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Common Fixed Point Techniques in Bipolar Orthogonal Metric Space With Applications to Economic Problem and Integral Equation

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Abstract. This article proves a new common fixed point theorems in bipolar orthogonal metric space in the context of the Meir-Keeler contraction type. We have given some suitable examples based on our obtain theorems. Finally, we provide an application to the integral equation and an application to the production-consumption equilibrium problem.

1. Introduction

The fixed point theorems have significant applications in the mathematics field. The most important and essential contribution to fixed point theory was provided by Banach in 1922 [1]. This concept is known as the Banach contraction principle. The majority of the authors in the fixed point theory generalized it. Using the Banach contraction theorem Mutlu et al. [2] generalized a metric space, also known as a bipolar metric space. Gunaseelan et al. [3] proved a unique fixed point theorem in fuzzy bipolar metric space. Srinuvasa et al. [4] proved a common fixed point theorems in bipolar metric space. Karapinar and Cvetković [5] have proposed bipolar metric space

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and proved a fixed point theorems. Soni [6] has established a common fixed point theorems in bipolar metric space. Ahmad et al. [7] have proved a fixed point theorem in graphical bipolar b-metric space. Murthy et al. [8] have proposed a common fixed point theorems in bipolar metric space. Meir and Keeler [9] have proposed a fixed point theorems by using a weakly contraction in complete metric spaces.

Sezen [10] has proved a fixed point theorems in orthogonal fuzzy bipolar metric space. Authors [11] investigated new common fixed point theorems in the context of bipolar fuzzy b-metric space. Javed et al. [12] have generalized orthogonal fuzzy metric space and proved a fixed point theorems. Gnanaprakasam et al. [13] have proved a fixed point theorems in orthogonal b-metric space. Janardhanan et al. [14] have proved a common fixed point theorem in orthogonal neutrosophic 2-metric space. Mani et al. [15] have proved orthogonal coupled fixed point theorems. Mani et al. [16] proved a common FPT in orthogonal Branciari metric spaces. Touail and Moutawakila [17] have proposed orthogonal complete metric space and proved a fixed point theorems. Murthy et al. [18] have proved a common fixed point theorems in bipolar metric space using Meir-Keeler contraction type. Kishore et al. [19] have proved common fixed point theorem in bipolar metric space. Nazama et al. [21] has proved fixed point theorem on orthogonal interpolative contraction. Mustafa Mudhesh et al. [22] has proved fixed point theorem in multi valued mappings. Hussain et al. [23] has proved fixed point theorem on interpolative convex contraction. Sharma et al. [24] has proposed orthogonal F-contraction mappings. Sharma et al. [25] has proved fixed point theorem on orthogonal F-metric space. Chandok et al. [26] has given fixed point theorem on orthogonal (τ, F) contraction mappings. Sharma et al. [27] has proved fixed point theorem on F-metric space. Okeke et al. [28] has proved common fixed point theorem on modular metric space. Okeke et al. [29] has proved fixed point theorem in Meir-Keeler contraction in modular extended b-metric space. Thirthar et al. [30] has investigated dynamical behavior of a fractional-order epidemic model in two fear effect function. Muthuvel et al. [31] has presented ψ-Caputo fractional delay control system. Nisar [32] has given numerical approach to solve the fractional equation.

The article's motivation, from [18], is to extend this work to bipolar orthogonal metric space and prove a new common fixed point theorems.

2. Bipolar $\mathcal{O}$- Metric Spaces

This part recalls some basic definitions as follows:

Definition 2.1. *[2] Let $\mathcal{Q}$ and $\mathcal{F}$ be two non-empty sets and $\wp : \mathcal{Q} \times \mathcal{F} \to [0, +\infty)$ be a function. Let $(\mathcal{Q}, \mathcal{F}, \wp)$ is known to be as bipolar metric space and $\wp$ is said to be a bipolar metric on $(\mathcal{Q}, \mathcal{F})$ then the conditions as follows:*

(i) $\wp(\chi, \varphi) = 0$ *iff* $\chi = \varphi$ *where* $(\chi, \varphi) \in \mathcal{Q} \times \mathcal{F}$,

(ii) If $\chi, \varphi \in \mathcal{Q} \cap \mathcal{F}$ *then* $\wp(\chi, \varphi) = \wp(\varphi, \chi)$,

(iii) $\wp(\chi_1, \varphi_2) \leq \wp(\chi_1, \varphi_1) + \wp(\chi_2, \varphi_1) + \wp(\chi_2, \varphi_2)$ *for all* $\chi_1, \chi_2 \in \mathcal{Q}$ *and* $\varphi_1, \varphi_2 \in \mathcal{F}$.

Now we define the notion of bipolar $\mathcal{O}$-metric space:

Definition 2.2. *Let Q and $\mathcal{F}$ be two non-empty sets and $\wp : Q\times\mathcal{F} \to [0, +\infty)$ be a function. Let $(Q, \mathcal{F}, \wp, \perp)$ is said to be a bipolar O-metric space and $\wp$ is said to be a bipolar metric on $(Q, \mathcal{F})$ and define a binary relation $\perp$ on $(Q, \mathcal{F})$ then the conditions as follows:*

(i) $\wp(\chi, \varphi) = 0$ *iff* $\chi = \varphi$ *where* $(\chi, \varphi) \in Q\times\mathcal{F}$*, such that* $\chi \perp \varphi$*,*

(ii) If $\chi, \varphi \in Q\cap\mathcal{F}$ *then* $\wp(\chi, \varphi) = \wp(\varphi, \chi)$*, such that* $\chi \perp \varphi$*,*

(iii) $\wp(\chi_1, \varphi_2) \leq \wp(\chi_1, \varphi_1) + \wp(\chi_2, \varphi_1) + \wp(\chi_2, \varphi_2)$ *for all* $\chi_1, \chi_2 \in Q$ *and* $\varphi_1, \varphi_2 \in \mathcal{F}$*, such that* $\chi_1 \perp \varphi_1$*,* $\chi_2 \perp \varphi_1$ *and* $\chi_2 \perp \varphi_2$ *.*

Definition 2.3. *[2] Let $(Q, \mathcal{F}, \wp, \perp)$ is said to be bipolar O-metric space. Elements of $Q, \mathcal{F}$ and $Q\cap\mathcal{F}$ are said to be left, right and central points respectively. The sequences Q and $\mathcal{F}$ are said to be left and right sequences respectively.*

A sequence $\{\eta_\aleph\}$ is called convergent at point t iff $\{\eta_\aleph\}$ is called left sequence, at right point η and $\lim_{\aleph\to+\infty} \wp(\eta_\aleph, \eta) = 0$ (or) $\{\eta_\aleph\}$ is called right sequence, at left point η and $\lim_{\aleph\to+\infty} \wp(\eta, \eta_\aleph) = 0$.

A sequence $\{(\chi_\aleph, \varphi_\aleph)\}$ in $Q\times\mathcal{F}$ is said to be a bisequence on $(Q, \mathcal{F})$. And the sequence is denoted by $(\chi_\aleph, \varphi_\aleph)$. Both sequences $\{\chi_\aleph\}$ and $\{\varphi_\aleph\}$ are converges, then the bisequence $(\chi_\aleph, \varphi_\aleph)$ is called convergent. Both sequences $\{\chi_\aleph\}$ and $\{\varphi_\aleph\}$ are converges at a point $v \in Q\cap\mathcal{F}$ then $(\chi_\aleph, \varphi_\aleph)$ is said to be biconvergent.

The bisequence $(\chi_\aleph, \varphi_\aleph)$ is called a Cauchy bisequence, if $\lim_{\aleph,\pi\to+\infty} \wp(\chi_\aleph, \varphi_\pi) = 0$. In every convergent Cauchy bisequence is called a biconvergent.

If every Cauchy bisequence is convergent, then it is biconvergent, in bipolar metric space is complete.

Every complete bipolar metric space is a complete bipolar O-metric space and converse need not be a true.

Example 2.1. *Let $Q = [0, \frac{1}{5}] \cup \{\frac{3\aleph}{5} : \aleph \in \mathbb{N}\}$ and $\mathcal{F} = [0, \frac{1}{5}] \cup \{\frac{3}{10}(2\aleph + 1) : \aleph \in \mathbb{N}\}$ and the distance $\wp : Q\times\mathcal{F} \to \mathbb{R}^+$ with Euclidean metric such that $\varrho \perp \varphi$ for all $\varrho \in Q$ and $\varphi \in \mathcal{F}$,*

$$\varrho \perp \varphi \Leftrightarrow \begin{cases} \varrho \leq \varphi \leq \frac{1}{5} \\ (or)\ \varrho = 0, \varphi = 0 \end{cases} .$$

Then $(Q, \mathcal{F}, \perp)$ is an O-set.

Clearly, Q and $\mathcal{F}$ with Euclidean metric are not complete bipolar metric space, but it is complete bipolar O-metric space. For this, $\{\varrho_k\}$ and $\{\varphi_k\}$ are an arbitrary Cauchy $\perp$-bisequence in Q and $\mathcal{F}$. Then there exists a bisubsequence $\{\varrho_{k_\aleph}\}$ of $\{\varrho_k\}$ and $\{\varphi_{k_\aleph}\}$ of $\{\varphi_k\}$ implies $\varrho_{k_\aleph} = 0, \varphi_{k_\aleph} = 0,\ \forall\ \aleph \geq 1$ or there exists a monotone bisubsequence $\{\varrho_{k_\aleph}\}$ of $\{\varrho_k\}$ and $\{\varphi_{k_\aleph}\}$ of $\{\varphi_k\}$ for which $\varrho_{k_\aleph} \leq \frac{1}{5}$ and $\varphi_{k_\aleph} \leq \frac{1}{5},\ \forall\ \aleph \geq 1$. Then $\{\varrho_{k_\aleph}\}$ and $\{\varphi_{k_\aleph}\}$ biconverges to a point $\varrho \in [0, \frac{1}{5}] \subseteq Q$ and $\varphi \in [0, \frac{1}{5}] \subseteq \mathcal{F}$. On the other hand, we know that every Cauchy bisequence with a biconvergent bisubsequence is biconvergent. Then $\{\varrho_k\}$ and $\{\varphi_k\}$is biconvergent.

