Int. J. Anal. Appl. (2025), 23:13

International Journal of Analysis and Applications

Chaos, Fractals and Bifurcation in Real Dynamics of Two-Parameter Family Associated to Logarithmic Functions

Shaurya Pratap Singh[1], **Mohammad Sajid**[2,*]

[1]*Department of Mathematics, University of Leeds, Leeds, LS2 9JT, UK*
[2]*Department of Mechanical Engineering, College of Engineering, Qassim University, Saudi Arabia*

**Corresponding author:* msajd@qu.edu.sa

Abstract. The purpose of this article is to provide bifurcation diagrams and observe chaotic behaviour in the real dynamics of two-parameter family of function $\Phi(x) = x + (1 - \lambda x)\ln(ax) : x > 0, \lambda > 0, a > 0$. We consider here parameter a is a positive and continuous real parameter while λ is positive but a discrete real parameter. The dynamical properties of this nonlinear system family analyze numerically as well as graphically by using fixed point iterative method. Bifurcation diagrams for the real dynamics of the function are plotted by varying the values of the parameter which are fractals in nature. Also, we show that chaos exists in the dynamics of the function by looking at period-doubling in the bifurcation diagram. Further, chaotic behaviour studies by simulation of the positive Lyapunov exponents which observe by varying the parameters similar to the bifurcation diagrams.

1. Introduction

A major focus of modern research is on studying dynamical systems. Dynamical systems are important in engineering and scientific systems. In the last few decades, the development and advancement in technology as well as mathematical theory allow us to understand and try more complex approaches for intricate nonlinear systems [35]. In the theory of dynamical systems (kind of iterative methods), in particular, chaos theory help us to understand the long-term qualitative behaviour of the system. Chaos theory has many applications in daily life as well as in many scientific disciplines extensively. We can see its application in various types of fields such as modelling [10, 30], optimization [20], stock market [11], photovoltaic plant [1] and so on. Population dynamics is studied in [17]. A detailed study of chaos in fuzzy dynamical systems is provided in [19]. The dynamic behaviour of a discrete memristor at different positions

Received: Oct. 26, 2024.

2020 *Mathematics Subject Classification.* 37C25, 37D45, 37G15, 37M15, 37M25.

Key words and phrases. bifurcation diagram; chaos; fractals; Lyapunov exponents; real periodic points; time series graph; iterative methods; simulation; nonlinear systems.

https://doi.org/10.28924/2291-8639-23-2025-13
ISSN: 2291-8639

can also be understood using chaotic attractor phase diagram, bifurcation diagrams, system state analysis etc [22]. Similarly, a discrete memristive Rulkov (m-Rulkov) neuron model has also been studied by looking at its bifurcation in [14]. Applicability of chaos can also be seen in the uncertainty quantification of epidemic SIR models [9]. Chaos has also been used to study particle motion in high energy physics [36]. Chaotic systems have also been applied to study digital image encyrption by using its properties like ergodicity, pseudo randomness and sensitivity to initial conditions, especially now with the use of two-dimensional Hénon-Sine map (2D-HSM) [34]. This map has better properties like ergodicity and randomness than the existing maps and so is ideal for use.

The fractal dynamics is a unique and systems-based approach of looking at and thinking about systems and organizations. When talking about fractals, they can be full of paradoxes, they are at many times a source of creativity, beauty, and surprise, while we use them at the same time as a powerful tool for simulation, analyzing and communicating complex ideas. These often lead to objects that are so complex that their dimension is not an integer while only start as a simple geometrical object. We can see fractals in the study of Banach, Hilbert, or Euclidean spaces [2] . We can also find its application in a variety of things like regional logistics [3], variation principles [33], ECG classification [32]. Fractals also have applications in nature [16] and even fields like digital imaging [31]. Some advances on fractals can be seen in recent surveys [6, 7] which provide interesting and in depth knowledge on the subject.

Another important definition to know about is of iteration. When we talk about iteration, we mean the repetition of a process in order to generate a sequence of outcomes. One single repetition of the process is considered a single iteration and we then use the outcome of each iteration as the starting point of the next iteration. One of our goals is to describe periodic points, which are the states of the system that repeat after several time steps. We study the dynamics to use it to understand long-term behaviour using the initial behaviour of trajectories. For a given initial value x_0, compute iterations as $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$, $\ldots$. The sequence x_0, $x_1, x_2, x_3, \ldots$ defines the trajectory. Points which come back to the same value after a finite number of iterations of function are called periodic points and a periodic point with period equal to one is known as a fixed point. For a periodic point x_f of period m, the orbit $x_f, x_1 = f(x_f), x_2 = f^2(x_f), \ldots, x_{m-1} = f^{m-1}(x_f)$ is called a cycle or a periodic cycle.

Our focus here is on the chaotic behaviour which is characterized in dynamical systems by the sensitive dependence on initial conditions. Chaos is defined by the sensitive dependence on the initial conditions of the system [13]. By studying chaos, we can observe that simple systems can exhibit complex and unpredictable behaviour [12, 13]. By looking at chaos in dynamical systems, we can try and make an attempt to understand and show the relationships between order and disorder in the system or the simplicity and complexity of the system [18]. The chaotic behaviour in the Newton's iteration of one the iterative methods associated with Kepler's equation studied in [29] is an example. The importance of real dynamics of the functions is also seen mathematically which leads to some important results in the complex plane [8, 28].

