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Existence and Uniqueness Results for a Coupled System of Nonlinear Hadamard Fractional Differential Equations

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Abstract. In this paper, we examine the existence and uniqueness of solutions to a system of four-point non-separated Hadamard fractional differential equations. Applying Leray-Schauder's alternative yields the existence of solutions; Banach's contraction principle establishes the uniqueness of the solution. Lastly, we provide two examples to demonstrate our results.

1. Introduction

The use of fractional differential equations in many different areas, including physics, mechanics, chemistry, engineering, etc., has led to their increased significance. Significant advancements in the field of partial and ordinary differential equations with fractional derivatives have been made in recent years, see the monographs of Kilbas et al. [18], Das [13], Miller and Ross [20], Podlubny [22] and Sabatier et al. [27]. It is evident that the majority of Caputo-type and Riemann-Liouville fractional differential are the focus of this work [2–8, 12, 15, 17, 23, 26]. A large number of academics have recently examined the existence and uniqueness results for a novel class of fractional differential equation with nonlocal, non-separated boundary conditions. Alsulami [3], studied the existence and uniqueness results for a non-separated boundary conditions of FDE

$$\begin{cases} {}^{c}\mathfrak{D}^{\alpha}x(\kappa) = \mathfrak{f}(\kappa, x(\kappa), y(\kappa)),\ \kappa \in [0, \mathfrak{T}], 1 < \alpha \leq 2 \\ {}^{c}\mathfrak{D}^{\beta}y(\kappa) = \mathfrak{g}(\kappa, x(\kappa), y(\kappa)),\ \kappa \in [0, \mathfrak{T}], 1 < \beta \leq 2, \\ x(0) = \lambda_1 y(\mathfrak{T}),\ x'(0) = \lambda_2 y'(\mathfrak{T}), \\ y(0) = \mu_1 x(\mathfrak{T}),\ y'(0) = \mu_2 x'(\mathfrak{T}), \end{cases}$$

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where ${}^{c}\mathfrak{D}^{\alpha}, {}^{c}\mathfrak{D}^{\beta}$ denote the Caputo derivatives of fractional order α and β respectively and $\lambda_i, \mu_i, i = 1,2$ are real constants with $\lambda_i\mu_i \neq 1, i = 1,2$.

In [26] Rao etc. explored the existence and uniqueness results of FDE with nonlocal boundary conditions

$$\begin{cases} {}^{c}\mathfrak{D}^{\alpha}_{0^+}u(\omega) = \mathfrak{f}(\omega, u(\omega), v(\omega)),\ 0 < \omega < 1, \\ {}^{c}\mathfrak{D}^{\beta}_{0^+}v(\omega) = \mathfrak{g}(\omega, u(\omega), v(\omega)),\ 0 < \omega < 1, \\ u(0) = \lambda_1 v(1),\ {}^{c}\mathfrak{D}^{\gamma}_{0^+}u(1) = \lambda_2\, {}^{c}\mathfrak{D}^{\gamma}_{0^+}v(\xi), \\ v(0) = \mu_1 u(1),\ {}^{c}\mathfrak{D}^{\gamma}_{0^+}v(1) = \mu_2\, {}^{c}\mathfrak{D}^{\gamma}_{0^+}u(\xi), \end{cases}$$

where $\alpha, \beta \in (1,2]$, $\xi \in (0,1)$, and $\lambda_i, \mu_i,\ i = 1,2$ real constants with $\mu_1\lambda_1 \neq 1$ and $\mu_2\lambda_2\xi^{2(1-\gamma)} \neq 1$.

As is well known, Hadamard invented a different type of fractional derivative in 1892, and examples of it may be found in the literature [16]. Recently, a few studies on Hadamard fractional BVP have been published. see [11,19,21,25,28,30]. Ahmad and Ntouyas [9,10] and Ardjouni [1] discussed existence and uniqueness of Hadamard FDE by applying Schauder and Banach fixed point theorem. Recently, Zhai etc [32] establish the existence and uniqueness of solutions for the new Hadamard FDE with four-point boundary conditions dependent on two constants l_f, l_g

$$\begin{cases} {}^{H}\mathfrak{D}^{\sigma}u(\varsigma) + \mathfrak{f}(\varsigma, v(\varsigma)) = l_{\mathfrak{f}},\ \varsigma \in (1,e), \\ {}^{H}\mathfrak{D}^{\rho}v(\varsigma) + \mathfrak{g}(\varsigma, u(\varsigma)) = l_{\mathfrak{g}},\ \varsigma \in (1,e), \\ u^{(i)}(1) = v^{(i)}(1) = 0,\ 0 \leq i \leq \kappa - 2, \\ u(e) = cv(\xi),\ v(e) = du(\eta),\ \eta, \xi \in (1,e), \end{cases}$$

where c, d are two parameters with $0 < cd(\ln\eta)^{\sigma-1}(\ln\xi)^{\rho-1} < 1, \sigma, \rho \in (\kappa-1, \kappa]$ are two real numbers and $\kappa \geq 3, l_f, l_g$ are constants.

Inspired by the above stated works, the author discusses existence and uniqueness of the following Hadamard FDE

$$\begin{aligned} &-\mathfrak{D}^{\rho_1}_{1^+}\Big(\phi_{\alpha_1}(\mathfrak{D}^{\sigma_1}_{1^+}\text{ß}(\varsigma))\Big) = \mathfrak{f}(\varsigma, \text{ß}(\varsigma), \varpi(\varsigma)),\ \varsigma \in (1,e), \\ &-\mathfrak{D}^{\rho_2}_{1^+}\Big(\phi_{\alpha_2}(\mathfrak{D}^{\sigma_2}_{1^+}\varpi(\varsigma))\Big) = \mathfrak{g}(\varsigma, \text{ß}(\varsigma), \varpi(\varsigma)),\ \varsigma \in (1,e), \end{aligned} \tag{1.1}$$

satisfies with four-point boundary conditions

$$\begin{aligned} &\text{ß}(1) = \text{ß}'(1) = \text{ß}''(1) = 0,\ \lambda_1\mathfrak{D}^{\gamma_1}_{1^+}\text{ß}(e) = \mu_1\mathfrak{D}^{\gamma_2}_{1^+}\varpi(\eta),\ \mathfrak{D}^{\sigma_1}_{1^+}\text{ß}(1) = 0, \\ &\varpi(1) = \varpi'(1) = \varpi''(1) = 0,\ \lambda_2\mathfrak{D}^{\delta_1}_{1^+}\varpi(e) = \mu_2\mathfrak{D}^{\delta_2}_{1^+}\text{ß}(\xi),\ \mathfrak{D}^{\sigma_2}_{1^+}\varpi(1) = 0, \end{aligned} \tag{1.2}$$

where $\sigma_i, \rho_i, \gamma_i, \delta_i \in R$, $\sigma_i \in (3,4], \rho_i \in (0,1]$, $\mathfrak{D}^{\sigma_i}_{1^+}$ and $\mathfrak{D}^{\rho_i}_{1^+}, i = 1,2$ are the Hadamard fractional derivatives of fractional order σ_i and ρ_i respectively. $\eta, \xi \in (1,e)$, $\gamma_1, \delta_1 \in [1,2], \gamma_2 \in [1,\delta_1], \delta_2 \in [0,\gamma_1]$, $\lambda_1, \lambda_2, \mu_1, \mu_2$ are real positive constants and $\mathfrak{f}, \mathfrak{g} \in \mathfrak{C}\Big([1,e] \times R^{+^2}, R\Big)$. $\alpha_1, \alpha_2 > 1, \phi_{\alpha_k}(\kappa) = |\kappa|^{\alpha_k - 2}, \phi^{-1}_{\alpha_k} = \phi_{\beta_k}, \frac{1}{\alpha_k} + \frac{1}{\beta_k} = 1, k = 1,2$.