Definition 2.4. *[2] Let $Q_1, \mathcal{F}_1, Q_2$ and $\mathcal{F}_2$ be four sets. A function $\Lambda : Q_1 \cup \mathcal{F}_1 \to Q_2 \cup \mathcal{F}_2$ is said to be a covariant map if $\Lambda(Q_1) \subseteq Q_2$ and $\Lambda(\mathcal{F}_1) \subseteq \mathcal{F}_2$ and is denoted as $\Lambda : (Q_1, \mathcal{F}_1) \rightrightarrows (Q_2, \mathcal{F}_2)$. In particular, if $(Q_1, \mathcal{F}_1, \wp_1, \perp)$ and $(Q_2, \mathcal{F}_2, \wp_2, \perp)$ are two bipolar O-metric space then we use the notaion $\Lambda : (Q_1, \mathcal{F}_1, \wp_1, \perp) \rightrightarrows (Q_2, \mathcal{F}_2, \wp_2, \perp)$ for covariant map Λ.*

Definition 2.5. *Consider $(\mathcal{Q}_1, \mathcal{F}_1, \wp_1, \perp)$ and $(\mathcal{Q}_2, \mathcal{F}_2, \wp_2, \perp)$ to be two bipolar $\mathcal{O}$-metric spaces. An operator $\Lambda : (\mathcal{Q}_1, \mathcal{F}_1) \rightrightarrows (\mathcal{Q}_2, \mathcal{F}_2)$ is said to be $\perp$-continuous at a point $\chi_0 \in \mathcal{Q}_1$, at given $\varepsilon > 0$, we can find $\kappa > 0$ such that $\varphi \in \mathcal{F}_1$ and $\wp_1(\chi_0, \varphi) < \kappa$ implies that $\wp_2(\Lambda(\chi_0), \Lambda(\varphi)) < \varepsilon$. It is $\perp$-continuous at a point $\varphi_0 \in \mathcal{F}_1$ if for any given $\varepsilon > 0$, there exists $\kappa > 0$ such that $\chi \in \mathcal{Q}_1$ and $\wp_1(\chi, \varphi_0) < \kappa$ implies that $\wp_2(\Lambda(\chi), \Lambda(\varphi_0)) < \varepsilon$. If Λ is $\perp$-continuous at each point $\chi \in \mathcal{Q}_1 \cup \mathcal{F}_1$, then it is said to be $\perp$-continuous.*

A covariant map $\Lambda : (\mathcal{Q}_1, \mathcal{F}_1) \rightrightarrows (\mathcal{Q}_2, \mathcal{F}_2)$ is continuous if and only if $\{\eta_\aleph\}$ converges to η on $(\mathcal{Q}_1, \mathcal{F}_1, \wp_1)$ implies $\{\Lambda(\eta_\aleph)\}$ converges to $\Lambda(\eta)$ on $(\mathcal{Q}_2, \mathcal{F}_2, \wp_2)$.

Definition 2.6. *[20] Let Υ be non-empty set and $\perp \subseteq \Upsilon \times \Upsilon$ be a binary relation such that*

$$\exists \varrho_0 \in \Upsilon : (\forall \varrho \in \Upsilon, \varrho \perp \varrho_0) \quad (or) \quad (\forall \varrho \in \Upsilon, \varrho_0 \perp \varrho),$$

then it is called an orthogonal set (briefly O-set). We denote this O-set by $(\Upsilon, \perp)$.

Definition 2.7. *[20] Let $(\Upsilon, \perp)$ be an O-set. A sequence $\{\varrho_\aleph\}$ is called an orthogonal sequence (briefly, O-sequence) if*

$$(\forall \aleph \in \mathbb{N}, \varrho_\aleph \perp \varrho_{\aleph+1}) \quad (or) \quad (\forall \aleph \in \mathbb{N}, \varrho_{\aleph+1} \perp \varrho_\aleph).$$

Definition 2.8. *[20] Let $(\Upsilon, \perp)$ be an O-set. A map $\wp : \Upsilon \times \Upsilon \to \Upsilon$ is known to be $\perp$-preserving if $\wp(\varrho, \varphi) \perp \wp(\varphi, \varrho)$ whenever $\varrho \perp \varphi$ and $\varphi \perp \varrho$.*

Definition 2.9. *Let $(\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be a bipolar $\mathcal{O}$-metric space and let $\Theta, \Gamma : (\mathcal{Q}, \mathcal{F}) \rightrightarrows (\mathcal{Q}, \mathcal{F})$ be two covariant maps then (Θ, Γ) is called $\perp$-compatible iff $\wp(\Gamma\Theta\chi_\aleph, \Theta\Gamma\varphi_\aleph) \to 0$ and $\wp(\Theta\Gamma\chi_\aleph, \Gamma\Theta\varphi_\aleph) \to 0$, whenever $(\chi_\aleph, \varphi_\aleph)$ is a sequence in $\mathcal{Q} \times \mathcal{F}$ such that $\lim_{\aleph \to +\infty} \Theta\chi_\aleph = \lim_{\aleph \to +\infty} \Gamma\chi_\aleph = \lim_{\aleph \to +\infty} \Theta\varphi_\aleph = \lim_{\aleph \to +\infty} \Gamma\varphi_\aleph = \rho$ for some $\rho \in \mathcal{Q} \cap \mathcal{F}$.*

Definition 2.10. *The Θ and Γ is said to be $\perp$-weakly compatible, if Θ and Γ are its coincidence points .*

Definition 2.11. *Let $(\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be a bipolar $\mathcal{O}$-metric space and let $\mathcal{F}, \Theta, \mathcal{G}, \Gamma : (\mathcal{Q}, \mathcal{F}) \rightrightarrows (\mathcal{Q}, \mathcal{F})$ be four covariant maps then $(\mathcal{F}, \Theta, \mathcal{G}, \Gamma)$ are called $\perp$-compatible iff $\wp(\Gamma\mathcal{F}\chi_\aleph, \mathcal{F}\Gamma\varphi_\aleph)$ and $\wp(\mathcal{G}\Theta\chi_\aleph, \Theta\mathcal{G}\varphi_\aleph)$ converges to zero, then the sequence $(\chi_\aleph, \varphi_\aleph)$ in $\mathcal{Q} \times \mathcal{F}$ such that*

$$\lim_{\aleph \to +\infty} \mathcal{F}\chi_\aleph = \lim_{\aleph \to \infty} \Theta\chi_\aleph = \lim_{\aleph \to \infty} \mathcal{G}\varphi_\aleph = \lim_{\aleph \to \infty} \Gamma\varphi_\aleph = \rho,$$

for some $\rho \in \mathcal{Q} \cap \mathcal{F}$.

3. Main Results

Let us begin with some propositions as follows:

Proposition 3.1. *Consider $(\mathcal{Q}, \mathcal{F}, \wp, \perp)$ to be a bipolar $\mathcal{O}$-metric space(bipolar $\mathcal{O} - \mathcal{MS}$), and $\Lambda, \Omega, \Theta, \Gamma : (\mathcal{Q}, \mathcal{F}, \wp) \rightrightarrows (\mathcal{Q}, \mathcal{F}, \wp)$ be four covariant maps satisfies the axioms as follows:*
Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \text{ implies } \wp(\Lambda\varrho, \Omega\varphi) < \nu \tag{3.1}$$

$$\text{and } \Theta\varrho = \Gamma\varphi \text{ implies } \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi, \tag{3.2}$$

then

$$\wp(\Lambda\varrho, \Omega\varphi) < \wp(\Theta\varrho, \Gamma\varphi), \text{ if } \Theta\varrho \neq \Gamma\varphi, \ \ \varrho \perp \varphi \quad \text{and} \tag{3.3}$$

$$\wp(\Lambda\varrho, \Omega\varphi) \leq \wp(\Theta\varrho, \Gamma\varphi), \ \ \varrho \perp \varphi \quad \text{forall } \ \varrho \in \mathcal{Q}, \varphi \in \mathcal{F}. \tag{3.4}$$

Proof. Let $\Theta\varrho \neq \Gamma\varphi$ then $\wp(\Theta\varrho, \Gamma\varphi) = \nu$ for some $\nu > 0$ and from condition (4.4) we have $\wp(\Lambda\varrho, \Omega\varphi) < \nu$ and so (4.4) holds. From (4.4) and (4.4) we get (4.2). □

Remark 3.1. *If we take $\mathcal{Q} = \mathcal{F}$, then we get the following result:*

Corollary 3.1. *Consider $(\mathcal{Q}, \wp, \perp)$ to be a $\mathcal{O}$-metric space, and $\Lambda, \Omega, \Theta, \Gamma : \mathcal{Q} \to \mathcal{Q}$ be four maps satisfies the axioms as follows:*
Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \text{ implies } \wp(\Lambda\varrho, \Omega\varphi) < \nu$$
$$\text{and } \Theta\varrho = \Gamma\varphi \text{ implies } \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi,$$

then

$$\wp(\Lambda\varrho, \Omega\varphi) < \wp(\Theta\varrho, \Gamma\varphi), \text{ if } \Theta\varrho \neq \Gamma\varphi, \ \ \varrho \perp \varphi \quad \text{and}$$
$$\wp(\Lambda\varrho, \Omega\varphi) \leq \wp(\Theta\varrho, \Gamma\varphi), \ \ \varrho \perp \varphi \quad \text{for all } \ \varrho, \varphi \in \mathcal{Q}.$$

Proposition 3.2. *Consider $(\mathcal{Q}, \mathcal{F}, \wp, \perp)$ to be a bipolar $\mathcal{O} - \mathcal{MS}$, and $\Theta, \Gamma : (\mathcal{Q}, \mathcal{F}, \wp, \perp) \rightrightarrows (\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be two covariant maps satisfying the condition:*

$$\wp(\Gamma\varrho, \Gamma\varphi) \leq \wp(\Theta\varrho, \Theta\varphi) \text{ for all, } \ \varrho \perp \sigma, \quad \varrho \in \mathcal{Q}, \sigma \in \mathcal{F}. \tag{3.5}$$

If Θ be an $\perp$-continuous function, then Γ is also $\perp$-continuous function.

