Int. J. Anal. Appl. (2025), 23:47

International Journal of Analysis and Applications

Controllability of Intuitionistic Fuzzy Impulsive Neutral Integro-Differential Equations with Nonlocal Conditions

T. Gunasekar[1], J. Thiravidarani[1], Rania Saadeh[2,*], Ahmad Qazza[2], Junaid Ahmad[3], Ghassan Abufoudeh[4]

[1]*Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai, Tamil Nadu, India*

[2]*Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13110, Jordan*

[3]*Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan*

[4]*Department of Mathematics, Faculty of Arts and Sciences, University of Petra, Amman,11196, Jordan*

[*]*Corresponding author:* rsaadeh@zu.edu.jo

Abstract. This paper explores the controllability of nonlocal intuitionistic fuzzy integro-differential equations using intuitionistic fuzzy semigroups and the contraction mapping principle. By establishing a clear theoretical approach, we show that it is possible to achieve controllability under specific conditions. This study offers new methods and significant insights into the analysis of fuzzy systems. The results demonstrate that, given the right conditions, controlling systems with nonlocal features is feasible, addressing important challenges in this area.

1. Introduction

In recent years, intuitionistic fuzzy sets and their applications have become a popular topic in mathematics and related areas [1–3]. These sets extend traditional fuzzy sets by including levels of inclusion, exclusion, and hesitation, making them a more flexible way to model uncertainty. This has led to the development of new mathematical structures and tools, like intuitionistic fuzzy metric spaces [4] and intuitionistic fuzzy semigroups [5].

The study of differential equations using fuzzy and intuitionistic fuzzy sets has been actively researched. Kaleva [6] established the basic principles for fuzzy differential equations, and Jeong [7] expanded this by adding nonlocal conditions, which increased the range of possible applications.

Received: Oct. 27, 2024.

2020 *Mathematics Subject Classification.* 03E72, 08A72.

Key words and phrases. intuitionistic fuzzy integro-differential equations; intuitionistic fuzzy solution; intuitionistic fuzzy semigroup; intuitionistic fuzzy number; mean-square analysis.

https://doi.org/10.28924/2291-8639-23-2025-47
ISSN: 2291-8639

The inclusion of random elements in fuzzy systems, as examined by Feng [8], shows how useful and adaptable fuzzy set theory is for dealing with real-world problems that involve uncertainty.

Distance measures between intuitionistic fuzzy sets are important for many applications. Szmidt and Kacprzyk [9], Chadli and Melliani [11], and Grzegorzewski [12] have all explored this topic. These measures help compare and analyze intuitionistic fuzzy sets, which is crucial for creating strong mathematical models.

Functional analysis offers a key foundation for understanding and solving differential equations. Brezis [10] has made significant contributions to finding exact solutions for intuitionistic fuzzy integro-differential equations. This work also covers important concepts such as functional analysis, Sobolev spaces, and partial differential calculus.

Fuzzy random variable theory, introduced by Puri and Ralescu [13], and the exploration of intuitionistic fuzzy topological spaces by Saadati and Park [14], demonstrate the wide-ranging applications and continuous development of fuzzy set theory and its extensions.

Acharya [20] investigated the controllability of fuzzy solutions for neutral impulsive functional differential equations with nonlocal conditions. Their work, published in Axioms, examines how controllability can be achieved in these complex systems and provides insights into managing fuzzy solutions effectively . Narayanamoorthy [21] focused on the existence and controllability of nonlinear first-order fuzzy neutral integrodifferential equations with nonlocal conditions. Their paper, published in the International Journal of Fuzzy Logic and Systems, highlights key results and methods for ensuring controllability in these types of equations . Murugesan [22] explored controllability results for nonlinear impulsive functional neutral integrodifferential equations within a fuzzy vector space. Their research, published in Applications and Applied Mathematics: An International Journal, provides a detailed analysis of controllability in multidimensional fuzzy systems [22, 27, 28].

Niazi [27] investigated controllability for fuzzy fractional evolution equations in credibility space. Abuasbeh [28] analyzed controllability of fractional functional random integroevolution equations with delay. Acharya [20] explored controllability of fuzzy solutions for neutral impulsive functional differential equations with nonlocal conditions. Georgieva [29] applied the Double Fuzzy Sumudu transform to solve partial Volterra fuzzy integro-differential equations. Dhandapani [30] provided numerical solutions for a differential system using pure hybrid fuzzy neutral delay theory. Gunasekar [31] studied non-linear impulsive neutral fuzzy delay differential equations with non-local conditions. Chou [32] examined controllability of a fractional-order particle swarm optimizer for heart disease classification. Ahmed [33] focused on existence solutions and controllability of Sobolev type delay nonlinear fractional integro-differential systems. Chalishajar and Kumar [34] investigated total controllability of second-order semi-linear differential equations with infinite delay and non-instantaneous impulses.

T. Gunasekar [23] have recently made significant advances in our knowledge of impulsive neutral functional integrodifferential systems with indefinite delay. Gunasekar [24] concentrated on these systems' presence and controllability outcomes. Damped second-order neutral integral equations with impulses were studied [25]. Controllability results for impulsive neutral stochastic functional integrodifferential inclusions were presented [26]. Also they investigated impulsive partial neutral functional integro-differential systems' controllability.

Johansyah [35] explored solutions to Riccati fractional differential equations in economic models. Telli [36] studied variable-order fractional equations with delays. Tunc [37]investigated integral equations with multiple delays. Ahmad et al. [38] examined fractional equations with non-conjugate boundary conditions, focusing on stability. Djaouti [39] analyzed fractional neutral stochastic equations using the Ψ-Caputo derivative, and Algahtani [40] studied Ulam stability in fractional hybrid equations.

This paper aims to contribute to this growing body of knowledge by proving the uniqueness and existence of solutions to fuzzy integro-differential equations with nonlocal IF solutions. The approach utilizes IF semigroups and the contraction mapping principle, offering a new perspective on solving such equations under specific conditions.

This paper investigates the existence and uniqueness of solutions to fuzzy integro-differential equations with IF nonlocal conditions, which are defined as follows:

$$\begin{aligned} &\dot{\varphi}(\zeta) = \Lambda\varphi(\zeta) + \vartheta(\zeta,\varphi(\zeta)) + \int_0^{\zeta} \eta(\zeta,\iota,\varphi(\iota))d\iota + \beta\rho(\zeta), \quad \zeta \in I = [0,a], \\ &\varphi(0) = \varphi_0 + \varpi\{\zeta_1,\zeta_2,\ldots,\zeta_p;\varphi(\cdot), \\ &\Delta\varphi(\zeta_k) = \mathcal{I}_k(\varphi(\zeta_k^-)), k = 1,2,...,m, \end{aligned} \tag{1.1}$$

where Λ generates an IF strongly continuous semigroup $(T(\zeta))_{\zeta\geq 0}$ on IF_1, $\varphi_0 \in \text{IF}_1$, and ϑ and ϖ are predefined functions.

ω, $\mathcal{I}_k$ are predefined,

$$\begin{aligned} \Delta\varphi(\zeta_k) &= \varphi(\zeta_k^+) - \varphi(\zeta_k^-), \\ \varphi(\zeta_k^+) &= \lim_{h\to 0^+} \varphi(\zeta_k + h), \text{ and} \\ \varphi(\zeta_k^-) &= \lim_{h\to 0^-} \varphi(\zeta_k - h) \end{aligned}$$

represent the left limit and right limit of $\varphi(\zeta_k)$ at$\delta = \zeta_k$, respectively $k = 1,2,...,m$.
Here, the notation $\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot))$ implies a substitution, wherein only elements from the set $\{\zeta_1,\zeta_2,\ldots,\zeta_p\}$ are considered. For instance, $\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot))$ can be expressed as

$$\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)) = c_1\varphi(t_1) + c_2\varphi(t_2) + \ldots + c_p\varphi(t_p)$$

where c_i (for $i = 1,2,\ldots,p$) are specified constants.

2. Preliminaries

Definition 2.1. Consider an arbitrary non-empty set X, and let $\text{IF}(X)$ denote the collection of intuitionistic fuzzy subsets of X, defined as

$$\mathrm{IF}(X) = \{(\rho, \sigma) \in I^X \times I^X : 0 \leq \rho(\varphi) + \sigma(\varphi) \leq 1, \text{ for all } \varphi \in X\}.$$

A function $d : \mathrm{IF}(X) \times \mathrm{IF}(X) \to \mathbb{R}$ is an IF metric on $\mathrm{IF}(X)$ if it adheres to the following properties:

(1) $d(\langle\rho_1, \sigma_1\rangle, \langle\rho_2, \sigma_2\rangle) \geq 0$, for every pair $\langle\rho_1, \sigma_1\rangle, \langle\rho_2, \sigma_2\rangle \in \mathrm{IF}(X)$.
(2) $d(\langle\rho_1, \sigma_1\rangle, \langle\rho_2, \sigma_2\rangle) = 0$ if and only if $\langle\rho_1, \sigma_1\rangle = \langle\rho_2, \sigma_2\rangle$.
(3) $d(\langle\rho_1, \sigma_1\rangle, \langle\rho_2, \sigma_2\rangle) = d(\langle\rho_2, \sigma_2\rangle, \langle\rho_1, \sigma_1\rangle)$, for all $\langle\rho_1, \sigma_1\rangle, \langle\rho_2, \sigma_2\rangle \in \mathrm{IF}(X)$.
(4) $d(\langle\rho_1, \sigma_1\rangle, \langle\rho_3, \sigma_3\rangle) \leq d(\langle\rho_1, \sigma_1\rangle, \langle\rho_2, \sigma_2\rangle) + d(\langle\rho_2, \sigma_2\rangle, \langle\rho_3, \sigma_3\rangle)$,
for all $\langle\rho_1, \sigma_1\rangle, \langle\rho_2, \sigma_2\rangle, \langle\rho_3, \sigma_3\rangle \in \mathrm{IF}(X)$.

The structure $(\mathrm{IF}(X), d)$ is termed an IF metric space.