We use the following notations for our convenience:

$$\nabla_1 = \left(\frac{1}{\Gamma(\rho_1+1)}\right)^{\beta_1-1} \int_1^e \mathfrak{H}_1(e,\kappa)(\ln\zeta)^{\rho_1(\beta_1-1)}\frac{d\kappa}{\kappa},$$
$$\nabla_2 = \left(\frac{1}{\Gamma(\rho_2+1)}\right)^{\beta_2-1} \int_1^e \mathfrak{H}_2(e,\kappa)(\ln\zeta)^{\rho_2(\beta_2-1)}\frac{d\kappa}{\kappa},$$
$$\nabla_3 = \left(\frac{1}{\Gamma(\rho_2+1)}\right)^{\beta_2-1} \int_1^e \mathfrak{H}_3(e,\kappa)(\ln\zeta)^{\rho_2(\beta_2-1)}\frac{d\kappa}{\kappa},$$
$$\nabla_4 = \left(\frac{1}{\Gamma(\rho_1+1)}\right)^{\beta_1-1} \int_1^e \mathfrak{H}_4(e,\kappa)(\ln\zeta)^{\rho_1(\beta_1-1)}\frac{d\kappa}{\kappa}.$$

Throughout the paper, we assume the following specific assumptions:

($\mathfrak{A}g$) The functions $\mathfrak{f},\mathfrak{g} : [1,e]\times R^{+^2} \to R^+$ are continuous, where R^+ is the set of positive real numbers,

($\mathfrak{A}$) $\nabla = \frac{\lambda_1\lambda_2\Gamma(\sigma_1)\Gamma(\sigma_2)}{\Gamma(\sigma_1-\gamma_1)\Gamma(\sigma_2-\delta_1)} - \frac{\mu_1\mu_2\Gamma(\sigma_1)\Gamma(\sigma_2)}{\Gamma(\sigma_1-\delta_2)\Gamma(\sigma_2-\gamma_2)}(\log\xi)^{\sigma_1-\delta_2-1}(\log\eta)^{\sigma_2-\gamma_2-1} > 0.$

The rest of the paper is organized as follows. In Section 2, we build the Green's functions and estimate the bounds for the corresponding linear Hadamard fractional order boundary value. In Section 3, Using Banach's contraction principle, Leray-Schauder's alternative, we proved the existence and uniqueness of Hadamard FDE (1.1)-(1.2). At the end, we provide an example to show that our theoretical result is feasible.

2. Preliminaries

We include certain Hadamard type fractional calculus notions for the reader's convenience in order to make the system analysis process easier (1.1)-(1.2).

Definition 2.1. *[18] For a function $f : [1,\infty) \to R$, the Hadamard fractional integrals of order δ is*

$${}^{H}I^{\delta}f(\tau) = \frac{1}{\Gamma(\delta)}\int_1^{\tau}\left(\log\frac{\tau}{\kappa}\right)^{\delta-1}\frac{f(\kappa)}{\kappa}d\kappa,\ \delta>0,$$

provided the integral exists.

Definition 2.2. *[18] For a function $f : [1,\infty) \to R$, the Hadamard fractional derivative of fractional order δ is*

$${}^{H}\mathfrak{D}^{\delta}f(\tau) = \frac{1}{\Gamma(m-\delta)}\left(\tau\frac{d}{d\tau}\right)^m\int_1^{\tau}\left(\log\frac{\tau}{\kappa}\right)^{m-\delta-1}\frac{f(\kappa)}{\kappa}d\kappa,\ m-1<\delta<m, m=\lceil\delta\rceil+1,$$

$\lceil\delta\rceil$ *where denotes the integral part of the real number δ and* $\log(\cdot)=\log_e(\cdot)$.

In this section, we construct the Green functions for the homogeneous BVP corresponding to (1.1)-(1.2) and estimate the bounds for the Green functions which are needed to establish the main results.

$$-\mathfrak{D}_{1^+}^{\sigma_1}\text{ß}(\varsigma)=0;\ -\mathfrak{D}_{1^+}^{\sigma_2}\varpi(\varsigma)=0,\ \varsigma\in(1,e), \tag{2.1}$$

$$\begin{aligned} &\text{ß}(1)=\text{ß}'(1)=\text{ß}''(1)=0,\ \lambda_1\mathfrak{D}_{1^+}^{\gamma_1}\text{ß}(e)=\mu_1\mathfrak{D}_{1^+}^{\gamma_2}\varpi(\eta),\\ &\varpi(1)=\varpi'(1)=\varpi''(1)=0,\ \lambda_2\mathfrak{D}_{1^+}^{\delta_1}\varpi(e)=\mu_2\mathfrak{D}_{1^+}^{\delta_2}\text{ß}(\xi)\ \eta,\xi\in(1,e),\end{aligned} \tag{2.2}$$

Lemma 2.1. *[24] Let $x, y \in C[1,e]$ be given functions and $\nabla \neq 0$. Then the system of Hadamard FDE*

$$\begin{aligned} &\mathfrak{D}_{1^+}^{\sigma_1} ß(\varsigma) + x(\varsigma) = 0,\ 1 < \varsigma < e, \\ &\mathfrak{D}_{1^+}^{\sigma_2} \varpi(\varsigma) + y(\varsigma) = 0,\ 1 < \varsigma < e, \end{aligned} \tag{2.3}$$

satisfying the boundary conditions (2.2)*, has a unique solution*

$$\begin{cases} ß(\varsigma) = \int_1^e \mathfrak{H}_1(\varsigma,\kappa)x(\kappa)\frac{d\kappa}{\kappa} + \int_1^e \mathfrak{H}_2(\varsigma,\kappa)y(\kappa)\frac{d\kappa}{\kappa}, \\ \\ \varpi(\varsigma) = \int_1^e \mathfrak{H}_3(\varsigma,\kappa)y(\kappa)\frac{d\kappa}{\kappa} + \int_1^e \mathfrak{H}_4(\varsigma,\kappa)x(\kappa)\frac{d\kappa}{\kappa}, \end{cases} \tag{2.4}$$

where

$$\begin{aligned} \mathfrak{H}_1(\varsigma,\kappa) &= \mathfrak{h}_1(\varsigma,\kappa) + \frac{(\ln\varsigma)^{\sigma_1-1}\mu_1\mu_2\Gamma(\sigma_2)(\ln\eta)^{\sigma_2-\gamma_2-1}}{\nabla\Gamma(\sigma_2-\gamma_2)}\mathfrak{h}_2(\xi,\kappa) \\ \mathfrak{H}_2(\varsigma,\kappa) &= \frac{(\ln\varsigma)^{\sigma_1-1}\mu_1\lambda_2\Gamma(\sigma_2)}{\nabla\Gamma(\sigma_2-\delta_1)}\mathfrak{h}_3(\eta,\kappa) \\ \mathfrak{H}_3(\varsigma,\kappa) &= \mathfrak{h}_4(\varsigma,\kappa) + \frac{(\ln\varsigma)^{\sigma_2-1}\lambda_1\lambda_2\Gamma(\sigma_1)(\ln\xi)^{\sigma_1-\gamma_2-1}}{\nabla\Gamma(\sigma_1-\gamma_2)}\mathfrak{h}_3(\eta,\kappa) \\ \mathfrak{H}_4(\varsigma,\kappa) &= \frac{(\ln\varsigma)^{\sigma_2-1}\mu_2\lambda_1\Gamma(\sigma_1)}{\nabla\Gamma(\sigma_1-\gamma_1)}\mathfrak{h}_2(\xi,\kappa), \forall\ \varsigma,\kappa \in [1,e], \end{aligned}$$