Proof. Assume a sequence $\{\chi_\aleph\}$ converges to a right point $\varphi \in \mathcal{F}$, then $\wp(\Theta\chi_\aleph, \Theta\varphi)$ tending to zero as Θ is an $\perp$-continuous, and from equation (3.5) $\wp(\Gamma\chi_\aleph, \Gamma\varphi)$ tending to zero, that is $\{\Gamma\chi_\aleph\}$ converges to $\Gamma\varphi$. Likewise, we find that if right sequence $\{\varphi_\aleph\}$ converges to left point $\chi \in \Upsilon$, then $\{\Gamma\varphi_\aleph\}$ converges to $\Gamma\chi$. Thus Γ is also $\perp$-continuous. □

Remark 3.2. *If we take $\mathcal{Q} = \mathcal{F}$, then we get the following result:*

Corollary 3.2. *Consider $(\mathcal{Q}, \wp, \perp)$ to be a $\mathcal{O}$-metric space, and $\Lambda, \Omega, \Theta, \Gamma : \mathcal{Q} \to \mathcal{Q}$ be four maps satisfies the axioms as follows:*
Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \text{ implies } \wp(\Lambda\varrho, \Omega\varphi) < \nu$$
$$\text{and } \Theta\varrho = \Gamma\varphi \text{ implies } \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi,$$

then

$$\wp(\Lambda\varrho, \Omega\varphi) < \wp(\Theta\varrho, \Gamma\varphi), \text{ if } \Theta\varrho \neq \Gamma\varphi, \ \ \varrho \perp \varphi \quad \text{and}$$

$$\wp(\Lambda\varrho, \Omega\varphi) \leq \wp(\Theta\varrho, \Gamma\varphi), \quad \varrho \perp \varphi \quad \text{for all} \quad \varrho, \varphi \in \mathcal{Q}.$$

Proposition 3.3. *Let $(\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be a bipolar $\mathcal{O} - \mathcal{MS}$ and let $\Theta, \Gamma : (\mathcal{Q}, \mathcal{F}, \wp, \perp) \rightrightarrows (\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be two covariant maps which are $\perp$-compatible. If ρ is a coincidence point of Θ and Γ (i.e., $\Gamma\rho = \Theta\rho$) then $\Gamma\Theta\rho = \Theta\Gamma\rho$. Then $\perp$-compatible map is $\perp$-weakly compatible.*

Proof. Taking $\chi_{\aleph} = \varphi_{\aleph} = \rho$, from the Definition 2.9 gives this clearly proved. □

Remark 3.3. *If we take $\mathcal{Q} = \mathcal{F}$, then we get the following result:*

Corollary 3.3. *Consider $(\mathcal{Q}, \wp, \perp)$ to be a $\mathcal{O}$-metric space, and $\Lambda, \Omega, \Theta, \Gamma : \mathcal{Q} \to \mathcal{Q}$ be four maps satisfies the axioms as follows:*
Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \;\text{ implies }\; \wp(\Lambda\varrho, \Omega\varphi) < \nu$$
$$\text{and }\; \Theta\varrho = \Gamma\varphi \;\text{ implies }\; \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi,$$

then

$$\wp(\Lambda\varrho, \Omega\varphi) < \wp(\Theta\varrho, \Gamma\varphi), \;\text{ if }\; \Theta\varrho \neq \Gamma\varphi, \quad \varrho \perp \varphi \quad \text{and}$$

$$\wp(\Lambda\varrho, \Omega\varphi) \leq \wp(\Theta\varrho, \Gamma\varphi), \quad \varrho \perp \varphi \quad \text{for all} \quad \varrho, \varphi \in \mathcal{Q}.$$

Proposition 3.4. *Let $(\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be a bipolar $\mathcal{O} - \mathcal{MS}$ and let $\Lambda, \Omega, \Theta, \Gamma : (\mathcal{Q}, \mathcal{F}, \wp, \perp) \rightrightarrows (\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be four covariant maps such that $(\Lambda, \Theta, \Omega, \Gamma)$ be an $\perp$-compatible. If ρ is a fixed point, then $\Gamma\Lambda\rho = \Lambda\Gamma\rho$ and $\Omega\Theta\rho = \Theta\Omega\rho$.*

Remark 3.4. *If we take $\mathcal{Q} = \mathcal{F}$, then we get the following result:*

Corollary 3.4. *Consider $(\mathcal{Q}, \wp, \perp)$ to be a $\mathcal{O}$-metric space, and $\Lambda, \Omega, \Theta, \Gamma : \mathcal{Q} \to \mathcal{Q}$ be four maps satisfies the axioms as follows:*
Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \;\text{ implies }\; \wp(\Lambda\varrho, \Omega\varphi) < \nu$$
$$\text{and }\; \Theta\varrho = \Gamma\varphi \;\text{ implies }\; \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi,$$

then

$$\wp(\Lambda\varrho, \Omega\varphi) < \wp(\Theta\varrho, \Gamma\varphi), \;\text{ if }\; \Theta\varrho \neq \Gamma\varphi, \quad \varrho \perp \varphi \quad \text{and}$$

$$\wp(\Lambda\varrho, \Omega\varphi) \leq \wp(\Theta\varrho, \Gamma\varphi), \quad \varrho \perp \varphi \quad \text{for all} \quad \varrho, \varphi \in \mathcal{Q}.$$

Below lemma is usefull for our findings in main theorems.

Lemma 3.1. *Consider* $(Q, \mathcal{F}, \wp, \perp)$ *to be a complete bipolar* $O-\mathcal{MS}$, *and* $Q\times\mathcal{F}$, $(\omega_{\aleph}, \theta_{\aleph})$ *is a bisequence satisfies the condition as follows:*
Given $\nu > 0$ *we can find* $\kappa > 0$ *and* $\omega_{\aleph} \perp \theta_{\aleph}$ *such that*

$$\nu \leq \wp(\omega_{\aleph}, \theta_{\pi}) < \nu + \kappa \text{ implies } \wp(\omega_{\aleph+1}, \theta_{\pi+1}) < \nu, \tag{3.6}$$

$$\text{and } \omega_{\aleph} = \theta_{\pi} \text{ implies } \omega_{\aleph+1} = \theta_{\pi+1}, \tag{3.7}$$

then a sequence $\{\omega_{\aleph}, \theta_{\aleph}\}$ *is Cauchy* $\perp$*-bisequence.*

Proof. Let $\alpha_{\aleph} = \wp(\omega_{\aleph}, \theta_{\aleph})$ and $\beta_{\aleph} = \wp(\omega_{\aleph}, \theta_{\aleph+1})$ then $\{\alpha_{\aleph}\}$ and $\{\beta_{\aleph}\}$ both are bounded below sequences. Thus,

$$\alpha_{\aleph} \to \nu^{+} \text{ forall } \nu \geq 0. \tag{3.8}$$

If $\nu > 0$, then ν we can find $\kappa > 0$ such that (3.6) satisfied.
Form equation (3.8) there exists, $\aleph_0 \in \mathbb{N}$ such that $\aleph \geq \aleph_0$

$$\begin{aligned} \nu &\leq \alpha_{\aleph} < \nu + \kappa \\ \nu &\leq \wp(\omega_{\aleph}, \theta_{\aleph}) < \nu + \kappa. \end{aligned}$$

This implies from (3.6) that

$$\begin{aligned} \wp(\omega_{\aleph+1}, \theta_{\aleph+1}) &< \nu \\ \alpha_{\aleph+1} &< \nu, \end{aligned}$$

which is contradiction of equation (3.8). Hence, $\nu = 0$ and

$$\alpha_{\aleph} \to 0^{+} \text{ as } \aleph \to +\infty. \tag{3.9}$$

Likewise

$$\beta_{\aleph} \to 0^{+} \text{ as } \aleph \to +\infty. \tag{3.10}$$

To find $(\omega_{\aleph}, \theta_{\aleph})$ be a Cauchy. Assume additionally, there exists $\nu > 0$ such that

$$\limsup_{\aleph, \pi \to +\infty} \wp(\omega_{\aleph}, \theta_{\pi}) > 2\nu. \tag{3.11}$$

For any ν we can find $\kappa > 0$ from equation (3.6) satified.
Let $\kappa' = min(\kappa, \nu)$. As we have,

$$\nu \leq \wp(\omega_{\aleph}, \theta_{\pi}) < \nu + \kappa' \text{ implies } \wp(\omega_{\aleph+1}, \theta_{\pi+1}) < \nu. \tag{3.12}$$

From equations (3.9), (3.10) and (3.11) there exists $\pi, \aleph, \mathcal{M}$ such that,

$$\pi, \aleph > \mathcal{M}, \alpha_{\mathcal{M}} = \wp(\omega_{\mathcal{M}}, \theta_{\mathcal{M}}) < \frac{\kappa'}{6} \text{ and } \beta_{\mathcal{M}} = \wp(\omega_{\mathcal{M}}, \theta_{\mathcal{M}+1}) < \frac{\kappa'}{6} \tag{3.13}$$

$$\wp(\omega_{\pi}, \theta_{\aleph}) > 2\nu \geq \nu + \kappa'. \tag{3.14}$$

Now, we considar two cases

If $\aleph > \pi$, then for $\sigma \in [\pi, \aleph] \cap \mathbb{N}$, we have by (B3)

$$\begin{aligned} \wp(\omega_\pi, \theta_\sigma) &\leq \wp(\omega_\pi, \theta_{\sigma+1}) + \wp(\omega_\sigma, \theta_{\sigma+1}) + \wp(\omega_\sigma, \theta_\sigma) \\ \wp(\omega_\pi, \theta_\sigma) - \wp(\omega_\pi, \theta_{\sigma+1}) &\leq \wp(\omega_\sigma, \theta_{\sigma+1}) + \wp(\omega_\sigma, \theta_\sigma) = \beta_\sigma + \alpha_\sigma. \end{aligned}$$

Using (3.13) and (3.14), the above inequality implies

$$\wp(\omega_\pi, \theta_\sigma) - \wp(\omega_\pi, \theta_{\sigma+1}) < \frac{\kappa'}{3}.$$

Similarly, we can prove that

$$\wp(\omega_\pi, \theta_{\sigma+1}) - \wp(\omega_\pi, \theta_\sigma) < \frac{\kappa'}{3}.$$

So that, we obtain

$$\left|\wp(\omega_\pi, \theta_\sigma) - \wp(\omega_\pi, \theta_{\sigma+1})\right| < \frac{\kappa'}{3}. \tag{3.15}$$

This implies, since $\wp(\omega_\pi, \theta_\pi) < \nu$, and $\wp(\omega_\pi, \theta_\aleph) > \nu + \kappa'$, that there exists $\sigma \in [\pi, \aleph] \cap \mathbb{N}$ such that

$$\nu + \frac{2\kappa'}{3} \leq \wp(\omega_\pi, \theta_\sigma) < \nu + \kappa'. \tag{3.16}$$