Definition 2.2. The IF zero is an IF set defined by:

$$\mathbf{0}_{(1,0)}(\varphi) = \begin{cases} (1,0) & \text{if } \varphi = 0 \\ (0,1) & \text{if } \varphi \neq 0 \end{cases}$$

3. Controllability of Intuitionistic Fuzzy Impulsive Integro-Differential Equations

Here, we make the following assumptions:

(H_1): The operator $\Lambda : D(\Lambda) \subseteq \mathrm{F}_1 \to \mathrm{F}_1$ produces a strongly continuous semigroup $(T(\zeta))_{\zeta \geq 0}$. There exist constants M and $\omega \in \mathbb{R}_+^*$ such that for $\zeta \geq 0$ and $\varphi, \upsilon \in L_2 \cap D(\Lambda)$, the following inequality is satisfied:

$$\aleph(T(\zeta)\varphi,\ T(\zeta)\upsilon) \leq Me^{\omega\zeta}\aleph(\varphi,\ \upsilon)\ .$$

(H_2): The mapping $\vartheta : I \times L_2 \to L_2$ Is the mean square constant in relation to ζ, and fulfills a generalized Lipschitz constraint. . This implies the existence of a constant K_1 such that

$$\aleph(\vartheta(\zeta,\ \varphi),\ \vartheta(\zeta,\ \upsilon)) \leq K_1\aleph(\varphi,\ \upsilon)\ .$$

(H_3): The mapping $\varpi : I^p \times L_2 \to L_2$ fulfills a generalized Lipschitz constraint. . For all $\zeta \in I$, $\varphi, \upsilon \in L_2$, and $\varphi_0 \in L_2$, there exists a constant K_2 such that

$$\aleph(\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi),\ \varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \upsilon)) \leq K_2\aleph(\varphi,\ \upsilon)\ .$$

(H_4): The mapping $\eta : I \times L_2 \to L_2$ is continuous in the mean square sense with respect to ζ and satisfies a generalized Lipschitz constraint. There exists a constant L_η such that

$$\aleph\Big(\int_0^\iota \eta(\zeta, \iota, \varphi(\iota))dt, \int_0^\iota \eta(\zeta, \iota, \upsilon(\iota))dt\Big) \leq L_\eta\aleph(\varphi, \upsilon)$$

(H_5): There exist a non-negative constant d_k such that

$$N(I_k(\varphi(\zeta_k^-))), I_k(\nu(\zeta_k^-)) \leq d_k N(\varphi, \nu),$$

$$\text{for } k = 1, 2, ..., m, \text{ for each } \varphi, \nu \in L_2 \text{ and } N(T(\zeta)) \leq E,\ \delta \in J$$

(H_6): The linear operator $W : L^2(J, \upsilon) \to \varphi$ is defined by

$$W\varphi = \in \zeta 0^T \varphi(T, h; \mu)\beta\varphi(h)dh$$

has an inverse operator $W^(-1)$, which takes values in $L^2(J, v)/KerW$ and there exists a positive constatnt D_0 such that $\aleph(\beta W^{-1}) \leq D_0$ for every $\mu \in \beta_r$

Definition 3.1. *The relationship $\varphi : I \to L_2$ is known as a mild solution of (2), if*

$$\begin{aligned}\varphi(\zeta) = T(\zeta)\left[\varphi_0 + \varpi(\zeta_1,\ \zeta_2,\ \ldots, \zeta_p,\ \varphi(\cdot))\right] \\ + \int_0^{\zeta} T(\zeta - \iota)\vartheta(\iota,\ \varphi(\iota))\, d\iota \\ + \int_0^{\zeta} T(\zeta - \iota) \int_0^{\iota} \eta(\iota, \rho, \varphi(\rho))\, d\rho, d\iota \\ + \sum_{0<\zeta_k<z} T(\zeta - \zeta_k) I_k(\varphi(\zeta_k^-)) + \int_0^{\zeta} T(\zeta - \iota)\beta u(\iota)\, d\iota\end{aligned}$$

for $0 \leq \zeta \leq a$.

Theorem 3.1. *Assuming that conditions (H_1) through (H_6) are satisfied, Equation (2)allows unique mild solution throughout the interval $[0,\ \xi]$, where*

$$\xi = \min\left\{a,\ \frac{D_0}{\omega}\log\left(\frac{b - \epsilon + \frac{N_1 M}{\omega}}{MN_2 + \frac{N_1 M}{\omega}}\right)^{D_0},\ \frac{D_0}{\omega}\log\left(\frac{1 + \frac{K_1 M}{\omega}}{K_2 M + \frac{K_1 M}{\omega}}\right)^{D_0}\right\}$$

subject to the conditions

$$\aleph(\vartheta(\zeta,\ \varphi),\ 0_{(1,0)}) \leq N_1, \quad \aleph(\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi,\ 0_{(1,0)}) \leq N_2.$$

proof: Let $B = \{\varphi \in L_2 | H(\varphi,\ \varphi_0) \leq b\}$ denote the space of mean square continuous IF mappings,, where

$$H(\varphi,\ v) = \sup_{0\leq\zeta\leq\xi} \aleph(\varphi(\zeta),\ v(\zeta))$$

and b is a non-negative number. We describe a mapping $P : B \to B$ as:
Using (H6) for an arbitary function $\rho(.)$ define the control

$$\begin{aligned}\rho(\zeta) = W^{-1}[\rho_1 - T(\zeta)[\varphi_0 + \varpi(\zeta_1, \zeta_2, \ldots, \zeta_p, \varphi(.))] \\ - \int_0^{\zeta} T(\zeta - \iota)\vartheta(\iota, \varphi(\iota))\, d\iota \\ - \int_0^{\zeta} T(\zeta - \iota)\left(\int_0^{\iota} \eta(\iota, \delta, \varphi(\delta))\, d\delta\right) d\iota \\ - \sum_{0<\zeta_k<\delta} T(\zeta - \zeta k) I_k(\varphi(\zeta k^-))]\end{aligned}$$

where

$$
\begin{aligned}
F(\eta,\rho) = BW^{-1}[\rho_2 1 - T(\zeta)[\varphi_0 + \varpi(\zeta 1, \zeta 2, \ldots, \zeta p, \varphi(.))] \\
+ \int_0^{\zeta} T(\zeta-\iota)\vartheta(\iota,\varphi(\iota))\,d\iota \\
- \int_0^{\zeta} T(\zeta-\iota)\left(\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota \\
- \sum_{0<\zeta k<\delta} T(\zeta-\zeta k) I_k(\varphi(\zeta k^-))]
\end{aligned}
$$

$$
\begin{aligned}
P\varphi(\zeta) &= T(\zeta)[\varphi_0 + \varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(.))] \\
&+ \int_0^{\zeta} T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))d\iota + \int_0^{\zeta} T(\zeta-\iota)\left(\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))d\delta\right)d\iota \\
&+ \sum_{0<\zeta_k<z} T(\zeta-\zeta_k) I_k(\varphi(\zeta_k^-)) + \int_0^{\zeta} T(\zeta-\iota)F(\eta,\varphi)d\iota
\end{aligned}
$$

Firstly, We show that P fulfills and is mean square continuous. $H(Px,\ \varphi_0) \le b$. *Since* ϑ *is the continuous mean square , we have*

$$
\begin{aligned}
&\aleph(P\varphi(\zeta+h),\ P\varphi(\zeta)) = \aleph(T(\zeta+h)[\varphi_0 + \varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot))] \\
&\quad + \int_0^{\zeta+h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota \\
&\quad + \int_0^{\zeta+h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota \\
&\quad + \sum_{0<(\zeta+h)_k<z} T(\zeta-\zeta_k) I_k(\varphi(\zeta_k^-)) + \int_0^{\zeta+h} T(\zeta+h-\iota)F(\eta,\rho)d\iota \\
&\quad + T(\zeta)[\varphi_0 + \varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot))] \\
&\quad + \int_0^{\zeta} T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota \\
&\quad + \int_0^{\zeta} T(\zeta-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota) \\
&\quad + \sum_{0<\zeta_k<z} T(\zeta-\zeta_k) I_k(\varphi(\zeta_k^-)) + \int_0^{\zeta} T(\zeta-\iota)F(\eta,\rho)d\iota \\
&\le \aleph(T(\zeta+h)\varphi_0, T(\zeta)\varphi_0) + \aleph(T(\zeta+h)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)),\ T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot))) \\
&\quad + \aleph\left(\int_0^{\zeta+h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ \int_0^{\zeta} T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota\right) \\
&\quad + \aleph\left(\int_0^{\zeta+h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ \int_0^{\zeta} T(\zeta-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\aleph\left(\sum_{0<(\zeta+h)_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right)\\
&+\aleph\left(\int_0^{\zeta+h} T(\zeta+h-\iota)F(\eta,\rho)d\iota,\int_0^{\zeta} T(\zeta-\iota)F(\eta,\rho)d\iota\right)\\
\leq{}&\aleph(T(\zeta+h)\varphi_0,T(\zeta)\varphi_0)+\aleph(T(\zeta+h)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(.)),\ T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(.)))\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ 0_{(1,0)}\right)\\
&+\aleph\left(\int_h^{\zeta+h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ \int_0^{\zeta} T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota\right)\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\delta,\iota,\ \varphi(\iota))\,d\delta d\iota,0_{(1,0)}\right)\\
&+\aleph\left(\int_h^{\zeta+h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ \int_0^{\zeta} T(\zeta-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)\\
&+\aleph\left(\sum_{0<(h)_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),0_{(1,0)}\right)\\
&+\aleph\left(\sum_{h<(\zeta+h)_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right)\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)F(\eta,\varphi)d\iota,0_{(1,0)}\right)\\
&+\aleph\left(\int_h^{\zeta+h} T(\zeta+h-\iota)F(\eta,\varphi)d\iota,\int_0^{t} T(\zeta-\iota)F(\eta,\varphi)d\iota\right)\\
\leq{}&Me^{\omega\zeta}[\aleph(T(h)\varphi_0,\ \varphi_0)+\aleph(T(h)\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\ T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)))\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ 0_{(1,0)}\right)\\
&+\int_0^{\zeta} Me^{\omega(\zeta-\iota)}\aleph(\vartheta(\iota+h,\ \varphi(\iota+h)),\vartheta(\iota,\ \varphi(\iota)))\,d\iota]\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ 0_{(1,0)}\right)\\
&+\int_0^{\zeta} Me^{\omega(\zeta-\iota)}\aleph\left(\int_0^{\iota}\eta(\iota,\delta+h,\varphi(\delta+h))\,d\delta,\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)F(\eta,\rho)d\iota,0_{(1,0)}\right)+\int_0^{\zeta} Me^{\omega(\zeta-\iota)}\aleph\left(F(\eta,\varphi),F(\eta,\upsilon)\right)d\iota\\
&+\aleph(I_k(\varphi(\zeta_k^-))),I_k(\nu(\zeta_k^-))
\end{aligned}
$$

It is obvious that $\aleph(T(h)\varphi_0,\ \varphi_0)\to 0,\aleph(T(h)\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(.)\ ,\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(.)\to 0$

and

$$\aleph(\int_0^h T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))d\iota,\ 0_{(1,0)})) \to 0$$

as $h \to 0$.