and

$$\begin{aligned} \mathfrak{h}_1(\varsigma,\kappa) &= \frac{1}{\Gamma(\sigma_1)}\begin{cases} (\ln\varsigma)^{\sigma_1-1}(1-\ln\kappa)^{\sigma_1-\gamma_1-1} - \left(\ln\frac{\varsigma}{\kappa}\right)^{\sigma_1-1},\ 1 \le \kappa \le \varsigma \le e, \\ (\ln\varsigma)^{\sigma_1-1}(1-\ln\kappa)^{\sigma_1-\gamma_1-1},\ 1 \le \varsigma \le \kappa \le e, \end{cases} \\ \mathfrak{h}_2(\varsigma,\kappa) &= \frac{1}{\Gamma(\sigma_1-\delta_2)}\begin{cases} (\ln\varsigma)^{\sigma_1-\delta_2-1}(1-\ln\kappa)^{\sigma_1-\gamma_1-1} - \left(\ln\frac{\varsigma}{\kappa}\right)^{\sigma_1-\delta_2-1},\ 1 \le \kappa \le \varsigma \le e, \\ (\ln\varsigma)^{\sigma_1-\delta_2-1}(1-\ln\kappa)^{\sigma_1-\gamma_1-1},\ 1 \le \varsigma \le \kappa \le e, \end{cases} \\ \mathfrak{h}_3(\varsigma,\kappa) &= \frac{1}{\Gamma(\sigma_2-\gamma_2)}\begin{cases} (\ln\varsigma)^{\sigma_2-\gamma_2-1}(1-\ln\kappa)^{\sigma_2-\delta_1-1} - \left(\ln\frac{\varsigma}{\kappa}\right)^{\sigma_2-\gamma_2-1},\ 1 \le \kappa \le \varsigma \le e, \\ (\ln\varsigma)^{\sigma_2-\gamma_2-1}(1-\ln\kappa)^{\sigma_2-\delta_1-1},\ 1 \le \varsigma \le \kappa \le e, \end{cases} \\ \mathfrak{h}_4(\varsigma,\kappa) &= \frac{1}{\Gamma(\sigma_2)}\begin{cases} (\ln\varsigma)^{\sigma_2-1}(1-\ln\kappa)^{\sigma_2-\delta_1-1} - \left(\ln\frac{\varsigma}{\kappa}\right)^{\sigma_2-1},\ 1 \le \kappa \le \varsigma \le e, \\ (\ln\varsigma)^{\sigma_2-1}(1-\ln\kappa)^{\sigma_2-\delta_1-1},\ 1 \le \varsigma \le \kappa \le e, \end{cases} \end{aligned} \tag{2.5}$$

Lemma 2.2. *Let $3 < \sigma_i \le 4, 0 < \rho_i \le 1$ for $i = 1,2$ and $h, k \in \mathfrak{C}[1,e]$. Then the unique solution of*

$$\begin{cases} \mathfrak{D}_{1^+}^{\rho_1}\Big(\phi_{\alpha_1}(\mathfrak{D}_{1^+}^{\sigma_1} ß(\varsigma))\Big) + h(\varsigma) = 0,\ \varsigma \in (1,e), \\ \mathfrak{D}_{1^+}^{\rho_2}\Big(\phi_{\alpha_2}(\mathfrak{D}_{1^+}^{\sigma_2} \varpi(\varsigma))\Big) + k(\varsigma) = 0,\ \varsigma \in (1,e), \\ ß(1) = ß'(1) = ß''(1) = 0,\ \lambda_1\mathfrak{D}_{1^+}^{\gamma_1} ß(e) = \mu_1\mathfrak{D}_{1^+}^{\gamma_2}\varpi(\eta),\ \mathfrak{D}_{1^+}^{\sigma_1} ß(1) = 0, \\ \varpi(1) = \varpi'(1) = \varpi''(1) = 0,\ \lambda_2\mathfrak{D}_{1^+}^{\delta_1}\varpi(e) = \mu_2\mathfrak{D}_{1^+}^{\delta_2} ß(\xi),\ \mathfrak{D}_{1^+}^{\sigma_2}\varpi(1) = 0, \end{cases} \tag{2.6}$$

is

$$\begin{cases} ß(\varsigma) = \int_1^e \mathfrak{H}_1(\varsigma,\kappa)\phi_{\beta_1}\Big(\frac{1}{\Gamma(\rho_1)}\int_1^\kappa \Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}h(\tau)\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\ \quad + \int_1^e \mathfrak{H}_2(\varsigma,\kappa)\phi_{\beta_2}\Big(\frac{1}{\Gamma(\rho_2)}\int_1^\kappa \Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}k(\tau)\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa}, \\ \\ \varpi(\varsigma) = \int_1^e \mathfrak{H}_3(\varsigma,\kappa)\phi_{\beta_2}\Big(\frac{1}{\Gamma(\rho_2)}\int_1^\kappa \Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}k(\tau)\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\ \quad + \int_1^e \mathfrak{H}_4(\varsigma,\kappa)\phi_{\beta_1}\Big(\frac{1}{\Gamma(\rho_1)}\int_1^\kappa \Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}h(\tau)\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa}. \end{cases} \tag{2.7}$$

Proof. Let $\phi = \mathfrak{D}^{\sigma_1}_{1^+}ß, \omega = \phi_{\alpha_1}(\phi)$ and $\psi = \mathfrak{D}^{\sigma_2}_{1^+}\varpi, \vartheta = \phi_{\alpha_2}(\psi)$. Then the solution of initial value problem,

$$\begin{cases} \mathfrak{D}^{\rho_1}_{1^+}\omega(t) + h(\varsigma) = 0,\ \varsigma \in (1,e) \\ \mathfrak{D}^{\rho_2}_{1^+}\vartheta(\varsigma) + k(\varsigma) = 0,\ \varsigma \in (1,e) \\ \omega(1) = 0.\ \vartheta(1) = 0. \end{cases} \tag{2.8}$$

And $0 < \rho_1, \rho_2 \le 1$. An equivalent integral equation for (2.8) is given by

$$\omega(t) = c_1(\ln\varsigma)^{\rho_1-1} - I^{\rho_1}_{1^+}h(\varsigma),\ \varsigma \in (1,e)$$
$$\vartheta(\varsigma) = d_1(\ln\varsigma)^{\rho_2-1} - I^{\rho_2}_{1^+}k(\varsigma),\ \varsigma \in (1,e).$$

From the relation $\omega(1) = \vartheta(1) = 0$, we get $c_1 = 0, d_1 = 0$ and consequently

$$\omega(\varsigma) = -I^{\rho_1}_{1^+}h(\varsigma),\ \varsigma \in (1,e),\ \vartheta(\varsigma) = -I^{\rho_2}_{1^+}k(\varsigma),\ \varsigma \in (1,e). \tag{2.9}$$

Noting that $\mathfrak{D}^{\sigma_1}_{1^+}ß = \phi,\ \phi = \phi^{-1}_{\alpha_1}(\omega)$ and $\mathfrak{D}^{\sigma_2}_{1^+}\varpi = \psi,\ \psi = \phi^{-1}_{\alpha_2}(\vartheta)$ we have from (2.9) that the solution of (2.8) satisfies

$$\begin{cases} \mathfrak{D}^{\sigma_1}_{1^+}ß = \phi^{-1}_{\alpha_1}(-I^{\rho_1}_{1^+}h(\varsigma)),\ \varsigma \in (1,e) \\ \mathfrak{D}^{\sigma_2}_{1^+}\varpi = \phi^{-1}_{\alpha_2}(-I^{\rho_2}_{1^+}k(\varsigma)),\ \varsigma \in (1,e) \\ ß(1) = ß'(1) = ß''(1) = 0,\ \lambda_1\mathfrak{D}^{\gamma_1}_{1^+}ß(e) = \mu_1\mathfrak{D}^{\gamma_2}_{1^+}\varpi(\eta), \\ \varpi(1) = \varpi'(1) = \varpi''(1) = 0,\ \lambda_2\mathfrak{D}^{\delta_1}_{1^+}\varpi(e) = \mu_2\mathfrak{D}^{\delta_2}_{1^+}ß(\xi). \end{cases} \tag{2.10}$$