We can quantify and understand chaos in dynamical systems in many ways, like it can be quantified by computing Lyapunov exponents or period-doubling in bifurcation diagrams [5, 26]. For a trajectory $\{x_i\}$, $i = 0, 1, 2, \ldots$ starting from x_0, the formula for Lyapunov exponent (LE) is given by

$$L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)|. \tag{1.1}$$

The 2-parameter logarithmic function, that we consider here, is $\Phi(x) = x + (1 - \lambda x)\ln(ax) : x > 0, \lambda > 0, a > 0$. We have to look at bifurcation diagrams which design to show convergence or periodicity or unpredictability that are nothing but the eventual behaviour of iterates. It shows the values approached or visited asymptotically in the system as a function of a bifurcation parameter in the system. We can observe period-doubling in the bifurcation diagram to see the route to chaotic behaviour of the system. In addition, a positive value of Lyapunov exponent shows us that chaotic behaviour presents in dynamical system. A very detailed study on chaotic behaviour and bifurcations about the same family of function using different parameter conditions is given in [27]. The Lyapunov exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories [23]. We can look at the chaotic behaviour in dislocation dynamics by calculating the positive Lyapunov exponent in [4]. Other than bifurcation diagrams and Lyapunov exponents, one can look at time series graphs to see the chaotic behaviour in dynamical systems. We have to look at time series graphs to observe the periodicity at different values of the parameters. For transcendental functions depending on two parameters, we can see both bifurcation and chaotic behaviour in real dynamics in [15]. Moreover, for two-parameter family of some special type of generating functions, a detailed study of the bifurcation as well as chaos in real dynamics can be seen in [25]. Recently, bifurcation diagrams in the real dynamics of entire transcendental function have been explored in [24]. Cobweb diagram shows observable chaos, detail about it is in [21].

In this paper, we have to look bifurcation diagrams which are fractals in nature as well as observe the chaotic behaviour in the real dynamics of 2-parameter family of functions $\Phi(x) = x + (1 - \lambda x)\ln(ax) : x > 0, \lambda > 0, a > 0$ which is one of the nonlinear systems. In Section 2, we generate and discuss bifurcation diagrams of $\Phi(x)$ for different parameter values of $\lambda > 0$ and $a > 0$. To understand the orbits in the dynamics of function $\Phi(x)$, the time series graphs are plotted in Section 3. By computing positive Lyapunov exponents in Section 4, we quantify the chaos in the real dynamics of $\Phi(x)$. In Section 5, we provide a short discussion about generation of graphical objects. At the end, the conclusion is given in Section 6.

2. Bifurcation and Periodic Points

A bifurcation diagram is very useful in understanding the dynamical behaviour of a system. Basically, a bifurcation diagram is a visual summary of the succession of period-doubling produced as a parameter increases/decreases. Bifurcation diagrams are fractals in nature and with them, we can graphically observe the periodicity of the orbits. Whenever period-doubling happens in the real dynamics of functions, we observe it as a route to chaos in dynamical systems. Bifurcation diagrams give many interesting information and are the integral part of nonlinear systems.

In our family of nonlinear functions with two parameters $\Phi(x) = x + (1 - \lambda x)\ln(ax) : x > 0, \lambda > 0, a > 0$; the parameter a is a continuous positive real parameter and λ is a discrete positive real parameter. The dynamics of our two-parameter family of functions changes when both values of parameters cross through certain values. We can see these changes in the bifurcation diagrams which obtain by iteration of $\Phi(x)$. For computation of the bifurcation points, we solve the equations associated to fixed points and neutral fixed point condition. So, we solve the following equations $\Phi^m(x) = x$ and $|(\Phi^m)'(x)| = 1$. The periodic points of $\Phi(x)$ are zeros of $\Phi^m(x) = x$, i.e.,

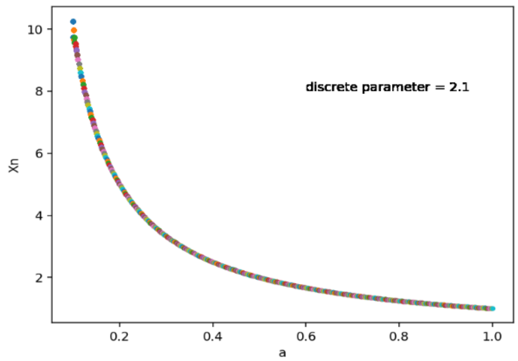
$$\Phi^{m-1}(x) + (1 - \lambda\Phi^{m-1}(x))\ln(a\Phi^{m-1}(x)) = x.$$

When parameter a increases beyond certain value, then the function $\Phi(x)$ exhibits periodic points of period more than or equal to 2. We provide here graphical analysis since the analytical computation of these periodic points of $\Phi(x)$ is significantly very difficult.

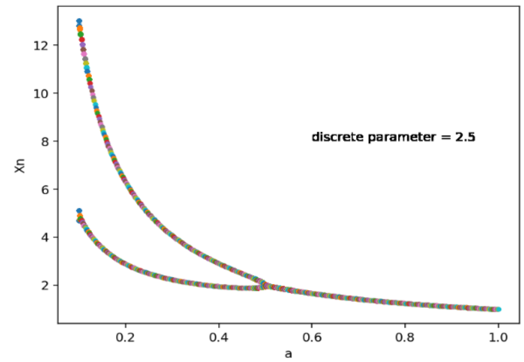
For achieving our purpose, we have considered parameter a as continuous and parameter λ as fix constant. What this means is that, for a particular graph, a varies between an arbitrary range and the values required are calculated for different values of a through that range while the parameter λ is kept constant. i.e a is the horizontal axis while the values are plotted against it on the vertical axis, while the value of λ is constant throughout a graph. We have chosen a as continuous and λ as constant and not vice a versa. Below we will look at graphs for different sets of values of a and λ.