$$\aleph(I_k(\varphi(\zeta_k^-))), I_k(\nu(\zeta_k^-)) \to 0$$

And according to the theorem of dominated convergence:

$$\int_0^\zeta Me^{\omega(\zeta-\iota)}\aleph(\vartheta(\iota+h,\ \varphi(\iota+h)),\ \vartheta(\iota,\ \varphi(\iota)))d\iota \to 0$$

$$\aleph\left(F(\eta,\varphi),F(\eta,v)\right) \to 0$$

The mapping P exhibits m.s continuity over I. Moreover,

$$\begin{aligned}
\aleph(P\varphi(\zeta),\ \varphi_0) &= \aleph\Big(T(\zeta)\Big[\varphi_0+\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot))\Big] \\
&\quad + \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota \\
&\quad + \int_0^\zeta T(\zeta-\iota)\int_\iota^\zeta \eta(\delta,\iota,\varphi(\iota))\,d\delta\,d\iota,\ \varphi_0\Big) \\
&\quad + \sum_{0<\zeta_k<\zeta} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),\varphi_0 \\
&\quad + \int_0^\zeta T(\zeta-\iota)F(\eta,\varphi)\,d\iota \\
&\le \aleph(T(\zeta)\varphi_0,\ \varphi_0)+\aleph(T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)),\ 0_{(1,0)}) \\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ 0_{(1,0)}\right) \\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\int_\iota^\zeta \eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ 0_{(1,0)}\right) \\
&\quad + \aleph\left(\sum_{0<\zeta_k<\zeta} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),\varphi_0\right)+\aleph\left(\int_0^\zeta T(\zeta-\iota)F(\eta,\varphi)d\iota\right) \\
&\le \epsilon+N_2Me^{\omega\zeta}+\frac{N_1M}{\omega}(e^{\omega\zeta}-1)+\frac{N_3M}{\omega}(e^{\omega\zeta}-1)+E.d_k \\
&\quad + D_0\left[N_2Me^{\omega\zeta}+\frac{N_1M}{\omega}(e^{\omega\zeta}-1)+\frac{N_3M}{\omega}(e^{\omega\zeta}-1)\right]
\end{aligned}$$

and so

$$\begin{aligned}
H(Px,\ \varphi_0) &= \sup_{0\le\zeta\le\xi}\aleph(P\varphi(\zeta),\ \varphi_0) \\
&\le b.
\end{aligned}$$

In $C([0,\ \xi],\ L_2)$*, denoted by* $C([0,\ \xi],\ L_2) = \varphi : [0,\xi] \to L_2 \mid \varphi(\zeta)$ *is the continuous mean square, completeness holds. We now demonstrate that B constitutes a closed subset of* $C([0,\ \xi],\ L_2)$*. Suppose* φ_n *is* *a sequence in B such that $\varphi_n \to \varphi \in C([0,\ \xi],\ L_2)$ as $n \to \infty$. Then,*

$$\begin{aligned}\aleph(\varphi(\zeta),\ \varphi_0) &\leq \aleph(\varphi(\zeta),\ \varphi_n(\zeta)) + \aleph(\varphi_n(\zeta),\ \varphi_0)\\ H(\varphi,\ \varphi_0) &= \sup_{0\leq\zeta\leq\xi} \aleph(\varphi(\zeta),\ \varphi_0)\\ &\leq H(\varphi,\ \varphi_n) + H(\varphi_n,\ \varphi_0)\\ &\leq \epsilon + b\end{aligned}$$

For adequately large n and for any $\epsilon > 0$, $\varphi \in B$. Consequently, B emerges as a closed subset of $C([0,\ \xi],\ L_2)$. Hence, B constitutes a complete metric space.Next, we will demonstrate the contraction property of P. Given $\varphi, v \in B$,

$$\begin{aligned}\aleph(P\varphi(\zeta),\ Py(\zeta)) &\leq \aleph(T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)), T(\zeta)\varpi(\zeta_1, \zeta_2,\ \ldots \zeta_p,\ v(\cdot))\\ &+ \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ v(\iota))\,d\iota\right)\\ &+ \aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,v(\delta))\,d\delta\,d\iota\right)\\ &+ \aleph\left(\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),\ \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(v(\zeta_k^-))\right)\\ &+ \aleph\left(\int_0^\zeta T(\zeta-\iota)F(z,\varphi)d\iota, \int_0^\zeta T(\zeta-\iota)F(z,v)d\iota\right)\\ &\leq K_2Me^{\omega\zeta}\aleph(\varphi,\ v) + K_1M\int_0^\zeta e^{\omega(\zeta-\iota)}\aleph(\vartheta(\iota,\ \varphi(\iota)),\ \vartheta(\iota,\ v(\iota)))\,d\iota\\ &+ L_2M\int_0^\zeta e^{\omega(\zeta-\iota)}\aleph(\int_0^\iota \eta(\eta(\iota,\delta,\varphi(\delta))d\delta, \int_0^\iota \eta(\iota,\delta,v(\delta))d\delta)d\iota\\ &+ \aleph\,(F(z,\varphi),F(z,v)) + \aleph(I_k(\varphi(\zeta_k^-))), I_k(\nu(\zeta_k^-))\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\leq\ &(K_2Me^{\omega\xi} + \frac{K_1M}{\omega}[e^{\omega\xi}-1] + \frac{L_2M}{\omega}[e^{\omega\xi}-1] + E.d_k)H(\varphi,\ v)\\ &+D_0\left[K_2Me^{\omega\zeta} + K_1M[e^{\omega\xi}-1] + \frac{L_2M}{\omega}[e^{\omega\xi}-1]\right]\end{aligned}$$

Due to the inequality $(K_2Me^{\omega\xi} + \frac{K_1M}{\omega}[e^{\omega\xi}-1] + \frac{L_2M}{\omega}[e^{\omega\xi}-1] + E.d_k + D_0\left[K_2Me^{\omega\zeta} + K_1M[e^{\omega\xi}-1] + \frac{L_2M}{\omega}[e^{\omega\xi}-1]\right]) < 1$, the mapping P qualifies as a contraction map. Hence, P possesses a unique fixed point.

$$\begin{aligned}\varphi(\zeta) = T(\zeta)&\left[\varphi_0 + \varpi(\zeta_1,\ \zeta_2,\ \ldots, \zeta_p,\ \varphi(\cdot))\right]\\ &+ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota\\ &+ \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\rho,\varphi(\rho))\,d\rho, d\iota\end{aligned}$$

$$+\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)) + \int_0^\zeta T(\zeta-\iota)\beta u(\iota)\, d\iota$$

This completes the proof. □

Theorem 3.2. *Suppose ϑ and ϖ are as described in Theorem 3.1. Let $\varphi(\zeta,\ \varphi_0)$ and $\upsilon(\zeta,\ \upsilon_0)$ represent solutions of Equation (4) for initial values φ_0 and υ_0, respectively. Consequently, there exist constants c_1 and c_2 such that*

1. $H(\varphi(.,\ \varphi_0),\ \upsilon\ (.,\upsilon_0))) \leq c_1\aleph(\varphi_0,\ \upsilon_0)$ for any $\varphi_0, \upsilon_0 \in L_2$

2. $H(\varphi(.,\ \varphi_0),\ 0_{(1,0)}) \leq c_2(\aleph(\varphi_0, 0_{(1,0)}) + N_1 + N_3 + N_4 + N_5)$ where

$\aleph(\varpi(\zeta_1,\ \zeta_2,\ \ldots,\zeta_p,\ \varphi(.)\ , 0_{(1,0)}) \leq N_1\ ,\ \int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\ 0_{(1,0)}), 0_{(1,0)}) \leq N_3$, *and*

$\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta(\iota,\zeta,0_{(1,0)}),0_{(1,0)}\right)d\zeta, d\iota \leq N_4, \int_0^\zeta e^{-\omega\iota}\aleph\left(F(\eta,0_{(1,0)})\right) \leq N_5$

Proof: For any $\zeta \in [0,\ \xi]$ we have

$$\begin{aligned}
&\aleph(\varphi(\zeta,\varphi_0),\ \upsilon(\zeta,\upsilon_0))\\
&\leq \aleph(T(\zeta)\varphi_0,\ T(\zeta)\upsilon_0)\\
&\quad + \aleph(T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot,\varphi_0)),\ T(\zeta)\varpi(\zeta_1,\zeta_2,\ \ldots,\zeta_p,\ \upsilon(\cdot,\upsilon_0)))\\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota,\varphi_0))\, d\iota,\ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \upsilon(\iota,\upsilon_0))\, d\iota\right)\\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0))\, d\delta\, d\iota,\ \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\upsilon(\delta,\upsilon_0))\, d\delta\, d\iota\right)\\
&\quad + \aleph\left(\sum_{0<(\zeta+h)_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)) + \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right)\\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)F(\eta,\varphi_0)\, d\iota, \int_0^\zeta T(\zeta-\iota)F(\eta,\upsilon_0)\, d\iota\right)\\
&\leq Me^{\omega\zeta}[\aleph(\varphi_0,\ \upsilon_0) + \aleph(\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot,\varphi_0))),\ \varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \upsilon(\cdot,\upsilon_0))]\\
&\quad + Me^{\omega\zeta}\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\ \varphi(\iota,\varphi_0))\ ,\ \vartheta(\iota,\ \upsilon(\iota,\upsilon_0)))\, d\iota\\
&\quad + Me^{\omega\zeta}\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0))\, d\delta\ ,\ \int_0^\iota \eta(\iota,\delta,\upsilon(\delta,\upsilon_0)\, d\delta\ )d\iota\right)\\
&\quad + Me^{\omega\zeta}\int_0^\zeta e^{-\omega\iota}\aleph\left(F(\eta,\varphi_0),F(\eta,\upsilon_0)\right) d\iota\\
&\quad + \aleph(I_k(\varphi(\zeta_k^-)), I_k(\upsilon(\zeta_k^-)))\\
&\leq Me^{\omega\xi}[\aleph(\varphi_0,\ \upsilon_0) + K_2\aleph(\varphi(\cdot,\ \varphi_0),\ \upsilon(\cdot,\ \upsilon_0))]\\
&\quad + Me^{\omega\xi}[K_1\int_0^\zeta e^{-\omega\iota}\aleph(\varphi(\iota,\ \varphi_0),\ \upsilon(\iota,\ \upsilon_0))\, d\iota\\
&\quad + L_z\int_0^\zeta e^{-\omega\iota}\aleph(\varphi(\iota,\ \varphi_0),\ \upsilon(\iota,\ \upsilon_0))\ ]d\iota
\end{aligned}$$

$$+\aleph(I_k(\varphi(\zeta_k^-)),I_k(\upsilon(\zeta_k^-)))$$
$$+Me^{\omega\zeta}\int_0^\zeta e^{-\omega\iota}\aleph\left(F(\eta,\varphi_0),F(\eta,\upsilon_0)\right)d\iota$$
$$\leq Me^{\omega\xi}[\aleph(\varphi_0,\ \upsilon_0)+K_2\aleph(\varphi(\cdot,\ \varphi_0),\ \upsilon(\cdot,\ \upsilon_0))]$$
$$+\left[Me^{\omega\xi}e^{-\omega\iota}\right]\int_0^\zeta(K_1\aleph(\varphi(\iota,\ \varphi_0),\ \upsilon(\iota,\ \upsilon_0))+L_\eta\aleph(\varphi(\iota,\ \varphi_0),\ \upsilon(\iota,\ \upsilon_0))+F[(\eta,\varphi_0),(\eta,\upsilon_0)])d\iota$$
$$+\aleph(I_k(\varphi(\zeta_k^-)),I_k(\upsilon(\zeta_k^-)))$$