By Lemma 2.1, the solution of equation (2.10) can be written as

$$ß(\varsigma) = -\int_1^e \mathfrak{H}_1(\varsigma,\kappa)\phi^{-1}_{\alpha_1}(-I^{\rho_1}_{1^+}h(\kappa))\frac{d\kappa}{\kappa} - \int_1^e \mathfrak{H}_2(\varsigma,\kappa)\phi^{-1}_{\alpha_2}(-I^{\rho_2}_{1^+}k(\kappa))\frac{d\kappa}{\kappa},\ \varsigma \in (1,e)$$
$$\varpi(\varsigma) = -\int_1^e \mathfrak{H}_3(\varsigma,\kappa)\phi^{-1}_{\alpha_2}(-I^{\rho_2}_{1^+}k(\kappa))\frac{d\kappa}{\kappa} - \int_1^e \mathfrak{H}_4(\varsigma,\kappa)\phi^{-1}_{\alpha_1}(-I^{\rho_1}_{1^+}h(\kappa))\frac{d\kappa}{\kappa},\ \varsigma \in (1,e)$$

since $h(\kappa) \geq 0, k(\kappa) \geq 0, \kappa \in [1,e]$, we have $\phi_{\alpha_1}^{-1}(-I_{1^+}^{\rho_1}h(\kappa)) = -\phi_{\beta_1}(I_{1^+}^{\rho_1}h(\kappa))$,
$\phi_{\alpha_2}^{-1}(-I_{1^+}^{\rho_2}k(\kappa)) = -\phi_{\beta_2}(I_{1^+}^{\rho_2}k(\kappa)), \kappa \in [1,e]$ which implies that the solution of equation (2.8) is

$$\begin{aligned}
ß(\varsigma) &= \int_1^e \mathfrak{H}_1(\varsigma,\kappa)\phi_{\beta_1}\Big(\frac{1}{\Gamma(\rho_1)}\int_1^\kappa \Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}h(\tau)\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\
&\quad + \int_1^e \mathfrak{H}_2(\varsigma,\kappa)\phi_{\beta_2}\Big(\frac{1}{\Gamma(\rho_2)}\int_1^\kappa \Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}k(\tau)\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa}, \ \varsigma \in (1,e) \\
\varpi(\varsigma) &= \int_1^e \mathfrak{H}_3(\varsigma,\kappa)\phi_{\beta_2}\Big(\frac{1}{\Gamma(\rho_2)}\int_1^\kappa \Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}k(\tau)\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\
&\quad + \int_1^e \mathfrak{H}_4(\varsigma,\kappa)\phi_{\beta_1}\Big(\frac{1}{\Gamma(\rho_1)}\int_1^\kappa \Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}h(\tau)\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa}, \ \varsigma \in (1,e).
\end{aligned}$$

□

Lemma 2.3. *[24] Suppose that condition* $(\mathfrak{A}2)$ *satisfied. Then the Green's functions* $\mathfrak{H}_l(\varsigma,\kappa), l = 1,2,3,4$ *defined respectively by* (2.5) *have the following properties:*

($\mathfrak{C}g$) $\mathfrak{H}_l \in \Big([1.e]\times[1.e],[0,\infty)]\Big)$ *and* $\mathfrak{H}_l(\varsigma,\kappa) > 0$ *for* $\varsigma,\kappa \in (1,e)$
($\mathfrak{C}$) $\mathfrak{H}_l(\varsigma,\kappa) \leq \mathfrak{H}_l(e,\kappa), l = 1,2,3,4$ *for all* $(\varsigma,\kappa) \in [1,e]\times[1,e]$,

By using the Green's functions $\mathfrak{H}_l(\varsigma,\kappa)$ for $l = 1,2,3,4$ an equivalent way to phrase our problem (1.1)-(1.2) is as the following nonlinear system of integral equations

$$\begin{cases}
ß(\varsigma) = \int_1^e \mathfrak{H}_1(\varsigma,\kappa)\phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}\mathfrak{f}(\tau,ß(\tau),\varpi(\tau))\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\
\qquad + \int_1^e \mathfrak{H}_2(\varsigma,\kappa)\phi_{\beta_2}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_2)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}\mathfrak{g}(\tau,ß(\tau),\varpi(\tau))\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa}, \ \varsigma \in (1,e), \\
\varpi(\varsigma) = \int_1^e \mathfrak{H}_3(\varsigma,\kappa)\phi_{\beta_2}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_2)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}\mathfrak{g}(\tau,ß(\tau),\varpi(\tau))\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\
\qquad + \int_1^e \mathfrak{H}_4(\varsigma,\kappa)\phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}\mathfrak{f}(\tau,ß(\tau),\varpi(\tau))\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \ \varsigma \in (1,e).
\end{cases}$$

3. Existence & Uniqueness of Solutions

We consider the Banach space $\mathfrak{X} = \mathfrak{C}[1,e]$ with the norm $\|\cdot\|$ and the Banach space $\mathfrak{Y} = \mathfrak{X}\times\mathfrak{X}$ with the norm $\|(ß,\varpi)\|_{\mathfrak{Y}} = \max\{\|ß\|,\|\varpi\|\}, \|ß\| = \max_{\varsigma\in[1,e]}|ß(\varsigma)|, \|\varpi\| = \max_{\varsigma\in[1,e]}|\varpi(\varsigma)|$.

We define the operators $\mathfrak{Q}_1,\mathfrak{Q}_2 : \mathfrak{Y}\to\mathfrak{X}$ and $\mathfrak{Q} : \mathfrak{Y}\to\mathfrak{Y}$ by

$$\mathfrak{Q}(ß,\varpi) = \Big(\mathfrak{Q}_1(ß,\varpi), \mathfrak{Q}_2(ß,\varpi)\Big),$$

where

$$\begin{aligned}
\mathfrak{Q}_1(ß,\varpi) &= \int_1^e \mathfrak{H}_1(\varsigma,\kappa)\phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}\mathfrak{f}(\tau,ß(\tau),\varpi(\tau))\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\
&\quad + \int_1^e \mathfrak{H}_2(\varsigma,\kappa)\phi_{\beta_2}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_2)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}\mathfrak{g}(\tau,ß(\tau),\varpi(\tau))\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa}, \ \varsigma \in (1,e),
\end{aligned}$$

$$\mathfrak{Q}_2(\text{ß},\varpi) = \int_1^e \mathfrak{H}_3(\varsigma,\kappa)\phi_{\beta_2}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_2)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}\mathfrak{g}(\tau,\text{ß}(\tau),\varpi(\tau))\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa}$$
$$+\int_1^e \mathfrak{H}_4(\varsigma,\kappa)\phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}\mathfrak{f}(\tau,\text{ß}(\tau),\varpi(\tau))\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa}\ \varsigma\in(1,e).$$

Lemma 3.1. *Leray-Schader's alternative [14]: Let $\mathfrak{F}:\mathfrak{E}\to\mathfrak{E}$ be a completely continuous operator (i.e., a map restricted to any bounded set in $\mathfrak{E}$ is compact). Let*

$$\varepsilon(\mathfrak{F}) = \Big\{x\in\mathfrak{E} : \mathfrak{X} = \lambda\mathfrak{F}(x) \text{ for some } 0<\lambda<1\Big\}.$$

Then either the set $\varepsilon(\mathfrak{F})$ is unbounded or $\mathfrak{F}$ has at least one fixed point.