Firstly, we look at the case of the continuous parameter a varying from 0.1 to 1. We observe that, in Figure 1, when the parameter λ (discrete parameter) is changed from 2.1 to 2.5 and then to 2.8 subsequently, then the point at which period-doubling begins also changes for a. For the small value of λ, the period doubling happens at a much smaller value of a, and here it does not happen at all when λ = 2.1 in our range of a. When the value of λ= 2.5, then period doubling happens at a greater value of a and only happens once in our range of a. Then, when we increase the value to λ = 2.8, then the point of a at which period doubling happens for the first time is the largest and we can see the period-doubling happening many times in this case showing us the presence of chaos.

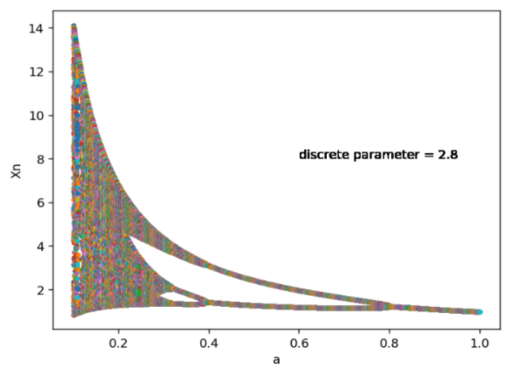
Next, we see the case when a varies from 1.0 to 2.0. Here also when the parameter λ is greater, then the first period-doubling occurs for the first time for greater values of a and when the value of λ is smaller, then period-doubling happens for the first time for smaller values of a. In Figure 2, we observe that as λ is increased from 3 to 3.8, then the first period-doubling happens at higher



(a) λ=2.1



(b) λ=2.5

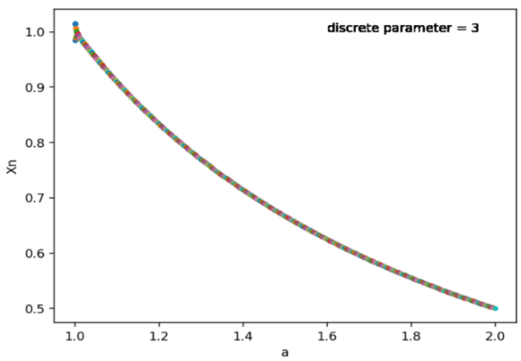


(c) λ=2.8

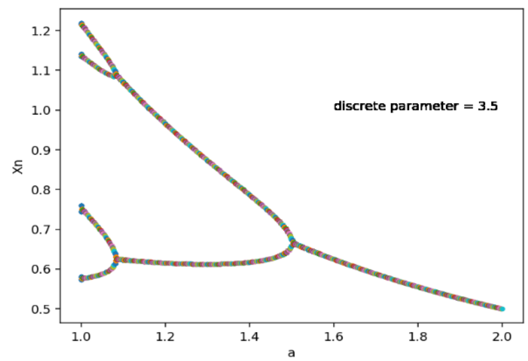
Figure 1. Bifurcation diagrams for $0.1 \leq a \leq 1$ and different parameter value of (a) $\lambda = 2.1$, $\lambda = 2.5$ and $\lambda = 2.8$

values of the variable a. In Figure 2(a) for λ=3, we observe only one period doubling which also occurs at a low value of a in range of 1.0 to 2.0. Then, in Figure 2(b), λ=3.5 we can see more than one period doubling scenario and the value of a for which the periodicity changes from 1 to 2 is relatively larger here compared to Figure 2(a). Finally, in Figure 2(c) for λ=3.8, we get multiple period doubling instances starting with a high value of a on the horizontal axis for the first split and eventually chaos is observed in our range. This indicates that, as the value of λ increases, we can see period doubling and chaotic behaviour for relatively larger values of a in range from 1.0 to 2.0

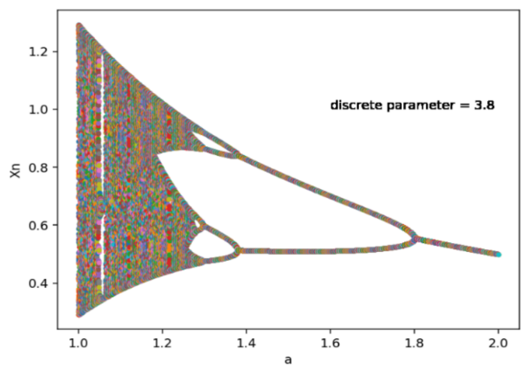
Lastly, we consider the case when a varies from 1.5 to 2.5. In this case , we can see that when the discrete parameter λ is greater, then the first period-doubling occurs for the first time for greater values of a and when λ is smaller, then the first period-doubling happens for smaller values of a. In Figure 3, as the value of λ increases from 3.8 to 4.3, the first period-doubling happens for larger values of the parameter a. In Figure 3(a) when λ=3.8 we can observe only one period doubling instance in our range of a. Figure 3(b) when λ=4 we can see multiple period doubling instances