Gronwall's inequality provides us with

$$\aleph(\varphi(\zeta,\ \varphi_0),\ \upsilon(\zeta,\ \upsilon_0))$$
$$\leq Me^{\omega\xi}[\aleph(\varphi_0,\ \upsilon_0)+K_2\aleph(\varphi(\cdot,\ \varphi_0),\ \upsilon(\cdot,\ \upsilon_0))]\exp((K_1+L\eta+E.d_k)Me^{\omega\xi}\int_0^\zeta e^{-\omega\varsigma}d\varsigma)$$
$$\leq Me^{\omega\xi}[\aleph(\varphi_0,\ \upsilon_0)+K_2\aleph(\varphi(\cdot,\ \varphi_0),\ \upsilon(\cdot,\ \upsilon_0))]\exp((K_1+L\eta+E.d_k)Me^{\omega\xi}\frac{1-e^{-\omega\zeta}}{\omega})$$

Thus we have

$$H(\varphi(.,\ \varphi_0),\ \upsilon(.,\upsilon_0))\leq Me^{\omega\xi}[\aleph(\varphi_0,\ \upsilon_0)+K_2H(\varphi(.,\ \varphi_0),\upsilon(.,\upsilon_0))]\exp([K_1+L_z+E.d_k]M\frac{e^{\omega\xi}-1}{\omega})$$

i.e,

$$(1-K_2Me^{\omega\zeta}E.d_k\exp\left(K_1M\frac{e^{\omega\xi}-1}{\omega}\right))H(\varphi(.,\ \varphi_0),\ \upsilon(.,\ \upsilon_0))$$
$$\leq Me^{\omega\xi}\exp\left([K_1+L_\eta+E.d_k]M\frac{e^{\omega\xi}-1}{\omega}\right)\aleph(\varphi_0,\ \upsilon_0)$$

Consequently, we obtain

$$H(\varphi(.,\ \varphi_0),\ \upsilon(.,\upsilon_0)\ )\leq\frac{Me^{\omega\xi}\exp([K_1+L_z+E.d_k]M\frac{e^{\omega\xi}-1}{\omega})}{(1-K_2Me^{\omega\xi}\exp([K_1+L_z+E.d_k]M\frac{e^{\omega\xi}-1}{\omega}))}\aleph(\varphi_0,\ \upsilon_0)$$

Taking $c_1=\dfrac{Me^{\omega\xi}\exp([K_1+L_z+E.d_k]M\frac{e^{\omega\xi}-1}{\omega})}{(1-K_2Me^{\omega\xi}\exp([K_1+L_z+E.d_k]M\frac{e^{\omega\xi}-1}{\omega}))}$ *we obtain* $H(\varphi(.,\ \varphi_0),\ \upsilon(.,\upsilon_0)\ )\leq c_1\aleph(\varphi_0,\ \upsilon_0)$

2. *For any* $\zeta\in[0,\ \xi]$ *we have*

$$\aleph(\varphi(\zeta,\varphi_0),\ 0_{(1,0)})$$
$$\leq\aleph(T(\zeta)\varphi_0,\ 0_{(1,0)})$$
$$+\aleph(T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot,\varphi_0),0_{(1,0)}))$$
$$+\aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota,\varphi_0))\,d\iota,0_{(1,0)}\right)$$
$$+\aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota\eta(\iota,\delta,\varphi(\delta,\varphi_0))\,d\delta\,d\iota,\ 0_{(1,0)}\right)$$

$$
\begin{aligned}
&+\aleph\left(\sum_{0<\zeta_k<\zeta} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),0_{(1,0)}\right)+\aleph\left(\int_0^\zeta T(\zeta-\iota)F(\eta,\varphi_0)d\iota,\ 0_{(1,0)}\right)\\
\leq\ & Me^{\omega\zeta}[\aleph(\varphi_0,0_{(1,0)})+\aleph(\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot,\varphi_0)),0_{(1,0)})]\\
&+Me^{\omega\zeta}\left[\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\varphi(\iota,\varphi_0)),0_{(1,0)})\,d\iota\right.\\
&\left.+\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0))\,d\delta,0_{(1,0)}\right)d\iota+\int_0^\zeta e^{-\omega\iota}\aleph(F(\eta,\varphi_0),0_{(1,0)})\right]\\
&+Me^{\omega\zeta}\sum_{0<\zeta_k<\zeta}\aleph\left(T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),0_{(1,0)})\right]\\
\leq\ & Me^{\omega\xi}[\aleph(\varphi_0,\ 0_{(1,0)})+\aleph(\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot,\varphi_0)),0_{(1,0)}))]\\
&+Me^{\omega\xi}[\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\varphi(\iota,\varphi_0)),\vartheta(\iota,\ 0_{(1,0)}))\,d\iota\\
&+\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\ 0_{(1,0)}),0_{(1,0)})\,]d\iota\\
&+Me^{\omega\xi}[\int_0^\zeta e^{-\omega\iota}\aleph(\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0)),\int_0^\iota \eta(\iota,\delta,0_{(1,0)}))d\delta,d\iota\\
&+\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0))\,d\delta,0_{(1,0)}\right)d\iota\\
&+Me^{\omega\xi}\left(\int_0^\zeta e^{-\omega\iota}\aleph(F(\eta,\varphi_0),0_{(1,0)}d\iota)+\int_0^\zeta e^{-\omega\iota}\aleph(F(\eta,0_{(1,0)}),0_{(1,0)}d\iota)\right)\\
&+\sum_{0<\zeta_k<\zeta} Me^{\omega\iota}\aleph\left(T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),0_{(1,0)}\right)\\
\leq\ & Me^{\omega\xi}\left[\aleph(\varphi_0,0_{(1,0)})+\aleph(\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot,\varphi_0)),0_{(1,0)})\right]\\
&+Me^{\omega\xi}\left[K_1\int_0^\zeta e^{-\omega\iota}\aleph(\varphi(\iota,\varphi_0),0_{(1,0)})\,d\iota+\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,0_{(1,0)}),0_{(1,0)})\,d\iota\right]\\
&+Me^{\omega\xi}\left[L_z\int_0^\zeta e^{-\omega\iota}\aleph(\varphi(\iota,\varphi_0),0_{(1,0)})\,d\iota+\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta,0_{(1,0)}\right)d\iota\right]\\
&+Me^{\omega\xi}\left(\int_0^\zeta e^{-\omega\iota}\aleph(F(\eta,\varphi_0),0_{(1,0)}d\iota)+\int_0^\zeta e^{-\omega\iota}\aleph(F(\eta,0_{(1,0)}),0_{(1,0)}d\iota)\right)\\
&+Me^{\omega\varsigma}\sum_{0<\zeta_k<\zeta}\aleph\left(T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),0_{(1,0)}\right)
\end{aligned}
$$

From Gronwall's inequality, we get

$$
\begin{aligned}
\aleph(\varphi(\zeta,\varphi_0),0_{(1,0)})\leq Me^{\omega\xi}&\Big[\aleph(\varphi_0,0_{(1,0)})+\aleph(\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot,\varphi_0)),0_{(1,0)})\\
&+\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,0_{(1,0)}),0_{(1,0)})\,d\iota
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta(\iota,\delta,0_{(1,0)}),0_{(1,0)}\right)d\delta,d\iota\\
&+\int_0^{\zeta} e^{-\omega\iota}\aleph(F(\eta,0_{(1,0)}),0_{(1,0)}))d\iota\\
&\aleph(I_k(\varphi(\zeta_k^-)),I_k(\upsilon(\zeta_k^-)))\Big]
\end{aligned}
$$

$$
\begin{aligned}
\aleph(\varphi(\zeta,\varphi_0),0_{(1,0)}) \leq Me^{\omega\xi}\Big[&\aleph(\varphi_0,0_{(1,0)})+\aleph(\varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot,\varphi_0)),0_{(1,0)})\\
&+\int_0^{\zeta} e^{-\omega\iota}\aleph(\vartheta(\iota,0_{(1,0)}),0_{(1,0)})\,d\iota\\
&+\int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta(\iota,\delta,0_{(1,0)})0_{(1,0)}\right)d\delta,d\iota+\int_0^{\zeta} e^{-\omega\iota}\aleph(F(\eta,0_{(1,0)}),0_{(1,0)}))d\iota\Big]\\
&\exp([K_1+L_\eta+E.d_k]Me^{\omega\xi}\int_0^{\zeta} e^{-\omega\iota}d\iota)\\
\leq\ & Me^{\omega\xi}[\aleph(\varphi_0,0_{(1,0)})+N_1+N_3+N_4+N_5]\exp([K_1+L_\eta+E.d_k]M\frac{e^{\omega\xi}-1}{\omega})
\end{aligned}
$$

Taking $c_2 = Me^{\omega\xi}\exp([K_1+L_\eta+E.d_k]M\frac{e^{\omega\xi}-1}{\omega})$ *, we get*

$$
\begin{aligned}
H(\varphi(.,\ \varphi_0),\ 0_{(1,0)}) &= \sup_{0\leq\zeta\leq\xi}\aleph(\varphi(\zeta,\ \varphi_0),\ 0_{(1,0)})\\
&\leq\ c_2[\aleph(\varphi_0,0_{(1,0)})+N_1+N_3+N_4+N_5]
\end{aligned}
$$