Theorem 3.1. *Assume that:*
$(\mathfrak{S}1)$ *$\mathfrak{f},\mathfrak{g} : [1,e]\times R^{+^2}\to R^+$ are continuous functions and there exist real constant $a_i, b_i\ge 0, i=1,2$ and $a_0, b_0>0$ such that $\forall a_i, b_i\in R, i=1,2,$ we have*

$$|\mathfrak{f}(\varsigma,\text{ß}_1,\text{ß}_2)| \le \phi_{\alpha_1}(a_0 + a_1|\text{ß}_1| + a_2|\text{ß}_2|),$$
$$|\mathfrak{g}(\varsigma,\text{ß}_1,\text{ß}_2)| \le \phi_{\alpha_2}(b_0 + b_1|\text{ß}_1| + b_2|\text{ß}_2|).$$

If $(\nabla_1+\nabla_4)a_1 + (\nabla_2+\nabla_3)b_1 < 1$ and $(\nabla_1+\nabla_4)a_2 + (\nabla_2+\nabla_3)b_2 < 1$, then the BVP (1.1)-(1.2) *has at least one positive solution.*

Proof. Firstly, we demonstrate the operator $\mathfrak{Q}:\mathfrak{Y}\to\mathfrak{Y}$ is completely continuous. Clearly, the operator $\mathfrak{Q}$ is continuous. Let $\Omega\subset\mathfrak{Y}$ be bounded. Then there exist k_1 and k_2 are two positive constants such that

$$|\mathfrak{f}(\varsigma,\text{ß}(\varsigma),\varpi(\varsigma))| \le \phi_{\alpha_1}(k_1),\ |\mathfrak{g}(\varsigma,\text{ß}(\varsigma),\varpi(\varsigma))| \le \phi_{\alpha_2}(k_2),\ \forall(\text{ß},\varpi)\in\Omega.$$

Then, for any $(\text{ß},\varpi)\in\Omega$, we have

$$\begin{aligned}|\mathfrak{Q}_1(\text{ß},\varpi)| &\le \int_1^e \mathfrak{H}_1(e,\kappa)\phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}|\mathfrak{f}(\tau,\text{ß}(\tau),\varpi(\tau))\frac{d\tau}{\tau}|\Big)\frac{d\kappa}{\kappa}\\ &\quad+\int_1^e \mathfrak{H}_2(e,\kappa)\phi_{\beta_2}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_2)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}|\mathfrak{g}(\tau,\text{ß}(\tau),\varpi(\tau))\frac{d\tau}{\tau}|\Big)\frac{d\kappa}{\kappa}\\ &\le k_1\Big(\frac{1}{\Gamma(\rho_1+1)}\Big)^{\beta_1-1}\int_1^e \mathfrak{H}_1(e,\kappa)(\ln\kappa)^{\rho_1(\beta_1-1)}\frac{d\kappa}{\kappa}\\ &\quad+k_2\Big(\frac{1}{\Gamma(\rho_2+1)}\Big)^{\beta_2-1}\int_1^e \mathfrak{H}_2(e,\kappa)(\ln\kappa)^{\rho_2(\beta_2-1)}\frac{d\kappa}{\kappa}\\ &= k_1\nabla_1 + k_2\nabla_2\end{aligned}$$

which implies that

$$\|\mathfrak{Q}_1(\text{ß},\varpi)\| \le k_1\nabla_1 + k_2\nabla_2.$$

Similarly, we get

$$\|\mathfrak{Q}_2(\text{ß},\varpi)\| \le k_2\nabla_3 + k_1\nabla_4.$$

Consequently, the operator $\mathfrak{Q}$ is uniformly bounded as a result of the aforementioned inequalities. We then demonstrate that $\mathfrak{Q}$ is equicontinuous. Let $\varsigma_1, \varsigma_2 \in [1, e]$ with $\varsigma_1 \leq \varsigma_2$. Then we have

$$\begin{aligned}
&|\mathfrak{Q}_1(ß(\varsigma_2), \varpi(\varsigma_2)) - \mathfrak{Q}_1(ß(\varsigma_1), \varpi(\varsigma_1))| \\
&\qquad \leq k_1 \Big| \int_1^e \Big(\mathfrak{H}_1(\varsigma_2, \kappa) - \mathfrak{H}_1(\varsigma_1, \kappa)\Big) \frac{d\kappa}{\kappa} \Big| \phi_{q_1}\Big(\int_1^e \frac{1}{\Gamma(\rho_1)} \Big(\ln \frac{\kappa}{\tau}\Big)^{\rho_1 - 1} \frac{d\tau}{\tau} \Big) \\
&\qquad + k_2 \Big| \int_1^e \Big(\mathfrak{H}_2(\varsigma_2, \kappa) - \mathfrak{H}_2(\varsigma_1, \kappa)\Big) \frac{d\kappa}{\kappa} \Big| \phi_{\beta_2}\Big(\int_1^e \frac{1}{\Gamma(\rho_2)} \Big(\ln \frac{\kappa}{\tau}\Big)^{\rho_2 - 1} \frac{d\tau}{\tau} \Big).
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}
&|\mathfrak{Q}_2(ß(\varsigma_2), \varpi(\varsigma_2)) - \mathfrak{Q}_2(ß(\varsigma_1), \varpi(\varsigma_1))| \\
&\qquad \leq k_2 \Big| \int_1^e \Big(\mathfrak{H}_2(\varsigma_2, \kappa) - \mathfrak{H}_2(\varsigma_1, \kappa)\Big) \frac{d\kappa}{\kappa} \Big| \phi_{q}\Big(\int_1^e \frac{1}{\Gamma(\rho_2)} \Big(\ln \frac{\kappa}{\tau}\Big)^{\rho_2 - 1} \frac{d\tau}{\tau} \Big) \\
&\qquad + k_1 \Big| \int_1^e \Big(\mathfrak{H}_1(\varsigma_2, \kappa) - \mathfrak{H}_1(\varsigma_1, \kappa)\Big) \frac{d\kappa}{\kappa} \Big| \phi_{q}\Big(\int_1^e \frac{1}{\Gamma(\rho_1)} \Big(\ln \frac{\kappa}{\tau}\Big)^{\rho_1 - 1} \frac{d\tau}{\tau} \Big).
\end{aligned}$$

Therefore, the operator $\mathfrak{Q}(ß, \varpi)$ is equicontinuous, and thus operator $\mathfrak{Q}(ß, \varpi)$ is completely continuous.

Finally, it will be verified that the set $\varepsilon = \{(ß, \varpi) \in \mathfrak{Y} | (ß, \varpi) = \lambda \mathfrak{Q}(ß, \varpi), 0 \leq \lambda \leq 1\}$ is bounded. Let $(ß, \varpi) \in \varepsilon$, then $(ß, \varpi) = \lambda \mathfrak{Q}(ß, \varpi)$. For any $\varsigma \in [1, e]$, we have

$$ß(\varsigma) = \lambda \mathfrak{Q}_1(ß, \varpi)(\varsigma), \ \varpi(\varsigma) = \lambda \mathfrak{Q}_2(ß, \varpi)(\varsigma).$$

From $(\mathfrak{S}1)$, we have

$$\begin{aligned}
|ß(\varsigma)| &= |\lambda \mathfrak{Q}_1(ß, \varpi)(\varsigma)| \leq |\mathfrak{Q}_1(ß, \varpi)(\varsigma)| \\
&\leq \int_1^e \mathfrak{H}_1(e, \kappa) \phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)} \Big(\ln \frac{\kappa}{\tau}\Big)^{\rho_1 - 1} |\mathfrak{f}(\tau, ß(\tau), \varpi(\tau)) \frac{d\tau}{\tau}| \Big) \frac{d\kappa}{\kappa} \\
&+ \int_1^e \mathfrak{H}_2(e, \kappa) \phi_{\beta_2}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_2)} \Big(\ln \frac{\kappa}{\tau}\Big)^{\rho_2 - 1} |\mathfrak{g}(\tau, ß(\tau), \varpi(\tau)) \frac{d\tau}{\tau}| \Big) \frac{d\kappa}{\kappa} \\
&\leq \Big(a_0 + a_1 |ß| + a_2 |\varpi|\Big) \Big(\frac{1}{\Gamma(\rho_1 + 1)} \Big)^{\beta_1 - 1} \int_1^e \mathfrak{H}_1(e, \kappa) (\ln \kappa)^{\rho_1(\beta_1 - 1)} \frac{d\kappa}{\kappa} \\
&+ \Big(b_0 + b_1 |ß| + b_2 |\varpi|\Big) \Big(\frac{1}{\Gamma(\rho_2 + 1)} \Big)^{\beta_2 - 1} \int_1^e \mathfrak{H}_2(e, \kappa) (\ln \kappa)^{\rho_2(\beta_2 - 1)} \frac{d\kappa}{\kappa} \\
&= \nabla_1 a_0 + a_1 |ß| + a_2 |\varpi|) + \nabla_2 (b_0 + b_1 |ß| + b_2 |\varpi|) \\
&= \nabla_1 a_0 + \nabla_2 b_0 + (\nabla_1 a_1 + \nabla_2 b_1) |ß| + (\nabla_1 a_2 + \nabla_2 b_2) |\varpi|
\end{aligned}$$