(a) λ=3



(b) λ=3.5



(c) λ=3.8

Figure 2. Bifurcation diagrams for $1 \leq a \leq 2$ and different parameter value of (a) $\lambda = 3$, $\lambda = 3.5$ and $\lambda = 3.8$

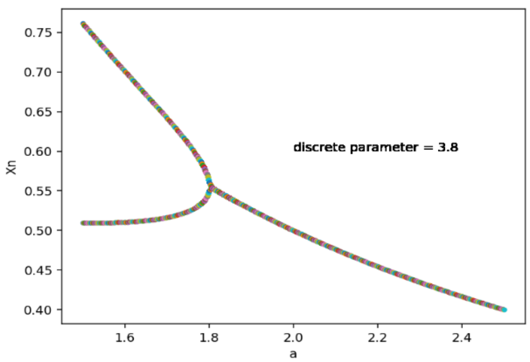
and also the first time when the diagram splits into 2, the value of a is higher comparatively to the value of a in Figure 3(a). Finally, in Figure 3(c) where λ=4.3, periodicity changes from 1 to 2 at higher value of a and we see multiple period doubling instances starting with a high value of a on the horizontal axis for the first split and eventually chaos is observed in our range.

Thus, we can see a pattern that in the above figures as the value of parameter λ is increased, then the values of a for which the period doubling happens for the first time also increases and subsequently, we observe the period-doubling happen multiple times for the same interval of a when the value of the parameter λ is larger.

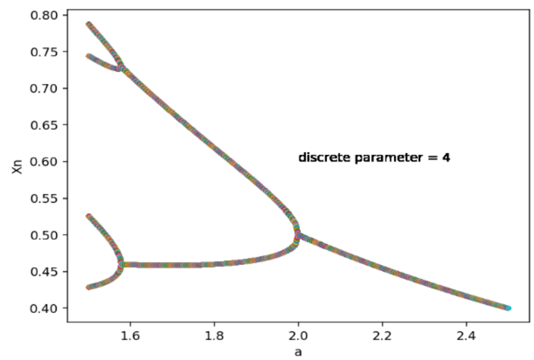
The presence of periodic points of periods more than or equal to 2 visualized by bifurcation diagrams leads to the route to chaos in our dynamical systems which can be quantified by Lyapunov exponents and we provide simulation of it later in the paper.

3. Time Series Graphs

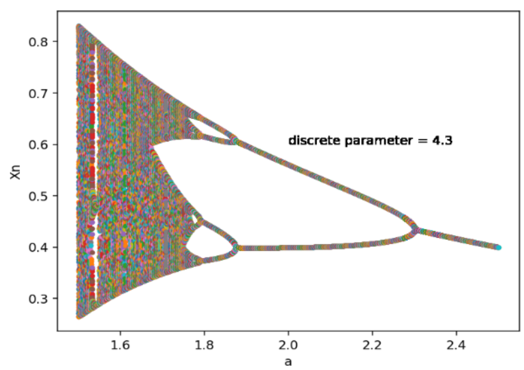
Time series graphs are very useful for us in understanding the dynamics of any function. We can explore the dynamics of our function by observing time series graphs. We can apply it to observe



(a) λ=3.8



(b) λ=4



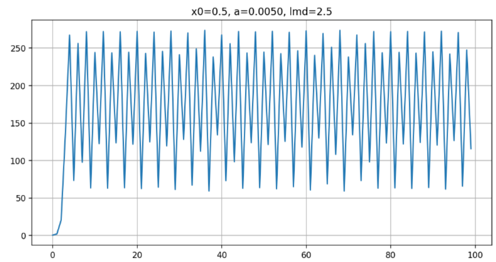
(c) λ=4.3

FIGURE 3. Bifurcation diagrams for $1.5 \leq a \leq 2.5$ and different parameter value of (a) $\lambda = 3.8$, $\lambda = 4$ and $\lambda = 4.3$

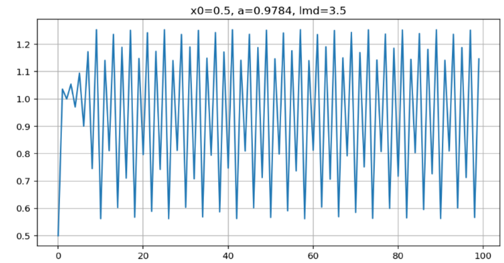
periodicity and period-doubling which are essential for understanding chaos. As we vary our parameters a and λ, the trajectories in the time series graph become stable or unstable. Depending on the value of our parameters, period-doubling can be initiated. This shows us the the existence of chaos in the dynamics of our family of functions. Because of the period-doubling, it is possible to look at graphs with periods 2, 4, 8, 16, 32, and so on.

We can see simulation of the time series graphs of period 32 in Figure 4. Because of the values of our parameters a and λ, we can observe a periodicity of 32 which happens due to the period-doubling. This indicates the presence of chaos in the dynamics of our family of functions. We can compare the periodicity of these time series graphs to the bifurcation diagrams for same parameter values from the above section.