This completes the proof. □

We investigate the following IF integro-differential equations with nonlocal conditions:

$$
\begin{aligned}
\varphi(\zeta) = T(\zeta)&\left[\varphi_0+\varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot))\right]\\
&+\int_0^{\zeta} T(\zeta-\iota)\vartheta(\iota,\varphi(\iota))\,d\iota\\
&+\int_0^{\zeta} T(\zeta-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta d\iota\\
&+\int_0^{\zeta} T(\zeta-\iota)F(\eta,\varphi)\,d\iota\\
&+\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\varphi_n(\zeta) = T(\zeta)&\left[\varphi_{n,0}+g_n(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi_n(\cdot))\right]\\
&+\int_0^{\zeta} T(\zeta-\iota)\vartheta_n(\iota,\varphi_n(\iota))\,d\iota\\
&+\int_0^{\zeta} T(\zeta-\iota)\left(\int_0^{\iota}\eta_n(\iota,\delta,\varphi_n(\delta))\,d\delta\right)d\iota,\quad n\geq 1.
\end{aligned}
$$

$$+ \int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi_n)\, d\iota$$
$$+ \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-)) \tag{4}$$

Theorem 3.3. *Assume that* ϑ, ϖ *are the same as in Thorem 3.1. If*

$$\aleph(\varphi_{n,0},\varphi_0) \to 0,$$
$$\aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot))) \to 0,$$
$$\aleph\left(\int_0^\iota \eta_n(\zeta,\iota,\varphi(\iota))\,dt, \int_0^\iota \eta(\zeta,\iota,\varphi(\iota))\,dt\right) \to 0,$$
$$\aleph(I_k(\varphi(\zeta_k^-)),I_k(\upsilon(\zeta_k^-))) \to 0, \aleph\left(\int_0^\zeta F_n(\eta,\varphi_n)d\iota, \int_0^\zeta F(\eta,\varphi)d\iota\right) \to 0$$

and

$$\sup_{0\le\zeta\le\xi} \aleph(\vartheta_n(\zeta,\upsilon),\vartheta(\zeta,\upsilon)) \to 0 \text{ as } n\to\infty \text{ for each } \upsilon\in L_2,$$

then

$$\sup_{0\le\zeta\le\xi} \aleph(\varphi_n(\zeta),\varphi(\zeta)) \to 0 \text{ as } n\to\infty.$$

proof: For any $\zeta \in [0,\ \xi]$ *we have*

$$\begin{aligned}
&\aleph(\varphi_n(\zeta),\ \varphi(\zeta))\\
&\le \aleph(T(\zeta)\varphi_{n,0},\ T(\zeta)\varphi_0)\\
&\quad + \aleph(T(\zeta)g_n(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)),\ T(\zeta)\varpi(\zeta_1,\zeta_2,\ \ldots\zeta_p,\ \varphi(\cdot)))\\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta_n(\iota,\ \varphi_n(\iota))\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota\right)\\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta_n(\iota,\delta,\varphi_n(\delta))\,d\delta\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)\\
&\quad + \aleph\left(\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-)), \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right)\\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi_n)\,d\iota, \int_0^\zeta T(\zeta-\iota)F(\eta,\varphi)\,d\iota\right)\\
&\le \aleph(T(\zeta)\varphi_{n,0},T(\zeta)\varphi_0)\\
&\quad + \aleph(T(\zeta)g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),T(\zeta)\varpi(\zeta_1,\zeta_2,\ \ldots\zeta_p,\varphi(\cdot)))\\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta_n(\iota,\varphi_n(\iota))\,d\iota, \int_0^\zeta T(\zeta-\iota)\vartheta_n(\iota,\varphi(\iota))\,d\iota\right)\\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta_n(\iota,\varphi_n(\iota))\,d\iota, \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\varphi(\iota))\,d\iota\right)
\end{aligned}$$

$$
\begin{aligned}
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta_n(\iota,\delta,\varphi_n(\delta))\,d\delta\,d\iota, \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta_n(\iota,\delta,\varphi_n(\delta))\,d\delta\,d\iota, \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)\\
&+\aleph\left(\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-)), \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-))\right)\\
&+\aleph\left(\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-)), \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi_n)\,d\iota, \int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi)\,d\iota\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi_n)\,d\iota, \int_0^\zeta T(\zeta-\iota)F(\eta,\varphi)\,d\iota\right)
\end{aligned}
$$

$$
\begin{aligned}
\aleph(\varphi_n(\zeta),\varphi(\zeta)) \le{}& Me^{\omega\xi}\Big[\aleph(\varphi_{n,0},\varphi_0)+\aleph(\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi_n(\cdot)), g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))\\
&+\aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))\Big]\\
&+\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))\,d\iota\\
&+\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta, \int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota\\
&+Me^{\omega\xi}\Big[\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi_n(\iota)),\vartheta_n(\iota,\varphi(\iota)))\,d\iota\\
&+\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta_n(\iota,\delta,\varphi_n(\delta))\,d\delta, \int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota\\
&+\int_0^\zeta e^{-\omega\iota}\aleph\left(F_n(\eta,\varphi_n)\,d\iota, F_n(\eta,\varphi)\,d\iota\right)\Big]+\aleph(I_k(\varphi(\zeta_k^-)),I_k(\upsilon(\zeta_k^-)))\\
\le{}& Me^{\omega\xi}[\aleph(\varphi_{n,0},\ \varphi_0)+K_2\aleph(\varphi_n(\cdot),\varphi(.))]\\
&+\aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))\\
&+\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))\,d\iota\\
&+\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta, \int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota\Big]\\
&+\int_0^\zeta e^{-\omega\iota}\aleph\left(F_n(\eta,\varphi_n)\,d\iota, F(\eta,\varphi)\,d\iota\right)\\
&+[K_1+L_\eta]Me^{\omega\xi}\int_0^\zeta e^{-\omega\iota}\aleph(\varphi_n(\iota)\varphi_\iota))\,]d\iota+E.d_k
\end{aligned}
$$

From Gronwall's inequality, we get

$$\begin{aligned}\aleph(\varphi_n(\zeta),\varphi(\zeta)) \le Me^{\omega\xi}\Big[&\aleph(\varphi_{n,0},\varphi_0)+K_2\aleph(\varphi_n(\cdot),\varphi(\cdot))+\aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))\\ &+\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))\,d\iota\\ &+\int_0^\zeta e^{-\omega\iota}\aleph\Big(\int_0^\iota \eta_n(\delta,\iota,\varphi(\delta))\,d\delta,\int_0^\iota \eta(\delta,\iota,\varphi(\delta))\,d\delta\Big)\,d\iota\\ &+\int_0^\zeta e^{-\omega\iota}\aleph(F_n(\eta,\varphi)),F(\eta,\varphi)d\iota\Big]\exp\Big((K_1+L_\eta+E.d_k)Me^{\omega\xi}\frac{1-e^{-\omega\zeta}}{\omega}\Big)\end{aligned}$$

That is,

$$\begin{aligned}&(1-K_2Me^{\omega\xi}\exp\Big((K_1+L_\eta)M\frac{e^{\omega\xi}-1}{\omega}\Big))\sup_{0\le\zeta\le\xi}\aleph(\varphi_n(\zeta),\varphi(\zeta))\\ &\le Me^{\omega\xi}\Big[\aleph(\varphi_{n,0},\varphi_0)+\aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))\\ &+\sup_{0\le\zeta\le\xi}\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))\,d\iota\\ &+\sup_{0\le\zeta\le\xi}\int_0^\zeta e^{-\omega\iota}\aleph\Big(\int_0^\iota \eta_n(\delta,\iota,\varphi(\delta))\,d\delta,\int_0^\iota \eta(\delta,\iota,\varphi(\delta))\,d\delta\Big)\,d\iota\\ &+\sup_{0\le\zeta\le\xi}\int_0^\zeta e^{-\omega\iota}\aleph\left((F_n(\eta,\varphi)),F(\eta,\varphi)\right)d\iota\Big]\exp\Big((K_1+L_\eta+E.d_k)M\frac{1-e^{-\omega\zeta}}{\omega}\Big)\,(5)\end{aligned}$$

And,

$$\begin{aligned}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota))) &\le \aleph\Big(\int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta,\int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta\Big)\\ &+\aleph\Big(\int_0^\iota \eta_n(\iota,\delta,0_{(1,0)})\,d\delta,\int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta\Big)\\ &+\aleph\Big(\int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta,\int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta\Big)\\ &+\aleph\Big(\int_0^\iota F_n(\eta,0_{(1,0)})\,d\iota,\int_0^\iota F(\eta,0_{(1,0)})\,d\iota\Big)\end{aligned}$$

$$\begin{aligned}&\le 2L_\eta\aleph(\varphi(\iota),0_{(1,0)})+\sup_{0\le\zeta\le\xi}\aleph\Big(\int_0^\iota \eta_n(\iota,\delta,0_{(1,0)})\,d\delta,\int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta\Big)\\ &\quad+\sup_{0\le\zeta\le\xi}\aleph\Big(\int_0^\iota F_n(\eta,0_{(1,0)})\,d\iota,\int_0^\iota F(\eta,0_{(1,0)})\,d\iota\Big)\\ &\le 2L_\eta C_2\Big(\aleph(\varphi_0,0_{(1,0)})+N_1+N_3+N_4+N_5\Big)+1\end{aligned}$$

As soon as n is sufficiently large, utilizing condition 2 of Theorem 3.1. Consequently, by utilizing the dominated convergence theorem in (5), we derive the theorem's conclusion. □

4. Controllability of Intuitionistic Fuzzy Impulsive Neutral Integro-Differential Equations

Consider the nonlocal IF neutral integro-differential equation:

$$\begin{cases} \frac{d}{dt}\Big(\varphi(\zeta) - \int_0^\zeta \eta(\zeta,\iota,\varphi(\iota)d\iota)\Big) = A\varphi(\zeta) + \vartheta(\zeta,\ \varphi(\zeta)) + \int_0^\zeta \eta(\zeta,\iota,\varphi(\iota)d\iota) + \beta\rho(\zeta), \quad \zeta \in I = [0,\ a] \\ \varphi(0) = \varphi_0 + \varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(.)) \\ \Delta\varphi(\zeta_k) = \mathcal{I}_k(\varphi(\zeta_k^-)), k = 1, 2, ..., m \end{cases}$$

where $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_p \le a$

Definition 4.1. *A function $\varphi : I \to L_2$ is known as a mild solution of (2), if*

$$\begin{aligned} \varphi(\zeta) = T(\zeta)\Big[\varphi_0 + \varpi(\zeta_1,\ \zeta_2,\ \zeta_p,\ \varphi(\cdot))\Big] &+ \int_0^\zeta \eta(\zeta,\iota,\varphi(\iota)d\iota) \\ &+ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\, d\iota \\ &+ \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\, d\delta, d\iota + \int_0^\zeta T(\zeta-\iota)\beta\rho(\iota)\, d\iota \\ &+ \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)) \end{aligned}$$

for $0 \le \zeta \le a$.