and

$$\begin{aligned}
|\varpi(\varsigma)| &= |\lambda \mathfrak{Q}_2(ß, \varpi)(\varsigma)| \leq |\mathfrak{Q}_2(ß, \varpi)(\varsigma)| \\
&\leq \int_1^e \mathfrak{H}_3(e, \kappa) \phi_{\beta_2}\Big(\int_1^s \frac{1}{\Gamma(\rho_2)} \Big(\ln \frac{\kappa}{\tau}\Big)^{\rho_2 - 1} |\mathfrak{g}(\tau, ß(\tau), \varpi(\tau)) \frac{d\tau}{\tau}| \Big) \frac{d\kappa}{\kappa} \\
&+ \int_1^e \mathfrak{H}_4(e, \kappa) \phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)} \Big(\ln \frac{\kappa}{\tau}\Big)^{\rho_1 - 1} |\mathfrak{f}(\tau, ß(\tau), \varpi(\tau)) \frac{d\tau}{\tau}| \Big) \frac{d\kappa}{\kappa}
\end{aligned}$$

$$\leq \Big(b_0 + b_1|ß| + b_2|\varpi|\Big)\Big(\frac{1}{\Gamma(\rho_2+1)}\Big)^{\beta_2-1} \int_1^e \mathfrak{H}_3(e,\kappa)(\ln\kappa)^{\rho_2(\beta_2-1)}\frac{d\kappa}{\kappa}$$
$$+ \Big(a_0 + a_1|ß| + a_2|\varpi|\Big)\Big(\frac{1}{\Gamma(\rho_1+1)}\Big)^{\beta_1-1} \int_1^e \mathfrak{H}_4(e,\kappa)(\ln\kappa)^{\rho_1(\beta_1-1)}\frac{d\kappa}{\kappa}$$
$$= \nabla_3\Big(b_0 + b_1|ß| + b_2|\varpi|\Big) + \nabla_4\Big(a_0 + a_1|ß| + a_2|\varpi|\Big)$$
$$= \nabla_3 b_0 + \nabla_4 a_0 + (\nabla_3 b_1 + \nabla_4 a_1)|ß| + (\nabla_3 b_2 + \nabla_4 a_2)|\varpi|.$$

Hence we have

$$||ß(\varsigma)|| \leq \nabla_1 a_0 + \nabla_2 b_0 + (\nabla_1 a_1 + \nabla_2 b_1)|ß| + (\nabla_1 a_2 + \nabla_2 b_2)|\varpi|,$$
$$||\varpi(\varsigma)|| \leq \nabla_3 b_0 + \nabla_4 a_0 + (\nabla_3 b_1 + \nabla_4 a_1)|ß| + (\nabla_3 b_2 + \nabla_4 a_2)|\varpi|,$$

which imply that

$$||ß|| + ||\varpi|| \leq (\nabla_1+\nabla_4)a_0 + (\nabla_2+\nabla_3)b_0 + \Big[(\nabla_1+\nabla_4)a_1 + (\nabla_2+\nabla_3)b_1\Big]||ß|| + \Big[(\nabla_1+\nabla_4)a_2 + (\nabla_2+\nabla_3)b_2\Big]||\varpi||.$$

Consequently,

$$||(ß,\varpi)|| \leq \frac{(\nabla_1+\nabla_4)a_0 + (\nabla_2+\nabla_3)b_0}{\mathfrak{M}},$$

where $\mathfrak{M} = \min\{1-(\nabla_1+\nabla_4)a_1 + (\nabla_2+\nabla_3)b_1,\ 1-(\nabla_1+\nabla_4)a_2 + (\nabla_2+\nabla_3)b_2\}$, for any $\varsigma \in [1,e]$, which proves that ε is bounded. Thus, by Lemma 3.1, the operator $\mathfrak{Q}$ has at least one fixed point. Hence, the boundary value problem (1.1)-(1.2) has at least one solution. □

In the second result, we use Banach's contraction principle to demonstrate the existence and uniqueness of solutions to the BVP (1.1)-(1.2).

Theorem 3.2. *Assume that:*
$(\mathfrak{S}2)$ $\mathfrak{f}, \mathfrak{g} : [1,e]\times R^2 \to R$ *are continuous functions and there exist constants m_1 and m_2 such that, for all $\varsigma \in [1,e]$ and $ß_i, \varpi_i \in R, i = 1,2,$ we have*

$$|\mathfrak{f}(\varsigma, ß_1, ß_2) - \mathfrak{f}(\varsigma,\varpi_1,\varpi_2)| \leq \phi_{\alpha_1}\Big[m_1|ß_1-\varpi_1| + m_2|ß_2-\varpi_2|\Big],$$
$$|\mathfrak{g}(\varsigma, ß_1, ß_2) - \mathfrak{g}(\varsigma,\varpi_1,\varpi_2)| \leq \phi_{\alpha_2}\Big[n_1|ß_1-\varpi_1| + n_2|ß_2-\varpi_2|\Big].$$

If $(\nabla_1+\nabla_4)(m_1+m_2) + (\nabla_2+\nabla_3)(n_1+n_2) < 1$, then the BVP (1.1)-(1.2) *has a unique solution.*

Proof. Define $\sup_{\varsigma\in[1,e]} \mathfrak{f}(\varsigma,0,0) = \phi_p(\mathfrak{M}_1) < \infty$ and $\sup_{\varsigma\in[1,e]} \mathfrak{g}(\varsigma,0,0) = \phi_p(\mathfrak{M}_2) < \infty$ such that

$$r \geq \frac{(\nabla_1+\nabla_4)\mathfrak{M}_1 + (\nabla_2+\nabla_3)\mathfrak{M}_2}{1-\Big[(\nabla_1+\nabla_4)(m_1+m_2) + (\nabla_2+\nabla_3)(n_1+n_2)\Big]}.$$

We show that $\mathfrak{Q}(\mathfrak{B}_r) \subset \mathfrak{B}_r$, where $\mathfrak{B}_r = \{(ß,\varpi) \in \mathfrak{Y} : ||(ß,\varpi)|| \leq r\}$. For $(ß,\varpi) \in \mathfrak{B}_r, \varsigma \in [1,e]$, we have

$$|\mathfrak{f}(\varsigma, ß(\varsigma), \varpi(\varsigma))| \leq |\mathfrak{f}(\varsigma, ß(\varsigma),\varpi(\varsigma)) - \mathfrak{f}(\varsigma,0,0)| + |\mathfrak{f}(\varsigma,0,0)|$$
$$\leq \phi_{\alpha_1}\Big[m_1|ß(\varsigma)| + m_2|\varpi(\varsigma)|\Big] + \phi_{\alpha_1}(\mathfrak{M}_1)$$
$$\leq \phi_{\alpha_1}\Big[m_1||ß|| + m_2||\varpi|| + \mathfrak{M}_1\Big]$$

and

$$|\mathfrak{g}(\varsigma, \text{ß}(\varsigma), \varpi(\varsigma))| \leq \phi_{\alpha_2}\Big[n_1\|\text{ß}\| + n_2\|\varpi\| + \mathfrak{M}_2\Big],$$