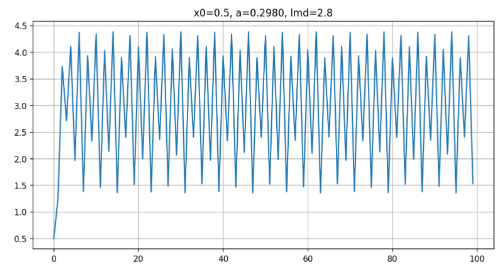
By varying the parameters, then the periodicity of the time series graph changes as can be seen in Figure 5 with time series graphs of periodicity of 8. In Figure 5(a), a=1.276 and λ=3.8 (both are constant here obviously), the periodicity of 8 is observed, this can also be understood from the bifurcation diagram in Figure 2(c) that period doubling has caused a periodicity of 8 for these



(a) x_0=0.5, a=0.005, λ=2.5



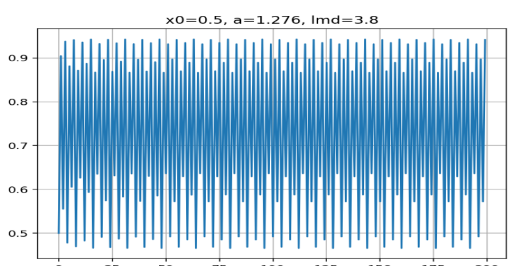
(b) x_0=0.5, a=0.9784, λ=3.5



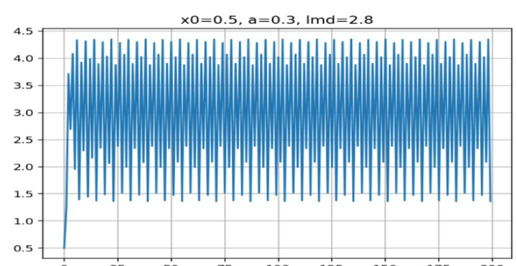
(c) x_0=0.5, a=0.2980, λ=2.8

Figure 4. Time series graphs for different parameter values of a and λ

parameter values. Then, in Figure 5(b) a=0.3 and λ=2.8 periodicity is 8 again, we can understand this from Figure 1(c) that period doubling has caused a periodicity of 8 for these parameter values. Hence, the periodicity depends on initial value as well as the parametric values both.



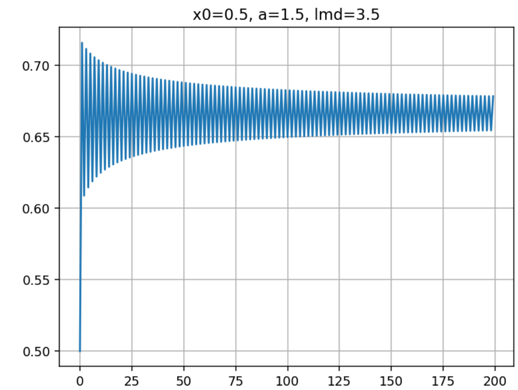
(a) x_0=0.5, a=1.276, λ=3.8



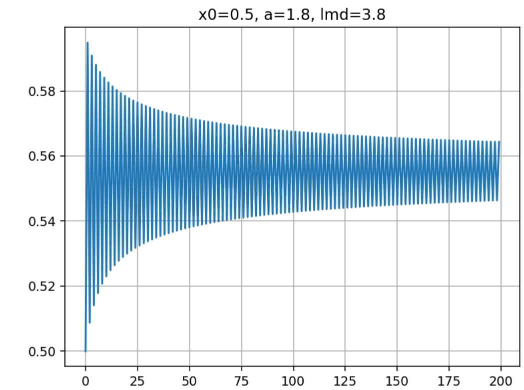
(b) x_0=0.5, a=0.3, λ=2.8

Figure 5. Time series graphs for different parameter values of a and λ

Similarly, when we change the parameters the periodicity changes again in Figure 6. This shows us that period doubling occurs in the dynamics of our family of function depending upon parameter values and depending upon these parameter values the periodicity can be easily changed. Thus, we can observe the presence of chaos in the dynamics of our family of function.



(a) x_0=0.5, a=1.5, λ=3.5



(b) x_0=0.5, a=1.8, λ=3.8

Figure 6. Time series graphs for different parameter values of a and λ

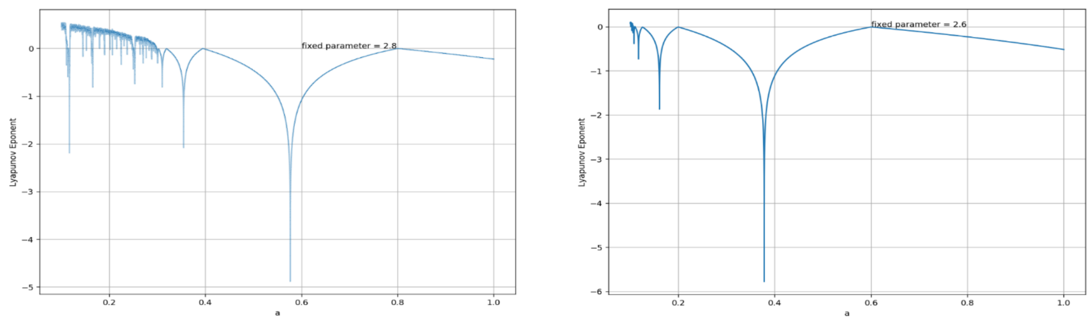
4. Lyapunov Exponents

A way to quantify chaos in the dynamics of the functions is to compute Lyapunov exponents. It characterizes the rate of separation of infinitesimally close trajectories. The Lyapunov exponent at a point measures the average loss of information during successive iterations of points near that point. In this section, we quantify chaos in the real dynamics of $\Phi(x)$ by calculating positive Lyapunov exponents. These are computed using Formula (1.1) and the graph is then plotted against the changing parameter values. Using Lyapunov exponent Formula (1.1), we can write it in our function as:

$$L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln|1 + 1/x_i - \lambda(1 + \ln(ax_i))|. \tag{4.1}$$