Theorem 4.1. *Assuming that conditions (H_1) through (H_6) are satisfied, Equation (2) allows a unique mild solution over the interval $[0,\ \xi]$, where*

$$\xi = \min\left\{a,\ \frac{D_0}{\omega}\log\left(\frac{b-\epsilon+\frac{N_1M}{\omega}}{MN_2+\frac{N_1M}{\omega}}\right)^{D_0},\ \frac{D_0}{\omega}\log\left(\frac{1+\frac{K_1M}{\omega}}{K_2M+\frac{K_1M}{\omega}}\right)^{D_0}\right\}$$

subject to the conditions

$$\aleph(\vartheta(\zeta,\ \varphi),\ 0_{(1,0)}) \le N_1, \quad \aleph(\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi\, 0_{(1,0)}) \le N_2.$$

proof: Let $B = \{\varphi \in L_2 | H(\varphi,\ \varphi_0) \le b\}$ represent the space of the continuous mean square IF mappings, where

$$H(\varphi,\ \upsilon) = \sup_{0\le\zeta\le\xi} \aleph(\varphi(\zeta),\ \upsilon(\zeta))$$

and b is a non-negative number. We describe a mapping $P : B \to B$ as: Using (H6) for an arbitary function $\rho(.)$ define the control

$$\begin{aligned} \rho(\zeta) = W^{-1}[\rho_2 1 - T(\zeta)[\varphi_0 + \varpi(\zeta 1, \zeta 2, \ldots, \zeta p, \varphi(.))] \\ + \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\varphi(\iota))\, d\iota \\ - \int_0^\zeta T(\zeta-\iota)\left(\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\, d\delta\right) d\iota \\ - \sum_{0<\zeta k<\delta} T(\zeta-\zeta k)I_k(\varphi(\zeta k^-)) \end{aligned}$$

where

$$\begin{aligned} F(\eta,\rho) =& BW^{-1}[\rho_2 1 - T(\zeta)[\varphi_0 + \varpi(\zeta 1, \zeta 2, \dots, \zeta p, \varphi(.))] \\ &+ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\varphi(\iota))\,d\iota \\ &- \int_0^\zeta T(\zeta-\iota)\left(\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota \\ &- \sum_{0<\zeta k<\delta} T(\zeta-\zeta k)I_k(\varphi(\zeta k^-))] \end{aligned}$$

$$\begin{aligned} P\varphi(\zeta) =& T(\zeta)\left[\varphi_0 + \varpi(\zeta_1,\ \zeta_2,\ \dots,\ \zeta_p,\ \varphi(.))\right] \\ &+ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota \\ &+ \int_0^\zeta T(\zeta-\iota)\left(\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota \\ &+ \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)) + \int_0^\zeta T(\zeta-\iota)F(\eta,\varphi)d\iota \end{aligned}$$

Firstly, we show that P fulfills and is mean square continuous. $H(Px,\ \varphi_0) \le b$. *Since* ϑ *is the continuous mean square , we have*

$$\begin{aligned} &\aleph(P\varphi(\zeta+h),\ P\varphi(\zeta)) \\ &= \aleph(T(\zeta+h)[\varphi_0 + \varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot))] \\ &\quad + \int_0^{\zeta+h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota + \int_0^{\zeta+h} \eta(\zeta,\iota,\varphi(\iota)d\iota) \\ &\quad + \int_0^{\zeta+h} T(\zeta+h-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,dz\,d\iota \\ &\quad + \sum_{0<(\zeta+h)_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)) + \int_0^{\zeta+h} T(\zeta+h-\iota)F(\eta,\rho)d\iota \\ &\quad + T(\zeta)[\varphi_0 + \varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot))] \\ &\quad + \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota + \int_0^\zeta \eta(\zeta,\iota,\varphi(\iota)d\iota) \\ &\quad + \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota) \\ &\quad + \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)) + \int_0^\zeta T(\zeta-\iota)F(\eta,\rho)d\iota \\ &\le \aleph(T(\zeta+h)\varphi_0, T(\zeta)\varphi_0) + \aleph(T(\zeta+h)\varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot)), T(\zeta)\varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot))) \end{aligned}$$

$$
\begin{aligned}
&+\aleph\left(\int_0^{\zeta+h} T(\zeta+h-\iota)\vartheta(\iota,\varphi(\iota))\,d\iota,\int_0^{\zeta} T(\zeta-\iota)\vartheta(\iota,\varphi(\iota))\,d\iota\right)\\
&+\aleph\left(\int_0^{\zeta+h} \eta(\zeta,\iota,\varphi(\iota))\,d\iota,\int_0^{\zeta} \eta(\zeta,\iota,\varphi(\iota))\,d\iota\right)\\
&+\aleph\left(\int_0^{\zeta+h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\int_0^{\zeta} T(\zeta-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)\\
&+\aleph\left(\int_0^{\zeta+h} T(\zeta+h-\iota)F(\eta,\rho)\,d\iota,\int_0^{\zeta} T(\zeta-\iota)F(\eta,\rho)\,d\iota\right)\\
&+\aleph\left(\sum_{0<\zeta_k<\zeta+h} T(\zeta+h-\zeta_k)I_k(\varphi(\zeta_k^-)),\sum_{0<\zeta_k<\zeta} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right)\\
\leq\ &\aleph(T(\zeta+h)\varphi_0,T(\zeta)\varphi_0)+\aleph(T(\zeta+h)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(.)),\ T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(.)))\\
&+\aleph\Big(\int_0^{\zeta+h}\eta(\zeta,\iota,\varphi(\iota)d\iota),\int_0^{\zeta}\eta(\zeta,\iota,\varphi(\iota)d\iota)\Big)+\aleph\left(\int_0^{h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ 0_{(1,0)}\right)\\
&+\aleph\left(\int_h^{\zeta+h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ \int_0^{\zeta} T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota\right)\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\delta,\iota,\ \varphi(\delta))\,d\delta d\iota,0_{(1,0)}\right)\\
&+\aleph\left(\int_h^{\zeta+h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ \int_0^{\zeta} T(\zeta-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)\\
&+\aleph\left(\sum_{0<(h)_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),0_{(1,0)}\right)\\
&+\aleph\left(\sum_{h<(\zeta+h)_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right)\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)F(\eta,\varphi)d\iota,0_{(1,0)}\right)\\
&+\aleph\left(\int_h^{\zeta+h} T(\zeta+h-\iota)F(\eta,\varphi)d\iota,\int_0^{t} T(\zeta-\iota)F(\eta,\varphi)d\iota\right)\\
\leq\ &Me^{\omega\zeta}[\aleph(T(h)\varphi_0,\ \varphi_0)+\aleph(T(h)\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\ T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)))\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ 0_{(1,0)}\right)+\aleph\Big(\int_0^{\zeta+h}\eta(\zeta,\iota,\varphi(\iota)d\iota),\int_0^{\zeta}\eta(\zeta,\iota,\varphi(\iota)d\iota)\Big)\\
&+\int_0^{\zeta} Me^{\omega(\zeta-\iota)}\aleph(\vartheta(\iota+h,\ \varphi(\iota+h)),\vartheta(\iota,\ \varphi(\iota)))\,d\iota]\\
&+\aleph\left(\int_0^{h} T(\zeta+h-\iota)\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ 0_{(1,0)}\right)
\end{aligned}
$$

$$+\int_0^{\zeta} Me^{\omega(\zeta-\iota)}\aleph\left(\int_0^{\iota}\eta(\iota,\delta+h,\varphi(\delta+h))\,d\delta,\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota$$
$$+\aleph\left(\int_0^{h}T(\zeta+h-\iota)F(\eta,\rho)d\iota,0_{(1,0)}\right)+\int_0^{\zeta} Me^{\omega(\zeta-\iota)}\aleph\left(F(\eta,\varphi),F(\eta,\upsilon)\right)d\iota$$
$$+\aleph(I_k(\varphi(\zeta_k^-))),I_k(\nu(\zeta_k^-))$$

It is obvious that $\aleph(T(h)\varphi_0,\ \varphi_0)\rightarrow 0,\aleph(T(h)\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(.)\ ,\ \varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(.)\rightarrow 0,\ +$ $\aleph\left(\int_0^{\zeta+h}\eta(\zeta,\iota,\varphi(\iota)d\iota),\int_0^{\zeta}\eta(\zeta,\iota,\varphi(\iota)d\iota)\right)\rightarrow 0$
and

$$\aleph(\int_0^{h}T(\zeta+h-\iota)\vartheta(\iota,\ \varphi(\iota))d\iota,\ 0_{(1,0)}))\rightarrow 0$$

as $h\rightarrow 0$.

$$\aleph(I_k(\varphi(\zeta_k^-))),I_k(\nu(\zeta_k^-))\rightarrow 0$$
$$\aleph\left(F(\eta,\varphi),F(\eta,\upsilon)\right)\rightarrow 0$$

And according to the theorem of dominated convergence :

$$\int_0^{\zeta} Me^{\omega(\zeta-\iota)}\aleph(\vartheta(\iota+h,\ \varphi(\iota+h)),\ \vartheta(\iota,\ \varphi(\iota)))d\iota\rightarrow 0$$

The mapping P exhibits m.s continuity over I. Moreover,

$$\begin{aligned}\aleph(P\varphi(\zeta),\varphi_0)=\aleph\Bigg(&T(\zeta)\Big[\varphi_0+\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot))\Big]\\&+\int_0^{\zeta}\eta(\zeta,\iota,\varphi(\iota))\,d\iota\\&+\int_0^{\zeta}T(\zeta-\iota)\vartheta(\iota,\varphi(\iota))\,d\iota\\&+\int_0^{\zeta}T(\zeta-\iota)\int_{\iota}^{\zeta}\eta(\delta,\iota,\varphi(\delta))\,d\delta\,d\iota,\varphi_0\Bigg)\\&+\sum_{0<\zeta_k<\zeta}T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),\varphi_0\\&+\int_0^{\zeta}T(\zeta-\iota)F(\eta,\varphi)\,d\iota\end{aligned}$$

$$\leq\aleph(T(\zeta)\varphi_0,\ \varphi_0)+\aleph(T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)),\ 0_{(1,0)})+\aleph(T(\zeta)\int_0^{\zeta}\eta(\zeta,\iota,\varphi(\iota)d\iota))$$
$$+\aleph\left(\int_0^{\zeta}T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ 0_{(1,0)}\right)+\aleph\left(\int_0^{\zeta}T(\zeta-\iota)\int_{\iota}^{\zeta}\eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ 0_{(1,0)}\right)$$
$$+\aleph\left(\sum_{0<\zeta_k<\zeta}T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),\varphi_0\right)+\aleph\left(\int_0^{\zeta}T(\zeta-\iota)F(\eta,\varphi)d\iota\right)$$

$$\leq \epsilon + N_2Me^{\omega\zeta} + \frac{N_1M}{\omega}(e^{\omega\zeta}-1) + \frac{N_3M}{\omega}(e^{\omega\zeta}) + E.d_k$$
$$+ D_0\left[N_2Me^{\omega\zeta} + \frac{N_1M}{\omega}(e^{\omega\zeta}-1) + \frac{N_3M}{\omega}(e^{\omega\zeta}-1)\right]$$

and so

$$H(Px,\ \varphi_0) = \sup_{0\leq\zeta\leq\xi} \aleph(P\varphi(\zeta),\ \varphi_0)$$
$$\leq b.$$