This implies by (3.12) that

$$\wp(\omega_{\pi+1}, \theta_{\sigma+1}) < \nu.$$

Now,

$$\begin{aligned} \wp(\omega_\pi, \theta_\sigma) &\leq \wp(\omega_\pi, \theta_{\pi+1}) + \wp(\omega_{\pi+1}, \theta_{\pi+1}) + \wp(\omega_{\pi+1}, \theta_{\sigma+1}) + \wp(\omega_\sigma, \theta_{\sigma+1}) \\ &+ \wp(\omega_\sigma, \theta_\sigma) \\ &< \frac{\kappa'}{6} + \frac{\kappa'}{6} + \nu + \frac{\kappa'}{6} + \frac{\kappa'}{6} = \nu + \frac{2\kappa'}{3}, \end{aligned}$$

which is contradiction of equation (3.16). This is the contradiction if $\aleph \leq \pi$. Hence, $(\omega_\aleph, \theta_\aleph)$ is Cauchy. □

Remark 3.5. *If we take $Q = \mathcal{F}$, then we get the following result:*

Corollary 3.5. *Consider $(Q, \wp, \perp)$ to be a O-metric space, and $\Lambda, \Omega, \Theta, \Gamma : Q \to Q$ be four maps satisfies the axioms as follows:*

Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \text{ implies } \wp(\Lambda\varrho, \Omega\varphi) < \nu$$
$$\text{and } \Theta\varrho = \Gamma\varphi \text{ implies } \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi,$$

then

$$\wp(\Lambda\varrho, \Omega\varphi) < \wp(\Theta\varrho, \Gamma\varphi), \text{ if } \Theta\varrho \neq \Gamma\varphi, \ \varrho \perp \varphi \text{ and}$$
$$\wp(\Lambda\varrho, \Omega\varphi) \leq \wp(\Theta\varrho, \Gamma\varphi), \ \varrho \perp \varphi \text{ for all } \varrho, \varphi \in Q.$$

Next, we prove our first result as follows:

Theorem 3.1. *Consider $(\mathcal{Q},\mathcal{F},\wp,\perp)$ to be a complete bipolar $\mathcal{O}-\mathcal{MS}$, and $\Theta,\Gamma : (\mathcal{Q},\mathcal{F},\wp,\perp) \rightrightarrows (\mathcal{Q},\mathcal{F},\wp,\perp)$ be two covariant maps satisfies the conditions as follows:*

(i) Θ and Γ are $\perp$-compatible mappings.
(ii) Θ and Γ are $\perp$-continuous and $\perp$-preserving.
(iii) $\Gamma(\mathcal{Q}\cup\mathcal{F}) \subseteq \Theta(\mathcal{Q}\cup\mathcal{F})$.
(iv) For any given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\chi,\Theta\varphi) < \nu+\kappa \text{ implies } \wp(\Gamma\chi,\Gamma\varphi) < \nu \tag{3.17}$$

$$\text{and } \Theta\chi = \Theta\varphi \text{ implies } \Gamma\chi = \Gamma\varphi, \tag{3.18}$$

where $\chi \in \mathcal{Q}$, $\varphi \in \mathcal{F}$, and $\chi \perp \varphi$.

Then, the mappings Θ and Γ have $\mathcal{UCFP}$.

Proof. Let $\chi_0 \in \mathcal{Q}$, $\varphi_0 \in \mathcal{F}$ and choose $\chi_1 \in \mathcal{Q}$ and $\varphi_1 \in \mathcal{F}$ such that $\Gamma\chi_0 = \Theta\chi_1 = \omega_1$ and $\Gamma\varphi_0 = \Theta\varphi_1 = \theta_1$. Since $\Gamma(\mathcal{Q}\cup\mathcal{F}) \subseteq \Theta(\mathcal{Q}\cup\mathcal{F})$.

$$\begin{array}{cc}
\chi_0 \perp \chi_1 & \varphi_0 \perp \varphi_1 \\
\Gamma\chi_0 \perp \Gamma\chi_1 & \Gamma\varphi_0 \perp \Gamma\varphi_1 \\
\Theta\chi_1 \perp \Theta\chi_2 & \Theta\varphi_1 \perp \Theta\varphi_2 \\
\Theta^{-1}\Theta\chi_1 \perp \Theta^{-1}\Theta\chi_2 & \Theta^{-1}\Theta\varphi_1 \perp \Theta^{-1}\Theta\varphi_2 \\
\chi_1 \perp \chi_2 & \varphi_1 \perp \varphi_2 \\
\vdots & \vdots \\
\chi_{\aleph-1} \perp \chi_\aleph & \varphi_{\aleph-1} \perp \varphi_\aleph.
\end{array}$$

In general, we can choose $(\chi_\aleph,\varphi_\aleph) \in \mathcal{Q}\times\mathcal{F}$ such that $\Gamma\chi_{\aleph-1} = \Theta\chi_\aleph = \omega_\aleph$ and $\Gamma\varphi_{\aleph-1} = \Theta\varphi_\aleph = \theta_\aleph$ for all $\aleph \in \mathbb{N}$.

Now, if $\Theta\chi_\aleph = \omega_\aleph = \theta_\pi = \Theta\varphi_\pi$ for some $\aleph,\pi \in \mathbb{N}$, then by condition that $\omega_{\aleph+1} = \Gamma\chi_\aleph = \Gamma\chi_\pi = \theta_{\pi+1}$ and if $\nu \leq \wp(\omega_\aleph,\theta_\pi) = \wp(\Theta\chi_\aleph,\Theta\varphi_\pi) < \nu+\kappa$, which implies from equation (3.17) that $\wp(\omega_{\aleph+1},\theta_{\pi+1}) < \nu$. By Lemma 3.1, $(\alpha_\aleph,\beta_\aleph)$ is a Cauchy $\perp$-bisequence, and $(\mathcal{Q},\mathcal{F},\wp,\perp)$ is complete $(\omega_\aleph,\theta_\aleph)$ converges and biconverges to $\rho \in \mathcal{Q}\cap\mathcal{F}$. Thus,

$$\lim_{\aleph\to+\infty}\Theta\chi_\aleph = \lim_{\aleph\to+\infty}\Gamma\chi_\aleph = \lim_{\aleph\to+\infty}\Theta\varphi_\aleph = \lim_{\aleph\to+\infty}\Gamma\varphi_\aleph = \rho.$$

Since Θ and Γ are $\perp$-compatible, hence

$$\wp(\Gamma\Theta\chi_\aleph,\Theta\Gamma\varphi_\aleph) \to 0 \text{ and } \wp(\Theta\Gamma\chi_\aleph,\Gamma\Theta\varphi_\aleph) \to 0.$$

From Proposition 3.2, both the functions Θ and Γ are $\perp$-continuous , we obtain

$$\Gamma\chi_\aleph \to \rho \quad \text{implies} \quad \Theta\Gamma\chi_\aleph \to \Theta\rho \text{ and}$$
$$\Theta\varphi_\aleph \to \rho \quad \text{implies} \quad \Gamma\Theta\varphi_\aleph \to \Gamma\rho.$$

By $\perp$-compatibility of Θ and Γ, we have

$$\begin{aligned} \wp(\Theta\rho,\Gamma\rho) &= \lim_{\aleph\to\infty}\wp(\Theta\Gamma\chi_{\aleph},\Gamma\Theta\varphi_{\aleph}) = 0 \\ \text{and this implies}\quad & \Theta\rho = \Gamma\rho, \\ \text{this implies}\quad & \Gamma\Theta\rho = \Theta\Gamma\rho. \end{aligned}$$

Let $\Theta\rho = \Gamma\rho = \omega$, then Θ and $\Gamma$is a $\mathcal{CFP}$ of ω.
Let $\Theta\omega \neq \omega$, then

$$\begin{aligned} \wp(\Gamma\omega,\omega) &= \wp(\Gamma\Theta\rho,\Gamma\rho) < \wp(\Theta\Theta\rho,\Theta\rho) \\ &= \wp(\Theta\omega,\omega) = \wp(\Theta\Gamma\rho,\Theta\rho) \\ &= \wp(\Gamma\Theta\rho,\Theta\rho) = \wp(\Gamma\omega,\omega), \end{aligned}$$

which is a contradiction. So $\Theta\omega = \omega$,

$$\text{i.e. } \Theta\Theta\rho = \Theta\Gamma\rho = \Theta\rho = \Gamma\rho = \Gamma\Theta\rho \text{ implies } \Gamma\omega = \omega.$$

Then, Θ and Γ is a $\mathcal{CFP}$ of ω.

Uniqueness: Consider that ω and θ be another $\mathcal{CFP}$ of Θ and Γ.

$$\omega_0 \perp \omega \text{(or) } \omega_0 \perp \theta.$$

Since Γ and Θ are $\perp$-preserving,

$$(\Gamma\omega_0 \perp \Gamma\omega \text{ and } \Theta\omega_0 \perp \Theta\omega),$$
$$(\Gamma\omega_0 \perp \Gamma\theta \text{ and } \Theta\omega_0 \perp \Theta\theta).$$

If $\Theta\omega \neq \Theta\theta$, then

$$\begin{aligned} \wp(\Gamma\omega,\Gamma\theta) &< \wp(\Theta\omega,\Theta\theta) \\ \Rightarrow \quad \wp(\omega,\theta) &< \wp(\omega,\theta). \end{aligned}$$

This contradicts. So $\Theta\omega = \Theta\theta$, which implies $\omega = \theta$. □

The following corollary results from using Θ as an identity mapping in the above theorem.

Corollary 3.6. *Let $(\mathcal{Q},\mathcal{F},\wp,\perp)$ be a complete bipolar $\mathcal{O}-\mathcal{MS}$ and let $\Gamma : (\mathcal{Q},\mathcal{F},\wp,\perp) \rightrightarrows (\mathcal{Q},\mathcal{F},\wp,\perp)$ be a covariant map satisfies the condition as follows:*
Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\chi,\varphi) < \nu + \kappa \text{ implies } \wp(\Gamma\chi,\Gamma\varphi) < \nu,$$

then the map Γ has a Unique fixed point.

Note that if we apply $\mathcal{Q} = \mathcal{F}$ to the above corollary, we obtain Meir and Keeler [9].

Our next outcome, we does not shows the $\perp$-continuity of Θ and instead of $\perp$-compatible maps we use weakly compatible maps.