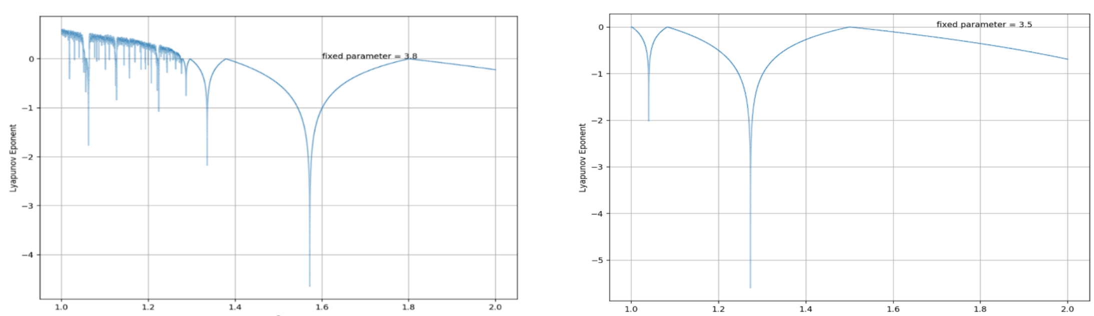
The parameter values that have been used are similar to the ones used in the bifurcation diagrams for better understanding. We can see in Figure 7 the values of Lyapunov exponent are positive using (4.1) for the same parameter values, the bifurcation diagram has a dark region, and negative in other places. Actually, what happens that the Lyapunov exponent takes a positive value when period-doubling happens. We can easily observe in figures that when period-doubling happens in bifurcation diagrams for some values of the parameter, the Lyapunov exponent has a positive value for the same values of the parameter. Also, when period-doubling is not happening, then the Lyapunov exponent is negative. Thus, it is clear visualization of the existence of chaos by looking at the positive Lyapunov exponent.

Corresponding to the bifurcation diagrams for the same values of parameter, we can see that in Figure 8 and Figure 9; that in the region of positive values for the Lyapunov exponent, the bifurcation diagram has a dark region, where period-doubling is happening and the Lyapunov exponent has negative values elsewhere. Hence, we can observe the presence of chaos in the dynamics of the family of functions. In Figure 8(a), λ=3.8, we can compare this to the bifurcation diagram with same parameters in Figure 2(c) and observe that at the points of period doubling in the bifurcation diagram, the LE has a positive value and the LE is negative otherwise. Similarly, for λ=3.5 we can look at the bifurcation diagram in Figure 2(b) and observe the same phenomenon. In Figure 9(a) for λ=4.3, we can compare this to the bifurcation diagram with same parameters in Figure 3(c) and observe that at the points of period doubling in the bifurcation diagram, the LE has a positive value and the LE is negative otherwise. Similarly, for λ=4 we can look at the bifurcation diagram in Figure 3(b) and observe the same phenomenon.



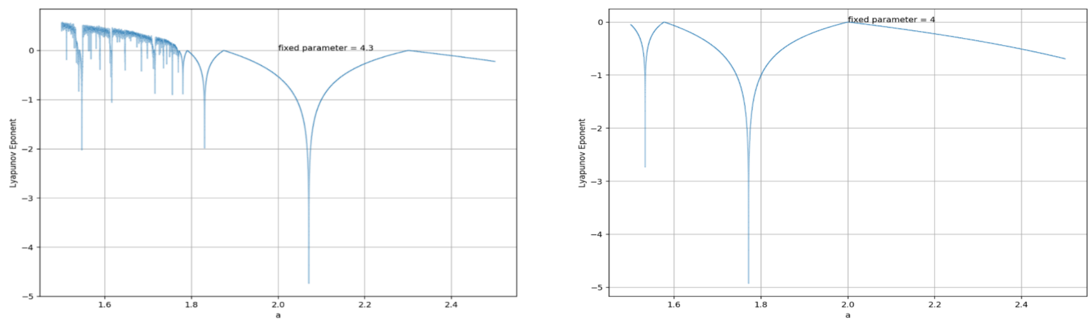
(a) λ=2.8 (b) λ=2.6

Figure 7. Lyapunov exponents for $0.1 \leq a \leq 1$ and different parameter value of (a) $\lambda = 2.8$ and $\lambda = 2.6$



(a) λ=3.8 (b) λ=3.5

Figure 8. Lyapunov exponents for $1 \leq a \leq 2$ and different parameter value of (a) $\lambda = 3.8$ and $\lambda = 3.5$



(a) λ=4.3 (b) λ=4

Figure 9. Lyapunov exponents for $1.5 \leq a \leq 2.5$ and different parameter value of (a) $\lambda = 4.3$ and $\lambda = 4$

5. Discussion

Python programming language is used here for the generation of all graphs and images. For the generation of bifurcation diagrams, two parameters are used where one is kept constant (λ) and the other is continuously varied (a) between certain values. The initial value of the variable $x = 0.5$ is taken. A value $N = 500$ is taken to indicate how many number of times the value of x_n changes using the given function. Then, using our function and the values declared for our continuous parameter a and discrete parameter λ and initial value $x = 0.5$, we plot the bifurcation diagram. As the function repeats itself $N(500)$ a number of times and the value of the continuous parameter a is changed in an interval on the horizontal axis, graph is obtained against the changing value of the parameter a. This process is repeated multiple times by different sets of the values of a and λ.