In $C([0,\ \xi],\ L_2)$, denoted by $C([0,\ \xi],\ L_2) = \varphi : [0,\xi] \to L_2 \mid \varphi(\zeta)$ is mean square continuous, completeness holds. We demonstrate that B constitutes a closed subset of $C([0,\ \xi],\ L_2)$. Suppose φ_n is a sequence in B such that $\varphi_n \to \varphi \in C([0,\ \xi],\ L_2)$ as $n \to \infty$. Then,

$$\aleph(\varphi(\zeta),\ \varphi_0) \leq \aleph(\varphi(\zeta),\ \varphi_n(\zeta)) + \aleph(\varphi_n(\zeta),\ \varphi_0)$$
$$H(\varphi,\ \varphi_0) = \sup_{0\leq\zeta\leq\xi} \aleph(\varphi(\zeta),\ \varphi_0)$$
$$\leq H(\varphi,\ \varphi_n) + H(\varphi_n,\ \varphi_0)$$
$$\leq \epsilon + b$$

For adequately large n and for any $\epsilon > 0$, $\varphi \in B$. Consequently, B emerges as a closed subset of $C([0,\ \xi],\ L_2)$. Hence, B constitutes a complete metric space. Next, we will demonstrate the contraction property of P. Given $\varphi, \upsilon \in B$,

$$\aleph(P\varphi(\zeta),\ P\upsilon(\zeta)) \leq \aleph(T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)), T(\zeta)\varpi(\zeta_1,\zeta_2,\ \ldots,\zeta_p,\ \upsilon(\cdot))$$
$$+ \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \upsilon(\iota))\,d\iota\right)$$
$$+ \aleph\Big(\int_0^\zeta \eta(\zeta,\iota,\varphi(\iota)d\iota), \int_0^\zeta \eta(\zeta,\iota,\upsilon(\iota)d\iota)\Big)$$
$$+ \aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\upsilon(\delta))\,d\delta\,d\iota\right)$$
$$+ \aleph\left(\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)), \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\upsilon(\zeta_k^-))\right)$$
$$+ \aleph\left(\int_0^\zeta T(\zeta-\iota)F(z,\varphi)d\iota, \int_0^\zeta T(\zeta-\iota)F(z,\upsilon)d\iota\right)$$
$$\leq K_2Me^{\omega\zeta}\aleph(\varphi,\ \upsilon) + K_1M\int_0^\zeta e^{\omega(\zeta-\iota)}\aleph(\vartheta(\iota,\ \varphi(\iota)),\ \vartheta(\iota,\ \upsilon(\iota)))\,d\iota$$
$$+ L_2M\int_0^\zeta e^{\omega(\zeta-\iota)}\aleph(\int_0^\iota \eta(\eta(\iota,\delta,\varphi(\delta))d\delta, \int_0^\iota \eta(\iota,\delta,\upsilon(\delta))d\delta)d\iota$$
$$+ \aleph\,(F(z,\varphi),F(z,\upsilon)) + \aleph(I_k(\varphi(\zeta_k^-))), I_k(\nu(\zeta_k^-))$$

Thus, we obtain

$$\leq \ (K_2Me^{\omega\xi} + \frac{K_1M}{\omega}[e^{\omega\xi} - 1] + \frac{L_2M}{\omega}[e^{\omega\xi} - 1] + L_\eta)H(\varphi,\ \upsilon) + E.d_k + D_0\Big[K_2Me^{\omega\zeta} + K_1M[e^{\omega\xi} - 1] + \frac{L_2M}{\omega}[e^{\omega\xi} - 1]\Big]$$

Due to the inequality $(K_2Me^{\omega\xi} + \frac{K_1M}{\omega}[e^{\omega\xi} - 1] + \frac{L_2M}{\omega}[e^{\omega\xi} - 1] + L_\eta + E.d_k + D_0\Big[K_2Me^{\omega\zeta} + K_1M[e^{\omega\xi} - 1] + \frac{L_2M}{\omega}[e^{\omega\xi} - 1]\Big]) < 1$, *the mapping P qualifies as a contraction map. Hence, P possesses a unique fixed point. This completes the proof.* □

Theorem 4.2. *Suppose* ϑ *and* ϖ *are as described in Theorem 3.1. Let* $\varphi(\zeta,\ \varphi_0)$ *and* $\upsilon(\zeta,\ \upsilon_0)$ *represent solutions of Equation (2) for initial values* φ_0 *and* υ_0*, respectively. Then, there exist constants* c_1 *and* c_2 *such that*

1. $H(\varphi(.,\ \varphi_0),\ \upsilon_0)) \leq c_1\aleph(\varphi_0,\ \upsilon_0)$ *for any* $\varphi_0, \upsilon_0 \in L_2$
2. $H(\varphi(.,\ \varphi_0),\ 0_{(1,0)}) \leq c_2(\aleph(\varphi_0, 0_{(1,0)}) + N_1 + N_3 + N_4 + N_5)$ *where* $\aleph(\varpi(\zeta_1,\ \zeta_2,\ \zeta_p,\ \varphi\ 0_{(1,0)}) \leq N_1$, $\displaystyle\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\ 0_{(1,0)}), 0_{(1,0)}) \leq N_3$*, and* $\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta(\iota,\zeta,0_{(1,0)}), 0_{(1,0)}\right) dt, d\iota\Big] \leq N_4$, $\int_0^\zeta e^{-\omega\iota}\aleph\left(F(\eta, 0_{(1,0)})\right) \leq N_5$

Proof: For any $\zeta \in [0,\ \xi]$ *we have*

$$\begin{aligned}
&\aleph(\varphi(\zeta,\varphi_0),\ \upsilon(\zeta,\upsilon_0)) \\
&\leq \aleph(T(\zeta)\varphi_0,\ T(\zeta)\upsilon_0) \\
&\quad + \aleph(T(\zeta)\varpi(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot,\varphi_0)),\ T(\zeta)\varpi(\zeta_1,\zeta_2,\ \ldots,\zeta_p,\ \upsilon(\cdot,\upsilon_0))) \\
&\quad + \aleph\Big(\int_0^\zeta \eta(\zeta,\iota,\varphi(\iota,\varphi_0)d\iota), \int_0^\zeta \eta(\zeta,\iota,\varphi(\iota)d\iota,\upsilon_0)\Big) \\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota,\varphi_0))\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \upsilon(\iota,\upsilon_0))\,d\iota\right) \\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0))\,d\delta\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\upsilon(\delta,\upsilon_0))\,d\delta\,d\iota\right) \\
&\quad + \aleph\left(\sum_{0<(\zeta+h)_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)) + \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right) \\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)F(\eta,\varphi_0)\,d\iota, \int_0^\zeta T(\zeta-\iota)F(\eta,\upsilon_0)\,d\iota\right) \\
&\leq Me^{\omega\zeta}\left[\aleph(\varphi_0,\upsilon_0) + \aleph(\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot,\varphi_0)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\upsilon(\cdot,\upsilon_0)))\right] \\
&\quad + \aleph\left(\int_0^\zeta \eta(\zeta,\iota,\varphi(\iota,\varphi_0))\,d\iota, \int_0^\zeta \eta(\zeta,\iota,\varphi(\iota,\upsilon_0))\,d\iota\right) \\
&\quad + Me^{\omega\zeta}\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\varphi(\iota,\varphi_0)),\vartheta(\iota,\upsilon(\iota,\upsilon_0)))\,d\iota
\end{aligned}$$

$$
\begin{aligned}
&+ Me^{\omega\zeta}\int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta(\iota,\delta,\varphi(\delta,\varphi_0))\,d\delta, \int_0^{\iota}\eta(\iota,\delta,v(\delta,v_0))\,d\delta\right)\\
&+ Me^{\omega\zeta}\int_0^{\zeta} e^{-\omega\iota}\aleph\left(F(\eta,\varphi_0),F(\eta,v_0)\right)\,d\iota\\
&+ \aleph(I_k(\varphi(\zeta_k^-)),I_k(v(\zeta_k^-)))\\
\leq\ & Me^{\omega\xi}[\aleph(\varphi_0,\ v_0)+K_2\aleph(\varphi(\cdot,\ \varphi_0),\ v(\cdot,\ v_0))]+L_\eta\\
&+ Me^{\omega\xi}[K_1\int_0^{\zeta} e^{-\omega\iota}\aleph(\varphi(\iota,\ \varphi_0),\ v(\iota,\ v_0))\,d\iota\\
&+ L_\eta\int_0^{\zeta} e^{-\omega\iota}\aleph(\varphi(\iota,\ \varphi_0),\ v(\iota,\ v_0))\,]d\iota\\
&+ \aleph(I_k(\varphi(\zeta_k^-)),I_k(v(\zeta_k^-)))\\
&+ Me^{\omega\zeta}\int_0^{\zeta} e^{-\omega\iota}\aleph\left(F(\eta,\varphi_0),F(\eta,v_0)\right)d\iota\\
\leq\ & Me^{\omega\xi}[\aleph(\varphi_0,\ v_0)+K_2\aleph(\varphi(\cdot,\ \varphi_0),\ v(\cdot,\ v_0))]\\
&+ \int_0^{\zeta}\left[Me^{\omega\xi}e^{-\omega\iota}\right](K_1\aleph(\varphi(\iota,\ \varphi_0),\ v(\iota,\ v_0)))+L_\eta\Big(\aleph(\varphi(\iota,\ \varphi_0),\ v(\iota,\ v_0))\,d\iota+1\Big)\\
&+ F[(\eta,\varphi_0),(\eta,v_0)])d\iota+\aleph(I_k(\varphi(\zeta_k^-)),I_k(v(\zeta_k^-)))
\end{aligned}
$$

