neq \{1, m\} \\ 2 & \text{if } i = j-1 \text{ or } i = j+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$A[i,j] = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } i = j-1 \\ 3 & \text{if } i = j-2 \\ 0 & \text{otherwise.} \end{cases}$$

$$b_i = 1 \text{ and } c_i = 0, \ \forall i, j = 1, .., n.$$

The obtained results are reported in Table 4 below.

dim(n)	nMF		AGls	
	iter	CPU(s)	iter	CPU(s)
200	5	0.0131	18	0.3244
400	3	0.0565	17	1.4199
600	3	0.6544	17	2.9025
800	3	0.9240	17	5.9867
1000	3	1.6548	17	10.6652
1500	3	1.8892	17	15.0438

TABLE 4. Numerical results for Problem 4.4.

Problem 4.5. *Consider the following quadratic problem with $n = 2m$, where the matrix Q is defined by:*

$$Q[i,j] = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i = j = n \\ 1 & \text{if } i = 2, ..., n \\ 4 & \text{if } i = 2, ..., n-1 \end{cases}$$

$$A[i,j] = \begin{cases} 0 & \text{if } i \neq j \text{ or } i+1 \neq j \\ 1 & \text{if } i = j \text{ or } i = j+m, \end{cases}$$

$$b_i = 4 \text{ and } c_i = \frac{i+1}{2}. \forall i, j = 1, .., n.$$

The obtained results are reported in Table 5 below.

dim($m \times n$)	nMF		AGls	
	iter	CPU(s)	iter	CPU(s)
200×400	6	1.9958	18	12.2549
300×600	8	2.0565	21	123.9912
600×1200	19	41.0584	53	495.9028
1000×2000	27	99.9878	67	1990.3652
1500×3000	42	206.6985	132	2995.5691

Table 5. Numerical results for Problem 4.5.

From the numerical results, we can see clearly that, the number of iterations and the computing time are considerably reduced using the proposed approximate approach in comparison with the line search method. These outcomes demonstrate that the approximate approach outperform line search method.

5. Conclusion

This paper introduces a new logarithmic barrier method for solving convex quadratic minimization under inequality constraints. In the proposed method, we have suggested a new minorant function that aims to provide a step-length faster and easier than traditional line search methods. Mathematical results concerning the convergence of the algorithm are established. The performance of the algorithm is examined on five test problems with different dimensions. Moreover, a comparison with the Wolfe line search technique is carried out, and the numerical results indicate the new minorant function speeds up the algorithm and outperforms the Wolfe line search technique in terms of efficiency.

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