which leads to

$$\begin{aligned}
|\mathfrak{Q}_1(\text{ß}, \varpi)| &\leq \int_1^e \mathfrak{H}_1(e,\kappa)\phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}|\mathfrak{f}(\tau, \text{ß}(\tau), \varpi(\tau))\frac{d\tau}{\tau}|\Big)\frac{d\kappa}{\kappa} \\
&\quad + \int_1^e \mathfrak{H}_2(e,\kappa)\phi_{\beta_2}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_2)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}|\mathfrak{g}(\tau, \text{ß}(\tau), \varpi(\tau))\frac{d\tau}{\tau}|\Big)\frac{d\kappa}{\kappa} \\
&\leq (m_1\|\text{ß}\| + m_2\|\varpi\| + \mathfrak{M}_1)\Big(\frac{1}{\Gamma(\rho_1+1)}\Big)^{\beta_1-1}\int_1^e \mathfrak{H}_1(e,\kappa)(\ln\kappa)^{\rho_1(\beta_1-1)}\frac{d\kappa}{\kappa} \\
&\quad + (n_1\|\text{ß}\| + n_2\|\varpi\| + \mathfrak{M}_2)\Big(\frac{1}{\Gamma(\rho_2+1)}\Big)^{\beta_2-1}\int_1^e \mathfrak{H}_2(e,\kappa)(\ln\kappa)^{\rho_2(\beta_2-1)}\frac{d\kappa}{\kappa} \\
&= \nabla_1(m_1\|\text{ß}\| + m_2\|\varpi\| + \mathfrak{M}_1) + \nabla_2(n_1\|\text{ß}\| + n_2\|\varpi\| + \mathfrak{M}_2) \\
&= (\nabla_1 m_1 + \nabla_2 n_1)\|\text{ß}\| + (\nabla_1 m_2 + \nabla_2 n_2)\|\varpi\| + \nabla_1\mathfrak{M}_1 + \nabla_2\mathfrak{M}_2 \\
&\leq [\nabla_1(m_1+m_2) + \nabla_2(n_1+n_2)]r + \nabla_1\mathfrak{M}_1 + \nabla_2\mathfrak{M}_2
\end{aligned}$$

Hence

$$\|\mathfrak{Q}_1(\text{ß}, \varpi)\| \leq [\nabla_1(m_1+m_2) + \nabla_2(n_1+n_2)]r + \nabla_1\mathfrak{M}_1 + \nabla_2\mathfrak{M}_2.$$

In the same way, we obtain

$$\|\mathfrak{Q}_2(\text{ß}, \varpi)\| \leq \Big[\nabla_3(n_1+n_2) + \nabla_4(m_1+m_2)\Big]r + \nabla_3\mathfrak{M}_2 + \nabla_4\mathfrak{M}_1.$$

Consequently,

$$\begin{aligned}
\mathfrak{Q}(\text{ß}, \varpi) &\leq [\nabla_1(m_1+m_2) + \nabla_2(n_1+n_2)]r + \nabla_1\mathfrak{M}_1 + \sigma_2\mathfrak{M}_2 \\
&\quad + \Big[\nabla_3(n_1+n_2) + \nabla_4(m_1+m_2)\Big]r + \nabla_3\mathfrak{M}_2 + \nabla_4\mathfrak{M}_1 \\
&\leq r.
\end{aligned}$$

Now for $(\text{ß}_2, \varpi_2), (\text{ß}_1, \varpi_1) \in \mathfrak{Y}$ and for any $\varsigma \in [1, e]$, we get

$$\begin{aligned}
&|\mathfrak{Q}_1(\text{ß}_2, \varpi_2)(\varsigma) - \mathfrak{Q}_1(\text{ß}_1, \varpi_1)(\varsigma)| \\
&\leq \int_1^e \mathfrak{H}_1(e,\kappa)\phi_{\beta_1}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_1)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_1-1}\Big|\mathfrak{f}(\tau, \text{ß}_2(\tau), \varpi_2(\tau)) - \mathfrak{f}(\tau, \text{ß}_1(\tau), \varpi_1(\tau))\Big|\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\
&\quad + \int_1^e \mathfrak{H}_2(e,\kappa)\phi_{\beta_2}\Big(\int_1^\kappa \frac{1}{\Gamma(\rho_2)}\Big(\ln\frac{\kappa}{\tau}\Big)^{\rho_2-1}\Big|\mathfrak{g}(\tau, \text{ß}_2(\tau), \varpi_2(\tau)) - \mathfrak{g}(\tau, \text{ß}_1(\tau), \varpi_1(\tau))\Big|\frac{d\tau}{\tau}\Big)\frac{d\kappa}{\kappa} \\
&\leq \nabla_1\Big(m_1\|\text{ß}_2 - \text{ß}_1\| + m_2\|\varpi_2 - \varpi_1\|\Big) + \nabla_2\Big(n_1\|\text{ß}_2 - \text{ß}_1\| + n_2\|\varpi_2 - \varpi_1\|\Big) \\
&= (\nabla_1 m_1 + \nabla_2 n_1)\|\text{ß}_2 - \text{ß}_1\| + (\nabla_1 m_2 + \nabla_2 n_2)\|\varpi_2 - \varpi_1\|
\end{aligned}$$

and consequently we obtain

$$\|\mathfrak{Q}_1(\text{ß}_2, \varpi_2) - \mathfrak{Q}_1(\text{ß}_1, \varpi_1)\| \leq (\nabla_1(m_1+m_2) + \nabla_2(n_1+n_2))\Big(\|\text{ß}_2 - \text{ß}_1\| + \|\varpi_2 - \varpi_1\|\Big). \tag{3.1}$$

Similarly

$$\|\mathfrak{Q}_2(\text{ß}_2, \varpi_2) - \mathfrak{Q}_2(\text{ß}_1, \varpi_1)\| \leq (\nabla_3(n_1+n_2) + \nabla_4(m_1+m_2))\Big(\|\text{ß}_2 - \text{ß}_1\| + \|\varpi_2 - \varpi_1\|\Big). \tag{3.2}$$

It follows from (3.1) and (3.2)

$$\|\mathfrak{Q}(ß_2,\varpi_2)-\mathfrak{Q}(ß_1,\varpi_1)\| \leq \Big[(\nabla_1+\nabla_4)(m_1+m_2)+(\nabla_2+\nabla_3)(n_1+n_2)\Big]\Big(\|ß_2-ß_1\|+\|\varpi_2-\varpi_1\|\Big).$$

Since $(\nabla_1+\nabla_4)(m_1+m_2)+(\nabla_2+\nabla_3)(n_1+n_2)<1$, therefore, $\mathfrak{Q}$ is a contraction operator. Thus, the operator $\mathfrak{Q}$ has a unique fixed point according to Banach's fixed point theorem, which is the unique solution of BVP (1.1)-(1.2). □

Example

In this section, we'll provide an example to demonstrate the main results.
Consider the following coupled system of Hadamard FBVP

$$\begin{aligned} &-\mathfrak{D}^{0.5}_{1^+}\Big(\phi_{\alpha_1}(\mathfrak{D}^{3.5}_{1^+}ß(\varsigma))\Big)=\mathfrak{f}(\varsigma,ß(\varsigma),\varpi(\varsigma)),\ \varsigma\in(1,e),\\ &-\mathfrak{D}^{0.33}_{1^+}\Big(\phi_{\alpha_2}(\mathfrak{D}^{3.7}_{1^+}\varpi(\varsigma))\Big)=\mathfrak{g}(\varsigma,ß(\varsigma),\varpi(\varsigma)),\ \varsigma\in(1,e),\end{aligned} \tag{3.3}$$

$$\begin{aligned} &ß(1)=ß'(1)=ß''(1)=0,\ \mathfrak{D}^{1.5}_{1^+}ß(e)=\mathfrak{D}^{1.5}_{1^+}\varpi(1.6),\ \mathfrak{D}^{0.5}_{1^+}ß(1)=0,\\ &\varpi(1)=\varpi'(1)=\varpi''(1)=0,\ 0.33\ \mathfrak{D}^{1.67}_{1^+}\varpi(e)=0.5\ \mathfrak{D}^{1.4}_{1^+}ß(1.75),\ D^{0.33}_{1^+}\varpi(1)=0.\end{aligned} \tag{3.4}$$

Here $\sigma_1=3.5,\sigma_2=3.7,\rho_1=0.5,\rho_2=0.33,\gamma_1=1.5,\delta_1=1.67,\gamma_2=1.5,\delta_2=1.4,\xi=1.6,\eta=1.75,\lambda_1=1,\lambda_2=0.33,\mu_1=1,\mu_2=0.5,\alpha_1=\alpha_2=2$.