Theorem 3.2. *Let $(\mathcal{Q},\mathcal{F},\wp,\perp)$ be a bipolar $\mathcal{O}-\mathcal{MS}$ and let $\Theta,\Gamma:(\mathcal{Q},\mathcal{F},\wp,\perp)\rightrightarrows(\mathcal{Q},\mathcal{F},\wp,\perp)$ be two covariant maps satisfies the conditions as follows:*

(i) Θ and Γ are $\perp$-weakly compatible maps,
(ii) $\Theta(\mathcal{Q}\cup\mathcal{F})$ is $\perp$-complete,
(iii) Θ and Γ are $\perp$-continuous and $\perp$-preserving,
(iv) Θ is injective,
(v) $\Gamma(\mathcal{Q}\cup\mathcal{F})\subseteq\Theta(\mathcal{Q}\cup\mathcal{F})$,
(vi) For any given $\nu>0$ we can find $\kappa>0$ such that

$$\nu\leq\wp(\Theta\chi,\Theta\varphi)<\nu+\kappa \text{ implies } \wp(\Gamma\chi,\Gamma\varphi)<\nu \tag{3.19}$$

$$\text{and } \Theta\chi=\Theta\varphi \text{ implies } \Gamma\chi=\Gamma\varphi, \tag{3.20}$$

where $\chi\in\mathcal{Q}$ and $\varphi\in\mathcal{F}$. Then, the functions Θ and Γ have $\mathcal{UCFP}$.

Proof. The proof continues from theorem (3.1),

$$\begin{array}{cc}
\chi_0\perp\chi_1 & \varphi_0\perp\varphi_1\\
\Gamma\chi_0\perp\Gamma\chi_1 & \Gamma\varphi_0\perp\Gamma\varphi_1\\
\Theta\chi_1\perp\Theta\chi_2 & \Theta\varphi_1\perp\Theta\varphi_2\\
\Theta^{-1}\Theta\chi_1\perp\Theta^{-1}\Theta\chi_2 & \Theta^{-1}\Theta\varphi_1\perp\Theta^{-1}\Theta\varphi_2\\
\chi_1\perp\chi_2 & \varphi_1\perp\varphi_2\\
\vdots & \vdots\\
\chi_{\aleph-1}\perp\chi_\aleph & \varphi_{\aleph-1}\perp\varphi_\aleph.
\end{array}$$

Then, from theorem (3.1) the $\mathcal{O}$-bisequence $(\omega_\aleph,\theta_\aleph)$ is a Cauchy $\mathcal{O}$-bisequence and hence biconverges to a point $\eta\in\Theta(\mathcal{Q})\cap\Theta(\mathcal{F})=\Theta(\mathcal{Q}\cap\mathcal{F})$. Hence, $\rho=\Theta\omega$ for some $\omega\in\mathcal{Q}\cap\mathcal{F}$. So

$$\lim_{\aleph\to+\infty}\Theta\chi_\aleph=\lim_{\aleph\to+\infty}\Gamma\chi_\aleph=\lim_{\aleph\to+\infty}\Theta\varphi_\aleph=\lim_{\aleph\to+\infty}\Gamma\varphi_\aleph=\rho=\Theta\omega. \tag{3.21}$$

Now, by using Proposition 3.1, we have

$$\lim_{\aleph\to+\infty}\wp(\Gamma\chi_\aleph,\Gamma\omega)\leq\lim_{\aleph\to+\infty}\wp(\Theta\chi_\aleph,\Theta\omega)=0.$$

So,

$$\lim_{\aleph\to+\infty}\Gamma\chi_\aleph=\Gamma\omega. \tag{3.22}$$

By (3.21) and (3.22), we have

$$\Theta\omega=\Gamma\omega=\rho, \tag{3.23}$$

$$\text{implies}\quad \Theta\Gamma\omega=\Gamma\Theta\omega \ \text{ (by } \perp-\text{weakly compatibility of } \Theta \text{ and } \Gamma). \tag{3.24}$$

Again from (3.23) we have $\Theta\Gamma\omega = \Theta\rho$ and $\Gamma\Theta\omega = \Gamma\rho$. So $\Theta\rho = \Gamma\rho$. Thus ω and ρ are two fixed points of Θ and Γ. Next we prove that $\rho = \omega$. We take contradiction. Then $\Theta\rho \neq \Theta\omega$ and we get

$$\wp(\Gamma\rho, \Gamma\omega) < \wp(\Theta\rho, \Theta\omega)$$
$$\wp(\Gamma\rho, \Gamma\omega) < \wp(\Gamma\rho, \Gamma\omega),$$

which is contradiction. So $\rho = \omega$ and hence $\Theta\omega = \Gamma\omega = \omega$.
Thus, Θ and Γ is a $\mathcal{CFP}$ of ω. Likewise, the $\mathcal{UCFP}$ can be proved from Theorem (3.1). □

Remark 3.6. *If we take $\mathcal{Q} = \mathcal{F}$, then we get the following result:*

Corollary 3.7. *Consider $(\mathcal{Q}, \wp, \perp)$ to be a $\mathcal{O}$-metric space, and $\Lambda, \Omega, \Theta, \Gamma : \mathcal{Q} \to \mathcal{Q}$ be four maps satisfies the axioms as follows:*
Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \text{ implies } \wp(\Lambda\varrho, \Omega\varphi) < \nu$$
$$\text{and } \Theta\varrho = \Gamma\varphi \text{ implies } \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi,$$

then

$$\wp(\Lambda\varrho, \Omega\varphi) < \wp(\Theta\varrho, \Gamma\varphi), \text{ if } \Theta\varrho \neq \Gamma\varphi, \ \varrho \perp \varphi \quad \text{and}$$
$$\wp(\Lambda\varrho, \Omega\varphi) \leq \wp(\Theta\varrho, \Gamma\varphi), \ \varrho \perp \varphi \quad \text{for all } \varrho, \varphi \in \mathcal{Q}.$$

Now, we will see the $\mathcal{CFP}$ for four mappings.

Theorem 3.3. *Let $(\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be a complete bipolar $\mathcal{O}-\mathcal{MS}$ and let $\Theta, \Gamma, \Lambda, \Omega : (\mathcal{Q}, \mathcal{F}, \wp, \perp) \rightrightarrows (\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be four covariant maps satisfies the conditions as follows:*

(i) The map $(\Lambda, \Theta, \Omega, \Gamma)$ is $\perp$-compatible,
(ii) $\Theta, \Gamma, \Lambda, \Omega$ are $\perp$-continuous and $\perp$-preserving,
(iii) $\Lambda(\mathcal{Q} \cup \mathcal{F}) \subseteq \Theta(\mathcal{Q} \cup \mathcal{F})$ and $\Omega(\mathcal{Q} \cup \mathcal{F}) \subseteq \Gamma(\mathcal{Q} \cup \mathcal{F})$,
(iv) For any given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\chi, \Gamma\varphi) < \nu + \kappa \text{ implies } \wp(\Lambda\chi, \Omega\varphi) < \nu \tag{3.25}$$
$$\text{and } \Theta\chi = \Gamma\varphi \text{ implies } \Lambda\chi = \Omega\varphi, \tag{3.26}$$

where $\chi \in \mathcal{Q}$ and $\varphi \in \mathcal{F}$. Then the functions Θ, Γ, Λ and Ω have $\mathcal{UCFP}$.

Proof. Let $\chi_0 \in \mathcal{Q}$, $\varphi_0 \in \mathcal{F}$ and choose $\chi_1 \in \mathcal{Q}$ and $\varphi_1 \in \mathcal{F}$ such that $\Lambda\chi_0 = \Theta\chi_1 = \omega_0$ and $\Omega\varphi_0 = \Gamma\varphi_1 = \theta_0$. This can be done since $\Lambda(\mathcal{Q} \cup \mathcal{F}) \subseteq \Theta(\mathcal{Q} \cup \mathcal{F})$ and $\Omega(\mathcal{Q} \cup \mathcal{F}) \subseteq \Gamma(\mathcal{Q} \cup \mathcal{F})$.

$$\chi_0 \perp \chi_1 \qquad \varphi_0 \perp \varphi_1$$
$$\Lambda\chi_0 \perp \Lambda\chi_1 \qquad \Omega\varphi_0 \perp \Omega\varphi_1$$
$$\Theta\chi_1 \perp \Theta\chi_2 \qquad \Gamma\varphi_1 \perp \Gamma\varphi_2$$
$$\Theta^{-1}\Theta\chi_1 \perp \Theta^{-1}\Theta\chi_2 \qquad \Gamma^{-1}\Gamma\varphi_1 \perp \Gamma^{-1}\Gamma\varphi_2$$

$$\chi_1 \perp \chi_2 \qquad\qquad \varphi_1 \perp \varphi_2$$
$$\vdots \qquad\qquad \vdots$$
$$\chi_{\aleph-1} \perp \chi_{\aleph} \qquad\qquad \varphi_{\aleph-1} \perp \varphi_{\aleph}.$$

In general, we can choose $(\chi_{\aleph}, \varphi_{\aleph}) \in \mathcal{Q}\times\mathcal{F}$ such that $\Lambda\chi_{\aleph} = \Theta\chi_{\aleph+1} = \omega_{\aleph}$ and $\Omega\varphi_{\aleph} = \Gamma\varphi_{\aleph+1} = \theta_{\aleph}$ for all $\aleph \in \mathbb{N}\cup\{0\}$.

Let $\nu > 0$ and $\nu \leq \wp(\omega_{\aleph}, \theta_{\pi}) = \wp(\Theta\chi_{\aleph+1}, \Gamma\varphi_{\pi+1}) \leq \nu + \kappa$. Then by condition (iv) of the theorem we have, $\wp(\omega_{\aleph+1}, \theta_{\pi+1}) = \wp(\Lambda\chi_{\aleph+1}, \Omega\varphi_{\pi+1}) < \nu$ and if $\wp(\omega_{\aleph}, \theta_{\pi}) = \wp(\Theta\varrho_{\aleph+1}, \Gamma\varphi_{\pi+1}) = 0$, then again by condition (iv) of the theorem, we have $\wp(\omega_{\aleph+1}, \theta_{\pi+1}) = \wp(\Lambda\varrho_{\aleph+1}, \Omega\varphi_{\pi+1}) = 0$.