For the generation of time series graphs, similarly as the above we have defined both parameters a and λ but gave them both a constant value in this case. Then, similarly to the bifurcation diagram, an initial value is given to x. The value $N = 500$ is taken. As the value of N went from 0 to 500, x_n is calculated using the equation at each time interval. In this case, the value of x_n is plotted mapping the family of functions against the values of N (representing the time interval) instead of the parameter a. The values of a and λ are changed to obtain time series graphs of different periods.

Finally, in Lyapunov exponents, a fixed value is given to the parameter λ and a continuous value is given to a along the horizontal axis. Initial value is assigned to x. Then, $N = 500$ is used depending on the number of times which we want to iterate. Using the initial value assigned to x and using parameters a and λ, the value of Lyapunov exponent is calculated by the formula given in Equation (4.1) and it is plotted against the continuous parameter a which varies along the horizontal axis with the value assigned. Values of range of a and constant λ are changed accordingly to obtain different Lyapunov exponent graphs.

6. Conclusion

In this research work, we have observed the real dynamics of two-parameter family of functions and its chaotic behaviour. We have seen chaos in the real dynamics of the function $\Phi(x) = x + (1 - \lambda x)\ln(ax) : x > 0, \lambda > 0, a > 0$ by plotting and understanding bifurcation diagrams (fractals in nature). Then, we have looked at time series graphs and saw period-doubling and periodic cycles which are indicators of chaotic behaviour. Lastly, we have computed the positive Lyapunov exponents graphs corresponding to the values of the parameters in the bifurcation diagram and observed the chaotic nature. As a result of this work, the possibility of future work is to look further in this direction and extend our discussion to more than 2-parameters functions and family of functions having dimension 2 or greater.

Acknowledgment: The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2024-9/1). This work has been done by Shaurya Pratap Singh (BS-MS dual degree in Science as 3rd year student at IISER, Tirupati, India) to finish his Summer Internship (Online) under the guidance of Professor Mohammad Sajid during the period from 1st June to 1st September 2021.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Abdelmoula, B. Robert, Bifurcations and Chaos in a Photovoltaic Plant, Int. J. Bifurcation Chaos 29 (2019), 1950102. https://doi.org/10.1142/S0218127419501025.

[2] T. Banakh, M. Nowak, F. Strobin, Embedding Fractals in Banach, Hilbert or Euclidean Spaces, Journal of Fractal Geometry, Math. Fractals Relat. Top. 7 (2020), 351–386. https://doi.org/10.4171/jfg/94.

[3] Y. Bi, H. Yuan, S.-H. Chang, Dynamic Correlation Analysis of Regional Logistics from the Perspective of Multifractal Feature, Fractals 28 (2020), 2040015. https://doi.org/10.1142/S0218348X20400150.

[4] V.S. Deshpande, A. Needleman, E. Van Der Giessen, Dislocation Dynamics Is Chaotic, Scr. Mater. 45 (2001), 1047–1053. https://doi.org/10.1016/S1359-6462(01)01135-6.

[5] S.N. Elaydi, Discrete Chaos: With Applications in Science and Engineering, Chapman and Hall/CRC, 2011. https://doi.org/10.1201/9781420011043.

[6] A. Husain, M.N. Nanda, M.S. Chowdary, M. Sajid, Fractals: An Eclectic Survey, Part-I, Fractal Fract. 6 (2022), 89. https://doi.org/10.3390/fractalfract6020089.

[7] A. Husain, M.N. Nanda, M.S. Chowdary, M. Sajid, Fractals: An Eclectic Survey, Part II, Fractal Fract. 6 (2022), 379. https://doi.org/10.3390/fractalfract6070379.

[8] G.P. Kapoor, M. Guru Prem Prasad, Dynamics of $(e^z - 1)/z$: The Julia Set and Bifurcation, Ergodic Theory Dyn. Syst. 18 (1998), 1363–1383. https://doi.org/10.1017/S0143385798118011.

[9] N. Kaur, K. Goyal, Uncertainty Quantification of Stochastic Epidemic SIR Models Using B-Spline Polynomial Chaos, Regul. Chaotic Dyn. 26 (2021), 22–38. https://doi.org/10.1134/S1560354721010020.

[10] S. Khajanchi, M. Perc, D. Ghosh, The Influence of Time Delay in a Chaotic Cancer Model, Chaos 28 (2018), 103101. https://doi.org/10.1063/1.5052496.

[11] L. Dobrescu, M. Neamtu, D. Opris, Bifurcation and Chaos Analysis in a Discrete-Delay Dynamic Model for a Stock Market, Int. J. Bifurcat. Chaos 23 (2013), 1350155. https://doi.org/10.1142/S0218127413501551.