We obtain $\nabla\approx 3.25>0$, and the assumptions $(\mathfrak{S}1)$ and $(\mathfrak{S}2)$ are satisfied. In addition, we deduce

$$\mathfrak{h}_1(\varsigma,\kappa)=\frac{1}{\Gamma(3.5)}\begin{cases}(\log\varsigma)^{2.5}(1-\log\kappa)-\Big(\log\frac{\varsigma}{\kappa}\Big)^{2.5},\ 1\leq\kappa\leq\varsigma\leq e,\\ (\log\varsigma)^{2.5}(1-\log\kappa),\ 1\leq\varsigma\leq\kappa\leq e,\end{cases}$$

$$\mathfrak{h}_2(\varsigma,\kappa)=\frac{1}{\Gamma(2.1)}\begin{cases}(\log\varsigma)^{1.1}(1-\log\kappa)-\Big(\log\frac{\varsigma}{\kappa}\Big)^{1.1},\ 1\leq\kappa\leq\varsigma\leq e,\\ (\log\varsigma)^{1.1}(1-\log\kappa),\ 1\leq\varsigma\leq\kappa\leq e,\end{cases}$$

$$\mathfrak{h}_3(\varsigma,\kappa)=\frac{1}{\Gamma(2.2)}\begin{cases}(\log\varsigma)^{\alpha_2-\gamma_2-1}(1-\log\kappa)^{1.2}-\Big(\log\frac{\varsigma}{\kappa}\Big)^{1.3},\ 1\leq\kappa\leq\varsigma\leq e,\\ (\log\varsigma)^{\alpha_2-\gamma_2-1}(1-\log\kappa)^{1.2},\ 1\leq\varsigma\leq\kappa\leq e,\end{cases}$$

$$\mathfrak{h}_4(\varsigma,\kappa)=\frac{1}{\Gamma(3.7)}\begin{cases}(\log\varsigma)^{2.7}(1-\log\kappa)^{1.03}-\Big(\log\frac{\varsigma}{\kappa}\Big)^{2.7},\ 1\leq\kappa\leq\varsigma\leq e,\\ (\log\varsigma)^{2.7}(1-\log\kappa)^{1.03},\ 1\leq\varsigma\leq\kappa\leq e,\end{cases}$$

$$\mathfrak{H}_1(\varsigma,\kappa)=h_1(\varsigma,\kappa)+\frac{(\log\varsigma)^{2.5}(0.5)\Gamma(3.7)(\log 1.75)^{1.2}}{(3.25)\Gamma(2.2)}\mathfrak{h}_2(1.6,\kappa)$$

$$\mathfrak{H}_2(\varsigma,\kappa)=\frac{(\log\varsigma)^{2.5}(0.33)\Gamma(3.7)}{(3.25)\Gamma(2.03)}\mathfrak{h}_3(1.75,\kappa)$$

$$\mathfrak{H}_3(\varsigma,\kappa)=\mathfrak{h}_4(\varsigma,\kappa)+\frac{(\log\varsigma)^{2.7}(0.33)\Gamma(3.5)(\log 1.6)}{(3.25)\Gamma(2.0)}\mathfrak{h}_3(1.75,\kappa)$$

$$\mathfrak{H}_4(\varsigma,\kappa)=\frac{(\log\varsigma)^{2.7}(0.5)\Gamma(3.5)}{(3.25)\Gamma(2.0)}\mathfrak{h}_2(1.6,\kappa),\forall\ \varsigma,\kappa\in[1,e],$$

We also deduce $\nabla_1\approx 1.394,\nabla_2\approx 0.1743,\nabla_3,\approx 0.2512,\nabla_4\approx 0.06375$

Example 4.1**:** Let two nonlinear functions $\mathfrak{f}, \mathfrak{g} : [1, e] \times R \times R \to R$ be given by

$$\begin{aligned} \mathfrak{f}(\varsigma, ß, \varpi) &= \frac{e^{-3\varsigma}}{2} + \frac{ß^2 \cos^2 \varsigma}{32(1 + |ß|)} + \frac{|\varpi|^4 \sin^2 \varsigma}{42(1 + \varpi^3)}, \\ \mathfrak{g}(\varsigma, ß, \varpi) &= \frac{4}{\varsigma^2 + 2} + \frac{\sin ß}{10(\varsigma + 4)} + \frac{\tan^{-1} \varpi}{12(3 + \varsigma^2)}. \end{aligned} \tag{3.5}$$

Note that

$$\begin{aligned} |\mathfrak{f}(\varsigma, ß, \varpi)| &\leq \phi_{\alpha_1}\Big(\frac{1}{2} + \frac{1}{32}|ß| + \frac{1}{42}|\varpi|\Big), \\ |\mathfrak{g}(\varsigma, ß, \varpi)| &\leq \phi_{\alpha_2}\Big(\frac{4}{3} + \frac{1}{50}|ß| + \frac{1}{48}|\varpi|\Big). \end{aligned}$$

We get $a_1 = \frac{1}{32}, a_2 = \frac{1}{42}, b_1 = \frac{1}{50}, b_2 = \frac{1}{48}$. By simple calculation, we have $(\nabla_1 + \nabla_4)a_1 + (\nabla_2 + \nabla_3)b_1 \approx 0.05406 < 1$ and $(\nabla_1 + \nabla_4)a_2 + (\nabla_2 + \nabla_3)b_2 \approx 0.0437 < 1$ By Theorem 3.1, the coupled boundary value problem (3.3) -(3.5) has at least one positive solution on [1,e].

Example 4.2**:** Let two nonlinear functions $\mathfrak{f}, \mathfrak{g} : [1, e] \times R \times R \to R$ be given by

$$\begin{aligned} \mathfrak{f}(\varsigma, ß, \varpi) &= \frac{1}{4(1 + \varsigma)^2} \frac{|ß|}{1 + |ß(\varsigma)|} + \frac{1}{30 + 2\varsigma^2} \sin^2 \varpi(\varsigma) + \frac{1}{2}, \ \varsigma \in (1, e), \\ \mathfrak{g}(\varsigma, ß, \varpi) &= \frac{1}{12} \sin(4\pi ß(\varsigma)) + \frac{|\varpi(\varsigma)|}{8(1 + \varsigma)^2} + \frac{1}{\sqrt{1 + \varsigma^2}}, \ \varsigma \in (1, e), \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} |\mathfrak{f}(\varsigma, ß_1, ß_2) - \mathfrak{f}(\varsigma, \varpi_1, \varpi_2)| &\leq \phi_{\alpha_1}\Big(\frac{1}{16}|ß_1 - ß_2| + \frac{1}{32}|\varpi_1 - \varpi_2|\Big), \\ |\mathfrak{g}(\varsigma, ß_1, ß_2) - \mathfrak{g}(\varsigma, \varpi_1, \varpi_2)| &\leq \phi_{\alpha_2}\Big(\frac{1}{12}|ß_1 - ß_2| + \frac{1}{32}|\varpi_1 - \varpi_2|\Big). \end{aligned}$$

We get $m_1 = \frac{1}{16}, m_2 = \frac{1}{32}, n_1 = \frac{1}{12}, n_2 = \frac{1}{32}$. By simple calculation, we have $(\nabla_1 + \nabla_4)(1/16 + 1/32) + (\nabla_2 + \nabla_3)(1/12 + 1/32) \approx 0.185419 < 1$. Thus all the conditions of Theorem 3.2 are satisfied. Problem (3.3)-(3.5) has a unique solution on $[1, e]$.

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