Using Lemma 3.1, the sequence $(\omega_{\aleph}, \theta_{\aleph})$ is a Cauchy $\mathcal{O}$-bisequence. Since $(\mathcal{Q}, \mathcal{F}, \wp)$ is complete, hence $(\omega_{\aleph}, \theta_{\aleph})$ biconverges to some point $\rho \in \mathcal{Q}\cap\mathcal{F}$. So $\Lambda\chi_{\aleph}, \Theta\chi_{\aleph}, \Omega\varphi_{\aleph}$ and $\Gamma\varphi_{\aleph}$ converge to ρ. Since the quadruple $(\Lambda, \Theta, \Omega, \Gamma)$ is $\perp$-compatible, we have $\wp(\Gamma\Lambda\chi_{\aleph}, \Lambda\Gamma\varphi_{\aleph}) \to 0$, and $\wp(\Omega\Theta\chi_{\aleph}, \Theta\Omega\varphi_{\aleph}) \to 0$. As all the four mapping Λ, Ω, Θ and Γ are $\perp$-continuous, which implies $\wp(\Gamma\rho, \Lambda\rho) = 0$ and $\wp(\Omega\rho, \Theta\rho) = 0$. Thus, $\Gamma\rho = \Lambda\rho$ and $\Omega\rho = \Theta\rho$. Let $\Theta\rho \neq \Gamma\rho$ then $\wp(\Lambda\rho, \Omega\rho) < \wp(\Theta\rho, \Gamma\rho) = \wp(\Omega\rho, \Lambda\rho) = \wp(\Lambda\rho, \Omega\rho)$. This is a contradiction. So,

$$\Gamma\rho = \Theta\rho = \mathcal{F}\rho = \mathcal{Q}\rho = \omega \ \text{(say)}.$$

By $\perp$-compatibility, this implies $\Gamma\Lambda\rho = \Lambda\Gamma\rho$ and $\Omega\Theta\rho = \Theta\Omega\rho$ that is, $\Gamma\omega = \Lambda\omega$ and $\Omega\omega = \Theta\omega$. If $\Theta\omega \neq \Gamma\omega$ then $\wp(\Lambda, \Omega\omega) < \wp(\Theta\omega, \Gamma\omega) = \wp(\Omega\omega, \Lambda\omega) = \wp(\Lambda\omega, \Omega\omega)$, which is contradiction. So

$$\Theta\omega = \Gamma\omega = \Lambda\omega = \Omega\omega.$$

Now, let $\Theta\omega \neq \omega$, that is $\Theta\Gamma\rho \neq \Gamma\rho$ then $\wp(\Lambda\omega, \omega) = \wp(\Lambda\Gamma\rho, \Omega\rho) < \wp(\Theta\Gamma\rho, \Gamma\rho) = \wp(\Theta\omega, \omega) = \wp(\Omega\omega, \omega) = \wp(\Lambda\omega, \omega)$, which is contradiction. So, $\Theta\omega = \omega = \Gamma\omega = \Lambda\omega = \Omega\omega$. Thus, Λ, Ω, Θ and Γ is a $\mathcal{CFP}$ of ω.

Uniqueness: Consider that ω and θ be two fixed points of Λ, Ω, Θ and Γ, we have

$$\omega_0 \perp \omega \,\text{(or)}\, \omega_0 \perp \theta.$$

Since Γ and Θ are $\perp$-preserving,

$$(\Gamma\omega_0 \perp \Gamma\omega \ \text{ and } \ \Theta\omega_0 \perp \Theta\omega)$$
$$(\Gamma\omega_0 \perp \Gamma\theta \ \text{ and } \ \Theta\omega_0 \perp \Theta\theta).$$

If $\Theta\omega \neq \Gamma\theta$ such that with $\omega \neq \theta$. Then $\wp(\Lambda\omega, \Omega\theta) < \wp(\Theta\omega, \Gamma\theta)$. This implies $\wp(\omega, \theta) < \wp(\omega, \theta)$, a contradiction. So $\Theta\omega = \Gamma\theta$, that is $\omega = \theta$. □

Remark 3.7. *In above theorem, taking $\Theta = \Gamma$ and $\Lambda = \Omega$, Theorem 3.3 follows as a corollary.*

Example 3.1. *Let $\mathcal{Q} = [0, \frac{1}{5}] \cup \{\frac{3\aleph}{5} : \aleph \in \mathbb{N}\}$ and $\mathcal{F} = [0, \frac{1}{5}] \cup \{\frac{3}{10}(2\aleph + 1) : \aleph \in \mathbb{N}\}$ and the distance $\wp : \mathcal{Q}\times\mathcal{F} \to \mathbb{R}^+$ is defined by $\wp(\chi, \varphi) = \left|\chi - \varphi\right|$ such that $\chi \perp \varphi$ for all $\chi \in \mathcal{Q}$ and $\varphi \in \mathcal{F}$,*

$$\chi\perp\varphi \iff \chi, \varphi \geq 0.$$

Then $(\mathcal{Q},\mathcal{F},\wp,\perp)$ *is a complete bipolar* $\mathcal{O}-\mathcal{MS}$. *Assume that* $\Gamma,\Theta:(\mathcal{Q},\mathcal{F},\wp,\perp)\rightrightarrows(\mathcal{Q},\mathcal{F},\wp,\perp)$ *defined by:* $\Gamma\chi=\frac{3\chi}{7}$, $\Theta\chi=\frac{3\chi}{5}$, *for all* $\chi\in[0,\frac{1}{5}]$, *and* $\Gamma(\frac{3\aleph}{5})=\frac{3\aleph}{10(\aleph+1)}$, $\Theta(\frac{3\aleph}{5})=\frac{3\aleph}{5}$, $\Gamma(\frac{3}{10}(2\aleph+1))=0$, $\Theta(\frac{3}{10}(2\aleph+1))=\frac{3}{10}(2\aleph+1)$ *for all* $\aleph\in\mathbb{N}$. *Additionally, we can see that*

$$\Gamma(\mathcal{Q}\cup\mathcal{F})=[0,\frac{1}{10}]\cup\{\frac{3\aleph}{10(\aleph+1)}:\aleph\in\mathbb{N}\},$$

and

$$\Theta(\mathcal{Q}\cup\mathcal{F})=[0,\frac{3}{10}]\cup\{\frac{3\aleph}{5}:\aleph\in\mathbb{N}\}\cup\{\frac{3}{10}(2\aleph+1):\aleph\in\mathbb{N}\}.$$

So, $\Gamma(\mathcal{Q}\cup\mathcal{F})\subseteq\Theta(\mathcal{Q}\cup\mathcal{F})$, Θ *and* Γ *are* $\perp$*-continuous functions and* $\perp$*-preserving.*

Next we prove the $\perp$*-compatibility of* Θ *and* Γ.
Let $(\chi_\aleph,\varphi_\aleph)$ *be a bisequence in* $\mathcal{Q}\times\mathcal{F}$ *such that* $\lim_{\aleph\to+\infty}\Theta\chi_\aleph=\lim_{\aleph\to+\infty}\Gamma\chi_\aleph=\lim_{\aleph\to+\infty}\Theta\varphi_\aleph=\lim_{\aleph\to+\infty}\Gamma\varphi_\aleph=\rho$ *for some* $\rho\in\mathcal{Q}\cap\mathcal{F}=[0,\frac{1}{5}]$.
Without loss of generality, we can assume that $\chi_\aleph,\varphi_\aleph\in[0,\frac{1}{5}]$.
So, $\Theta\chi_\aleph=\frac{3\chi_\aleph}{5}$ *and* $\Gamma\chi_\aleph=\frac{3\chi_\aleph}{7}$. *Both* $\Theta\chi_\aleph$ *and* $\Gamma\chi_\aleph$ *biconverge to* ρ, *so* $\rho=0$.
Now, $\lim_{\aleph\to+\infty}\wp(\Gamma\Theta\chi_\aleph,\Theta\Gamma\varphi_\aleph)=\lim_{\aleph\to+\infty}\wp(\Gamma\rho,\Theta\rho)=0$. *Similarly, we have,* $\lim_{\aleph\to+\infty}\wp(\Theta\Gamma\chi_\aleph,\Theta\Gamma\varphi_\aleph)=0$. *Thus* Θ *and* Γ *are compatible.*

Next, to show that Θ *and* Γ *satisfies the condition (iv) of Theorem 3.1.*
Given $\nu>0$. *Then the maximum value of* κ *is defined by*

$$\kappa=\begin{cases}\frac{2\nu}{5}, & if\ \nu\in(0,\frac{1}{5}]\cup[\frac{3}{10},\frac{1}{2}]\cup[\frac{3}{5},\infty];\\ \frac{3}{10}-\nu, & if\ \nu\in(\frac{1}{5},\frac{3}{10});\\ \frac{3}{5}-\nu, & if\ \nu\in(\frac{1}{2},\frac{3}{5}).\end{cases}$$

Let us verify the above condition for $\nu\in(0,\frac{1}{5}]$. *For this* ν, *we take* $\kappa=\frac{2\nu}{5}$.
Let $\nu\le\wp(\Theta\chi,\Theta\varphi)<\nu+\kappa$. *This implies* $\nu\le\wp(\Theta\chi,\Theta\varphi)<\frac{7\nu}{5}$.
This is possible only if $\chi,\varphi\in[0,\frac{1}{5}]$ *so that* $\frac{3}{5}\left|\chi-\varphi\right|<\frac{7\nu}{5}$.
It gives $\frac{3}{7}\left|\chi-\varphi\right|<\nu$ *and hence* $\wp(\Gamma\chi,\Gamma\varphi)<\nu$. *Similarly, other value of* ν *can be verified easily. All the axioms of Theorem 3.1 are verified and zero is the* $\mathcal{UCFP}$ *of* Θ *and* Γ.

Example 3.2. *Let* $(\mathcal{Q},\mathcal{F},\wp,\perp)$ *be bipolar* $\mathcal{O}-\mathcal{MS}$. *And* $\mathcal{Q}=[0,\frac{1}{5}]\cup\{\frac{3\aleph}{5}:\aleph\in\mathbb{N}\}$ *and* $\mathcal{F}=[0,\frac{1}{5}]\cup\{\frac{3}{10}(2\aleph+1):\aleph\in\mathbb{N}\}$. *and the distance* $\wp:\mathcal{Q}\times\mathcal{F}\to\mathbb{R}^+$ *is defined by* $\wp(\chi,\varphi)=\left|\chi-\varphi\right|$ *such that* $\chi\perp\varphi$ *for all* $\chi\in\mathcal{Q}$ *and* $\varphi\in\mathcal{F}$,

$$\chi\perp\varphi\iff\chi,\varphi\ge 0.$$

Let us assume covariant maps Λ,Ω,Θ *and* $\Gamma:(\mathcal{Q},\mathcal{F},\wp,\perp)\rightrightarrows(\mathcal{Q},\mathcal{F},\wp,\perp)$ *defined by* $\Gamma\chi=\frac{\chi}{3}$, $\Theta\chi=\frac{3\chi}{5}$, $\Lambda\chi=\frac{3\chi}{7}$ *and* $\Omega\chi=\frac{5\chi}{21}$ *for all* $\chi\in[0,\frac{1}{5}]$ *and*

$$\Gamma(\frac{3\chi}{5})=\Omega(\frac{3\chi}{5})=\frac{3\chi}{10(\chi+1)},\Gamma(\frac{3\chi}{5})=\Theta(\frac{3\chi}{5})=\frac{3\chi}{5},$$

$$\Lambda(\frac{3}{10}(2\chi+1))=0=\Omega(\frac{3}{10}(2\chi+1)),$$

$$\Gamma(\frac{3}{10}(2\chi+1)) = \Theta(\frac{3}{10}(2\chi+1)) = \frac{3}{10}(2\chi+1) \text{ for all } \aleph \in \mathbb{N}.$$

Clearly, Λ, Ω, Θ and Γ are self-mapping with $\Lambda(\mathcal{Q}\cup\mathcal{F}) \subseteq \Theta(\mathcal{Q}\cup\mathcal{F})$ and $\Omega(\mathcal{Q}\cup\mathcal{F}) \subseteq \Gamma(\mathcal{Q}\cup\mathcal{F})$ and $\Lambda, \Omega, \Theta, \Gamma$ are $\perp$-continuous functions and $\perp$-preserving.