[12] L.A. Bunimovich, L.V. Vela-Arevalo, Some New Surprises in Chaos, Chaos 25 (2015), 097614. https://doi.org/10.1063/1.4916330.
[13] M. Lakshmanan, S. Rajasekar, Nonlinear Dynamics: Integrability, Chaos, and Patterns, Springer, Berlin, 2003.
[14] K. Li, H. Bao, H. Li, J. Ma, Z. Hua, B. Bao, Memristive Rulkov Neuron Model With Magnetic Induction Effects, IEEE Trans. Ind. Inform. 18 (2022), 1726–1736. https://doi.org/10.1109/TII.2021.3086819.
[15] D. Lim, Fixed Points and Dynamics on Generating Function of Genocchi Numbers, J. Nonlinear Sci. Appl. 09 (2016), 933–939. https://doi.org/10.22436/jnsa.009.03.22.
[16] B.B. Mandelbrot, The Fractal Geometry of Nature, W.H. Freeman, San Francisco, 1982.
[17] S. Manjunath, A. Podapati, G. Raina, Stability, Convergence, Limit Cycles and Chaos in Some Models of Population Dynamics, Nonlinear Dyn. 87 (2017), 2577–2595. https://doi.org/10.1007/s11071-016-3212-4.
[18] P. Manneville, Instabilities, Chaos and Turbulence: An Introduction to Nonlinear Dynamics and Complex Systems, Imperial College Press, London, 2004.
[19] F. Martínez-Giménez, A. Peris, F. Rodenas, Chaos on Fuzzy Dynamical Systems, Mathematics 9 (2021), 2629. https://doi.org/10.3390/math9202629.
[20] A. Naanaa, Fast Chaotic Optimization Algorithm Based on Spatiotemporal Maps for Global Optimization, Appl. Math. Comput. 269 (2015), 402–411. https://doi.org/10.1016/j.amc.2015.07.111.
[21] T. Onozaki, One-Dimensional Nonlinear Cobweb Model, in: Nonlinearity, Bounded Rationality, and Heterogeneity, Springer Japan, Tokyo, 2018: pp. 25–77. https://doi.org/10.1007/978-4-431-54971-0_2.
[22] Y. Peng, S. He, K. Sun, A Higher Dimensional Chaotic Map with Discrete Memristor, AEU - Int. J. Electron. Commun. 129 (2021), 153539. https://doi.org/10.1016/j.aeue.2020.153539.
[23] A. Pikovsky and A. Politi, Lyapunov Exponents: A Tool to Explore Complex Dynamics, Cambridge University Press, 2016.
[24] D.J. Prajapati, S. Rawat, A. Tomar, M. Sajid, R.C. Dimri, A Brief Study on Julia Sets in the Dynamics of Entire Transcendental Function Using Mann Iterative Scheme, Fractal Fract. 6 (2022), 397. https://doi.org/10.3390/fractalfract6070397.
[25] M. Sajid, Bifurcation and Chaos in Real Dynamics of a Two-Parameter Family Arising from Generating Function of Generalized Apostol-Type Polynomials, Math. Comput. Appl. 23 (2018), 7. https://doi.org/10.3390/mca23010007.
[26] M. Sajid, Chaotic Behavior in Real Dynamics and Singular Values of Family of Generalized Generating Function of Apostol-Genocchi Numbers, J. Math. Comput. Sci. 19 (2019), 41–50. https://doi.org/10.22436/jmcs.019.01.06.
[27] M. Sajid, Chaotic Behaviour and Bifurcation in Real Dynamics of Two-Parameter Family of Functions Including Logarithmic Map, Abstr. Appl. Anal. 2020 (2020), 7917184. https://doi.org/10.1155/2020/7917184.
[28] M. Sajid, G.P. Kapoor, Dynamics of a Family of Transcendental Meromorphic Functions Having Rational Schwarzian Derivative, J. Math. Anal. Appl. 326 (2007), 1356–1369. https://doi.org/10.1016/j.jmaa.2006.02.089.
[29] L. Stumpf, Chaotic Behaviour in the Newton Iterative Function Associated With Kepler Equation, Celest. Mech. Dyn. Astron. 74 (1999), 95-109. https://doi.org/10.1023/A:1008339416143.
[30] J.M.T. Thompson, Chaos, Fractals and Their Applications, Int. J. Bifurcation Chaos 26 (2016), 1630035. https://doi.org/10.1142/S0218127416300354.
[31] Y. Tian, G. Cui, H. Morris, Digital Imaging Based on Fractal Theory and Its Spatial Dimensionality, Fractals 28 (2020), 2040014. https://doi.org/10.1142/S0218348X20400149.
[32] J. Wang, W. Shao, J. Kim, ECG Classification Comparison between MF-DFA and MF-DXA, Fractals 29 (2021), 2150029. https://doi.org/10.1142/S0218348X21500298.
[33] K.-J. Wang, K.-L. Wang, Variational Principles for Fractal Whitham–Broer–Kaup Equations in Shallow Water, Fractals 29 (2021), 2150028. https://doi.org/10.1142/S0218348X21500286.
[34] J. Wu, X. Liao, B. Yang, Image Encryption Using 2D Hénon-Sine Map and DNA Approach, Signal Process. 153 (2018), 11–23. https://doi.org/10.1016/j.sigpro.2018.06.008.

[35] Y. Zhou, Z. Hua, C.-M. Pun, C.L.P. Chen, Cascade Chaotic System With Applications, IEEE Trans. Cybern. 45 (2015), 2001–2012. https://doi.org/10.1109/TCYB.2014.2363168.

[36] Z. Zhou, J.-P. Wu, Particle Motion and Chaos, Adv. High Energy Phys. 2020 (2020), 1670362. https://doi.org/10.1155/2020/1670362.