Next we prove the $\perp$-compatibility of Λ, Ω, Θ and Γ
Let $(\chi_\aleph, \varphi_\aleph)$ be a bisequence in $\mathcal{Q}\times\mathcal{F}$ such that $\lim_{\aleph\to+\infty} \Lambda\chi_\aleph = \lim_{\aleph\to+\infty} \Omega\chi_\aleph = \lim_{\aleph\to+\infty} \Theta\chi_\aleph = \lim_{\aleph\to+\infty} \Gamma\chi_\aleph = \lim_{\aleph\to+\infty} \Theta\varphi_\aleph = \lim_{\aleph\to+\infty} \Gamma\varphi_\aleph = \rho$ for some $\rho \in \mathcal{Q}\cap\mathcal{F} = [0, \frac{1}{5}]$.
Without loss of generality, we can assume that $\chi_\aleph, \varphi_\aleph \in [0, \frac{1}{5}]$.
So, $\Gamma\chi_\aleph = \frac{\chi_\aleph}{3}$, $\Theta\chi_\aleph = \frac{\chi_\aleph}{5}$, $\Lambda\chi_\aleph = \frac{\chi_\aleph}{500}$ and $\Omega\chi_\aleph$. Then $\Gamma\chi_\aleph$, $\Theta\chi_\aleph$, $\Lambda\chi_\aleph$ and $\Omega\chi_\aleph$ biconverge to ρ, so $\rho = 0$.
Now, $\lim_{\aleph\to+\infty} \wp(\Gamma\Lambda\chi_\aleph, \Lambda\Gamma\varphi_\aleph) = \lim_{\aleph\to+\infty} \wp(\Gamma\rho, \Lambda\rho) = 0$. Similarly, we have, $\lim_{\aleph\to+\infty} \wp(\Theta\Omega\chi_\aleph, \Omega\Theta\varphi_\aleph) = \lim_{\aleph\to+\infty} \wp(\Theta\rho, \Omega\rho) = 0$. Thus Λ, Ω, Θ and Γ are $\perp$-compatible.

Next, to show that Λ, Ω, Θ and Γ satisfies the condition (iv) of Theorem 3.3. Given $\nu > 0$. Then the maximum value of κ is defined by

$$\kappa = \begin{cases} \frac{2\nu}{5}, & \text{if } \nu \in (0, \frac{1}{5}] \cup [\frac{3}{10}, \frac{1}{2}] \cup [\frac{3}{5}, \infty]; \\ \frac{3}{10} - \nu, & \text{if } \nu \in (\frac{1}{5}, \frac{3}{10}); \\ \frac{3}{5} - \nu, & \text{if } \nu \in (\frac{1}{2}, \frac{3}{5}). \end{cases}$$

Let us verify the above condition for $\nu \in (0, \frac{1}{5}]$. For this ν, we take $\kappa = \frac{2\nu}{5}$. Let $\nu \le \wp(\Theta\chi, \Gamma\varphi) < \nu + \kappa$. This implies $\nu \le \wp(\Theta\chi, \Gamma\varphi) < \frac{7\nu}{5}$. This is possible only if $\chi, \varphi \in [0, \frac{1}{5}]$ so that $\left|\frac{3\chi}{5} - \frac{\varphi}{3}\right| < \frac{7\nu}{5}$. It gives $\left|\frac{3\chi}{7} - \frac{5\varphi}{21}\right| < \nu$ and hence $\wp(\Lambda\chi, \Omega\varphi) < \nu$. Similarly, other value of ν can be verified easily. All the axioms of Theorem 3.3 are verified and zero is the $\mathcal{UCFP}$ of Λ, Ω, Θ and Γ.

4. Applications

4.1. **Solution of Integral Equation.** As an application of Corollary 3.6, we examine the existence and uniqueness solution of an integral equation in this section.

Theorem 4.1. *Assume that the integral equation*

$$\chi(\varrho) = \mathfrak{h}(\varrho) + \int_{\pounds_1\cup\pounds_2} \mathcal{G}(\varrho, \phi, \chi(\phi))d\phi, \ \ \varrho \in \pounds_1 \cup \pounds_2,$$

where $\pounds_1 \cup \pounds_2$ is a Lebesgue measurable set. Let us assume

(T1) $\mathcal{G} : (\pounds_1^2 \cup \pounds_2^2) \times [0, \infty) \to [0, \infty)$ and $\mathfrak{h} \in L^\infty(\pounds_1) \cup L^\infty(\pounds_2)$,
(T2) A continuous function exists, $\theta : \pounds_1^2 \cup \pounds_2^2 \to [0, \infty)$ such that

$$|\mathcal{G}(\varrho, \phi, \chi(\phi)) - \mathcal{G}(\varrho, \phi, \varphi(\phi)| \le \frac{1}{3}|\theta(\varrho, \phi)|(|\chi(\phi) - \varphi(\phi)|,$$

for $\varrho, \phi \in \pounds_1^2 \cup \pounds_2^2$,

(T3) $\| \int_{\pounds_1\cup\pounds_2} \theta(\varrho, \phi)d\phi\|_\infty \le 1$, i.e $\sup_{\varrho\in\pounds_1\cup\pounds_2} \int_{\pounds_1\cup\pounds_2} |\theta(\varrho, \phi)|d\phi \le 1$.

Then, the integral equation has a unique solution in $\mathcal{L}^\infty(\pounds_1) \cup \mathcal{L}^\infty(\pounds_2)$.

Proof. Let two normed linear spaces be $\mathcal{Q} = \mathcal{L}^{\infty}(£_1)$ and $\mathcal{F} = \mathcal{L}^{\infty}(£_2)$, where $£_1, £_2$ are Lebesgue measurable sets and $\mathfrak{m}(£_1 \cup £_2) < \infty$.
Consider $\wp : \mathcal{Q} \times \mathcal{F} \to \mathbb{R}^+$ to be defined by $\wp(\chi, \varphi) = \|\chi - \varphi\|_\infty$ for all $(\chi, \varphi) \in \mathcal{Q} \times \mathcal{F}$. Define a binary relation $\perp$ on $\mathcal{Q} \times \mathcal{F}$ by $\chi \perp \varphi$ iff $\chi \perp \varphi \geq 0$. Then $(\mathcal{Q}, \mathcal{F}, \wp, \perp)$ be a complete bipolar $\mathcal{O}$-metric space. Let the covariant mapping be $\Gamma : \mathcal{L}^{\infty}(£_1) \cup \mathcal{L}^{\infty}(£_2) \to \mathcal{L}^{\infty}(£_1) \cup \mathcal{L}^{\infty}(£_2)$ by

$$\Gamma(\chi(\varrho)) = \mathfrak{h}(\varrho) + \int_{£_1 \cup £_2} \mathcal{G}(\varrho, \phi, \chi(\phi)) d\phi, \ \varrho \in £_1 \cup £_2.$$

For any given $\nu > 0$ we can find $\kappa = \frac{\nu}{3} > 0$ such that

$$\nu \leq \wp(\chi, \varphi) < \nu + \frac{\nu}{3} = \frac{4}{3}\nu.$$

Now, we have

$$\begin{aligned}
&|\Gamma\chi(\varrho) - \Gamma\varphi(\varrho)| \\
&= \left| \mathfrak{h}(\varrho) + \int_{£_1 \cup £_2} \mathcal{G}(\varrho, \phi, \chi(\phi)) d\phi - \left(\mathfrak{h}(\varrho) + \int_{£_1 \cup £_2} \mathcal{G}(\varrho, \phi, \varphi(\phi)) d\phi \right) \right| \\
&\leq \int_{£_1 \cup £_2} \left| \mathcal{G}(\varrho, \phi, \chi(\phi)) d\phi - \mathcal{G}(\varrho, \phi, \varphi(\phi)) d\phi \right| \\
&\leq \frac{1}{3}(|\chi(\phi) - \varphi(\phi)|) \int_{£_1 \cup £_2} |\theta(\varrho, \phi)| d\phi.
\end{aligned}$$

Taking supremum on both sides,

$$\begin{aligned}
\wp(\Gamma\chi, \Gamma\varphi) &\leq \frac{1}{3}(\|\chi - \varphi\|_\infty) \sup_{\varrho \in £_1 \cup £_2} \int_{£_1 \cup £_2} |\theta(\varrho, \phi)| d\phi \\
&\leq \frac{1}{3}\|\chi - \varphi\|_\infty \\
&\leq \frac{1}{3}\wp(\chi, \varphi) \\
&< \frac{1}{3}(\frac{4}{3}\nu) \\
&< \nu.
\end{aligned}$$

From Corollary 3.6, all the hypothesis are fulfilled and satisfied. Hence the integral equation has a unique solution in $\mathcal{Q} \times \mathcal{F}$.

□

Remark 4.1. *If we take $\mathcal{Q} = \mathcal{F}$, then we get the following result:*

Corollary 4.1. *Consider $(\mathcal{Q}, \wp, \perp)$ to be a complete $\mathcal{O}$-metric space, and $\Lambda, \Omega, \Theta, \Gamma : \mathcal{Q} \to \mathcal{Q}$ be four maps satisfies the axioms as follows:*
Given $\nu > 0$ we can find $\kappa > 0$ such that

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \ \text{ implies } \ \wp(\Lambda\varrho, \Omega\varphi) < \nu$$
$$\text{and } \ \Theta\varrho = \Gamma\varphi \ \text{ implies } \ \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi,$$

then

$$\wp(\Lambda\varrho,\Omega\varphi) < \wp(\Theta\varrho,\Gamma\varphi), \text{ if } \Theta\varrho \neq \Gamma\varphi, \ \varrho \perp \varphi \quad \text{and}$$

$$\wp(\Lambda\varrho,\Omega\varphi) \leq \wp(\Theta\varrho,\Gamma\varphi), \ \varrho \perp \varphi \quad \text{for all } \varrho,\varphi \in \mathcal{Q}.$$

4.2. **Production-Consumption Equilibrium.** For production ϱ_χ and consumption ϱ_φ, daily price patterns, and prices show a significant influence on markets, regardless of whether prices are rising or falling. As a result, $\chi(\phi)$ is interesting to the economist at this time. Assume that

$$\varrho_\chi = \mathfrak{o}_1 + \mathfrak{p}_1\chi(\phi) + \mathfrak{q}_1\frac{d\chi(\phi)}{d\phi} + \mathfrak{r}_1\frac{d^2\chi(\phi)}{d\phi^2},$$
$$\varrho_\varphi = \mathfrak{o}_2 + \mathfrak{p}_2\chi(\phi) + \mathfrak{q}_2\frac{d\chi(\phi)}{d\phi} + \mathfrak{r}_2\frac{d^2\chi(\phi)}{d\phi^2},$$

initially $\chi(0) = 0$, $\frac{d\chi}{d\phi}(0) = 0$, where $\mathfrak{o}_1, \mathfrak{o}_2, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{r}_1$ and $\mathfrak{r}_2$ are constants. A state of dynamic economic equilibrium occurs when market forces are in balance, meaning that the current gap between production and consumption stabilities, that is, $\varrho_\chi = \varrho_\varphi$.Thus,

$$\begin{aligned}
&\mathfrak{o}_1 + \mathfrak{p}_1\chi(\phi) + \mathfrak{q}_1\frac{d\chi(\phi)}{d\phi} + \mathfrak{r}_1\frac{d^2\chi(\phi)}{d\phi^2} = \mathfrak{o}_2 + \mathfrak{p}_2\chi(\phi) + \mathfrak{q}_2\frac{d\chi(\phi)}{d\phi} + \mathfrak{r}_2\frac{d^2\chi(\phi)}{d\phi^2},\\
&(\mathfrak{o}_1-\mathfrak{o}_2) + (\mathfrak{p}_1-\mathfrak{p}_2)\chi(\phi) + (\mathfrak{q}_1-\mathfrak{q}_2)\frac{d\chi(\phi)}{d\phi} + (\mathfrak{r}_1-\mathfrak{r}_2)\frac{d^2\chi(\phi)}{d\phi^2} = 0,\\
&\mathfrak{r}\frac{d^2\chi(\phi)}{d\phi^2} + \mathfrak{q}\frac{d\chi(\phi)}{d\phi} + \mathfrak{p}\chi(\phi) = -\mathfrak{o},\\
&\frac{d^2\chi(\phi)}{d\phi^2} + \frac{\mathfrak{q}}{\mathfrak{r}}\frac{d\chi(\phi)}{d\phi} + \frac{\mathfrak{p}}{\mathfrak{r}}\chi(\phi) = \frac{-\mathfrak{o}}{\mathfrak{r}},
\end{aligned}$$

where $\mathfrak{o} = \mathfrak{o}_1 - \mathfrak{o}_2$, $\mathfrak{p} = \mathfrak{p}_1 - \mathfrak{p}_2$, $\mathfrak{q} = \mathfrak{q}_1 - \mathfrak{q}_2$, $\mathfrak{r} = \mathfrak{r}_1 - \mathfrak{r}_2$. Our initial problem is now represented as

$$\chi''(\phi) + \frac{\mathfrak{q}}{\mathfrak{r}}\chi'(\phi) + \frac{\mathfrak{p}}{\mathfrak{r}}\chi(\phi) = \frac{-\mathfrak{o}}{\mathfrak{r}}, \quad \text{with} \quad \chi(0) = 0 \quad \text{and} \quad \chi'(0) = 0. \tag{4.1}$$

Studying the production and consumption duration time T yields the following problem (4.1) as follows:

$$\chi(\phi) = \int_0^T \mathcal{G}(\phi,\varrho)\mathcal{K}(\varrho,\phi,\chi(\phi))d\phi, \tag{4.2}$$

where the Green function $\mathcal{G}(\phi,\varrho)$ is given as follows:

$$\mathcal{G}(\phi,\varrho) = \begin{cases} \mathfrak{p}e^{\frac{\mathfrak{p}}{2\mathfrak{q}}}(\varrho-\phi), & 0 \leq \varrho \leq \phi \leq T \\ \mathfrak{q}e^{\frac{\mathfrak{p}}{2\mathfrak{q}}}(\varrho-\phi), & 0 \leq \phi \leq \varrho \leq T \end{cases},$$

and $\mathcal{K}\colon [0,T]\times\mathcal{Q}\times\mathcal{F} \to \mathbb{R}$ is a continuous function.
Now, let us assume an operator $\Gamma\colon \mathcal{Q}\times\mathcal{F} \to \mathbb{R}^+$ be described as

$$\Gamma\chi(\phi) = \int_0^T \mathcal{G}(\phi,\varrho)\mathcal{K}(\varrho,\phi,\chi(\phi))d\phi. \tag{4.3}$$

At this moment, a fixed point of Γ is the solution to the dynamic market equilibrium issue, which is represented by (4.1). Equation (4.1) controls the current price $\chi(\phi)$. Let represents the family of real continuous functions of $C[0,\Gamma]$ on $[0,T]$, and assume $\mathcal{Q}\times\mathcal{F} = C[0,T]$.

Next, let us define a distance function $\wp\colon \mathcal{Q}\times\mathcal{F} \to \mathbb{R}^+$ as $\wp(\chi,\varphi) = \|\chi-\varphi\|_\infty, \chi,\varphi \in \mathcal{Q}\times\mathcal{F}$ and $\phi \in [0,T]$. Then $(\mathcal{Q},\mathcal{F},\wp,\perp)$ is a complete bipolar $\mathcal{O}-\mathcal{MS}$.

Theorem 4.2. *Let us assume the map $\Gamma\colon \mathcal{Q}\times\mathcal{F} \to \mathbb{R}$ is a complete bipolar $\mathcal{O}$-metric space $(\mathcal{Q},\mathcal{F},\wp,\perp)$, such that*

(1) there exist $\phi \in [0,T]$ and $\chi,\varphi \in \mathcal{Q}\times\mathcal{F}$ such that $|\mathcal{K}(\varrho,\phi,\chi(\phi)) - \mathcal{K}(\varrho,\phi,\varphi(\phi))| \leq |\chi(\phi) - \varphi(\phi)|$,

(2) a continuous function $\mathcal{G}\colon \mathcal{Q}\times\mathcal{F} \to \mathbb{R}$ such that

$$\sup_{\phi\in[0,T]} \int_0^T \mathcal{G}(\phi,\varrho)d\phi \leq \frac{1}{3}.$$

Then, there exists a unique solution to the dynamic market equilibrium problem (4.1)*.*

Proof. For any given $\nu > 0$, we can find $\kappa > 0$ such that

$$\nu \leq \wp(\chi,\varphi) < \nu + \frac{\nu}{3} = \frac{4}{3}\nu.$$

Then, one has

$$\begin{aligned}
|\Gamma\chi(\phi) - \Gamma\varphi(\phi)| &= \left|\int_0^T \mathcal{G}(\phi,\varrho)\mathcal{K}(\varrho,\phi,\chi(\phi))d\phi - \int_0^T \mathcal{G}(\phi,\varrho)\mathcal{K}(\varrho,\phi,\varphi(\phi))d\phi\right| \\
&\leq \int_0^T \left|\mathcal{G}(\phi,\varrho)\mathcal{K}(\varrho,\phi,\chi(\phi))d\phi - \mathcal{G}(\phi,\varrho)\mathcal{K}(\varrho,\phi,\varphi(\phi))d\phi\right| \\
&\leq \int_0^T \left|\mathcal{G}(\phi,\varrho)(\mathcal{K}(\varrho,\phi,\chi(\phi)) - \mathcal{K}(\varrho,\phi,\varphi(\phi)))d\phi\right| \\
&\leq \int_0^T \mathcal{G}(\phi,\varrho)|(\mathcal{K}(\varrho,\phi,\chi(\phi)) - \mathcal{K}(\varrho,\phi,\varphi(\phi)))|d\phi \\
&\leq \int_0^T \mathcal{G}(\phi,\varrho)|\chi(\phi) - \varphi(\phi)|d\phi.
\end{aligned}$$

Taking supremum on both sides, we have

$$\begin{aligned}
\wp(\Gamma\chi,\Gamma\varphi) &\leq \frac{1}{3}\wp(\chi,\varphi) \\
&< \frac{1}{3}(\frac{4}{3}\nu) \\
&< \nu.
\end{aligned}$$

Thus, the mapping Γ has a $\mathcal{UFP}$. From Corollary 3.6, the equation (4.1) has a unique solution. □

Remark 4.2. *If we take $\mathcal{Q} = \mathcal{F}$, then we get the following result:*

Corollary 4.2. *Consider* $(\mathcal{Q}, \wp, \perp)$ *to be a complete* $\mathcal{O}$*-metric space, and* $\Lambda, \Omega, \Theta, \Gamma : \mathcal{Q} \to \mathcal{Q}$ *be four maps satisfies the axioms as follows:*
Given $\nu > 0$ *we can find* $\kappa > 0$ *such that*

$$\nu \leq \wp(\Theta\varrho, \Gamma\varphi) < \nu + \kappa \;\text{ implies }\; \wp(\Lambda\varrho, \Omega\varphi) < \nu$$
$$\text{and }\; \Theta\varrho = \Gamma\varphi \;\text{ implies }\; \Lambda\varrho = \Omega\varphi; \varrho \perp \varphi,$$

then

$$\wp(\Lambda\varrho, \Omega\varphi) < \wp(\Theta\varrho, \Gamma\varphi), \;\text{ if }\; \Theta\varrho \neq \Gamma\varphi, \;\; \varrho \perp \varphi \quad \text{and}$$
$$\wp(\Lambda\varrho, \Omega\varphi) \leq \wp(\Theta\varrho, \Gamma\varphi), \;\; \varrho \perp \varphi \quad \text{for all }\; \varrho, \varphi \in \mathcal{Q}.$$

5. Conclusion

Throughout this article, we have introduced some new $\mathcal{CFP}$ theorems by using the Meir-Keeler contraction type in the concept of bipolar orthogonal metric spaces. Based on our outcomes we have given some examples to strengthen our results. Also, we have given strong applications to an integral equation and economic problem. It is an interseting open problem to prove from [28] generalized modular metric space to generalized $\mathcal{O}$-modular bipolar metric space.

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