Gronwall's inequality provides us with

$$
\begin{aligned}
&\aleph(\varphi(\zeta,\ \varphi_0),\ v(\zeta,\ v_0))\\
&\leq Me^{\omega\xi}[\aleph(\varphi_0,\ v_0)+K_2\aleph(\varphi(\cdot,\ \varphi_0),\ v(\cdot,\ v_0))]\exp((K_1+L\eta+E.d_k)Me^{\omega\xi}\int_0^{\zeta}e^{-\omega\varsigma}d\varsigma)\\
&\leq Me^{\omega\xi}[\aleph(\varphi_0,\ v_0)+K_2\aleph(\varphi(\cdot,\ \varphi_0),\ v(\cdot,\ v_0))]\exp((K_1+L\eta+E.d_k)Me^{\omega\xi}\frac{1-e^{-\omega\zeta}}{\omega})
\end{aligned}
$$

Thus we have

$$
H(\varphi(.,\ \varphi_0),\ v(.,v_0))\leq Me^{\omega\xi}[\aleph(\varphi_0,\ v_0)+L_\eta+K_2H(\varphi(.,\ \varphi_0),v(.,v_0))]\exp([K_1+L_z+E.d_k]M\frac{e^{\omega\xi}-1}{\omega})
$$

i.e,

$$
\begin{aligned}
(1-K_2Me^{\omega\zeta}E.d_k\exp\left(K_1M\frac{e^{\omega\xi}-1}{\omega}\right))H(\varphi(.,\ \varphi_0),\ v(.,\ v_0))\\
\leq Me^{\omega\xi}E.d_k\exp\left([K_1+L_\eta]M\frac{e^{\omega\xi}-1}{\omega}+L_\eta\right)\aleph(\varphi_0,\ v_0)
\end{aligned}
$$

Consequently, we obtain

$$
H(\varphi(.,\ \varphi_0),\ v(.,v_0)\ )\leq\frac{Me^{\omega\xi}\exp([K_1+L_z+E.d_k]M\frac{e^{\omega\xi}-1}{\omega}+L_\eta)}{(1-K_2Me^{\omega\xi}\exp([K_1+L_z+E.d_k]M\frac{e^{\omega\xi}-1}{\omega}))}\aleph(\varphi_0,\ v_0)
$$

Taking $c_1 = \dfrac{Me^{\omega\xi}\exp([K_1 + L_z + E.d_k]M\frac{e^{\omega\xi}-1}{\omega})}{(1 - K_2 Me^{\omega\xi}\exp([K_1 + L_z + E.d_k]M\frac{e^{\omega\xi}-1}{\omega} + L_\eta))}$
we obtain $H(\varphi(., \varphi_0), \upsilon(., \upsilon_0)) \leq c_1 \aleph(\varphi_0, \upsilon_0)$

2. *For any* $\zeta \in [0, \xi]$ *we have*

$$
\begin{aligned}
&\aleph(\varphi(\zeta,\varphi_0), 0_{(1,0)}) \\
&\leq \aleph(T(\zeta)\varphi_0, 0_{(1,0)}) \\
&\quad + \aleph(T(\zeta)\varpi(\zeta_1, \zeta_2, \dots, \zeta_p, \varphi(\cdot,\varphi_0), 0_{(1,0)})) \\
&\quad + \aleph\left(\int_0^\zeta \vartheta(\iota, \varphi(\iota,\varphi_0))\, d\iota, 0_{(1,0)}\right) \\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta(\iota, \varphi(\iota,\varphi_0))\, d\iota, 0_{(1,0)}\right) \\
&\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0))\, d\delta\, d\iota, 0_{(1,0)}\right) \\
&\quad + \aleph\left(\sum_{0<\zeta_k<\zeta} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)), 0_{(1,0)}\right) + \aleph\left(\int_0^\zeta T(\zeta-\iota)F(\eta,\varphi_0)d\iota, 0_{(1,0)}\right) \\
&\leq Me^{\omega\zeta}\left[\aleph(\varphi_0, 0_{(1,0)}) + \aleph(\varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot,\varphi_0)), 0_{(1,0)})\right] \\
&\quad + Me^{\omega\zeta}\bigg[\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\varphi(\iota,\varphi_0)), 0_{(1,0)})\, d\iota \\
&\quad + \aleph\left(\int_0^\zeta \vartheta(\iota, \varphi(\iota,\varphi_0))\, d\iota, 0_{(1,0)}\right) \\
&\quad + \int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0))\, d\delta, 0_{(1,0)}\right) d\iota \\
&\quad + \int_0^\zeta e^{-\omega\iota}\aleph(F(\eta,\varphi_0), 0_{(1,0)})\, d\iota\bigg] \\
&\quad + Me^{\omega\zeta}\sum_{0<\zeta_k<\zeta}\aleph\left(T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)), 0_{(1,0)}\right) \\
&\leq Me^{\omega\xi}[\aleph(\varphi_0, 0_{(1,0)}) + \aleph(\varpi(\zeta_1, \zeta_2, \dots, \zeta_p, \varphi(\cdot,\varphi_0)), 0_{(1,0)}))] \\
&\quad + Me^{\omega\xi}[\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota,\varphi(\iota,\varphi_0)), \vartheta(\iota, 0_{(1,0)}))\, d\iota \\
&\quad + \int_0^\zeta e^{-\omega\iota}\aleph(\vartheta(\iota, 0_{(1,0)}), 0_{(1,0)})\,]d\iota + \aleph\left(\int_0^\zeta \vartheta(\iota, \varphi(\iota,\varphi_0))\, d\iota, 0_{(1,0)}\right) \\
&\quad + Me^{\omega\xi}[\int_0^\zeta e^{-\omega\iota}\aleph(\int_0^\iota \eta(\iota,\delta,\varphi(\delta,\varphi_0)), \int_0^\iota \eta(\iota,\delta,0_{(1,0)}))d\delta, d\iota \\
&\quad + \int_0^\zeta e^{-\omega\iota}\aleph(\int_0^\iota \eta(\iota,\delta,0_{(1,0)})d\delta, 0_{(1,0)}))\, d\iota]
\end{aligned}
$$

$$
\begin{aligned}
&+ Me^{\omega\xi}\left(\int_0^{\zeta} e^{-\omega\iota}\aleph(F(\eta,\varphi_0),0_{(1,0)}d\iota) + \int_0^{\zeta} e^{-\omega\iota}\aleph(F(\eta,0_{(1,0)}),0_{(1,0)}d\iota)\right)\\
&+ \sum_{0<\zeta_k<\zeta} Me^{\omega\iota}\aleph\Big(T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),0_{(1,0)}\Big)\\
\leq{}& Me^{\omega\xi}\Big[\aleph(\varphi_0,0_{(1,0)}) + \aleph(\varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot,\varphi_0)),0_{(1,0)})\Big]\\
&+ Me^{\omega\xi}\Big[K_1\int_0^{\zeta}\Big(e^{-\omega\iota}\aleph(\varphi(\iota,\varphi_0),0_{(1,0)})\Big) + \int_0^{\zeta} e^{-\omega\iota}\aleph(\vartheta(\iota,0_{(1,0)}),0_{(1,0)})\,d\iota\Big]\\
&+ \aleph\left(\int_0^{\zeta}\vartheta(\iota,\ \varphi(\iota,\varphi_0))\,d\iota,0_{(1,0)}\right)\\
&+ Me^{\omega\xi}\Big[L_z\int_0^{\zeta}\Big(e^{-\omega\iota}\aleph(\varphi(\iota,\varphi_0),0_{(1,0)})\Big)d\iota + \int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta(\iota,\delta,0_{(1,0)})d\delta,0_{(1,0)}\right)d\iota\\
&+ Me^{\omega\xi}\left(\int_0^{\zeta} e^{-\omega\iota}\aleph(F(\eta,\varphi_0),0_{(1,0)}d\iota) + \int_0^{\zeta} e^{-\omega\iota}\aleph(F(\eta,0_{(1,0)}),0_{(1,0)}d\iota)\right)\\
&+ Me^{\omega\varsigma}\sum_{0<\zeta_k<\zeta}\aleph\Big(T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-)),0_{(1,0)}\Big)\Big]
\end{aligned}
$$

From Gronwall's inequality, we get

$$
\begin{aligned}
\aleph(\varphi(\zeta,\varphi_0),0_{(1,0)}) \leq Me^{\omega\xi}\Big[&\aleph(\varphi_0,0_{(1,0)}) + \aleph(\varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot,\varphi_0)),0_{(1,0)})\\
&+ \int_0^{\zeta} e^{-\omega\iota}\aleph(\vartheta(\iota,0_{(1,0)}),0_{(1,0)})\,d\iota\\
&+ \int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta(\iota,\delta,0_{(1,0)}),0_{(1,0)}\right)d\delta,d\iota\Big]\\
&+ \aleph\left(\int_0^{\zeta}\vartheta(\iota,\ \varphi(\iota,\varphi_0))\,d\iota,0_{(1,0)}\right)\\
&+ \int_0^{\zeta} e^{-\omega\iota}\aleph(F(\eta,0_{(1,0)}),0_{(1,0)}))d\iota\\
&+ \aleph(I_k(\varphi(\zeta_k^-)),I_k(\upsilon(\zeta_k^-)))\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\aleph(\varphi(\zeta,\varphi_0),0_{(1,0)})\\
&\leq Me^{\omega\xi}\Big[\aleph(\varphi_0,0_{(1,0)}) + \aleph(\varpi(\zeta_1,\zeta_2,\dots,\zeta_p,\varphi(\cdot,\varphi_0)),0_{(1,0)})\\
&\quad+ \int_0^{\zeta} e^{-\omega\iota}\aleph(\vartheta(\iota,0_{(1,0)}),0_{(1,0)})\,d\iota\\
&\quad+ \int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta(\iota,\delta,0_{(1,0)})0_{(1,0)}\right)d\delta,d\iota + \int_0^{\zeta} e^{-\omega\iota}\aleph(F(\eta,0_{(1,0)}),0_{(1,0)})d\iota\Big]
\end{aligned}
$$

$$\exp([K_1 + L_\eta + E.d_k]Me^{\omega\xi}\int_0^\zeta e^{-\omega\iota}d\iota)\Big]$$

$$\leq\ L_\eta + Me^{\omega\xi}[\aleph(\varphi_0, 0_{(1,0)}) + N_1 + N_3 + N_4 + N_5]\exp([K_1 + L_\eta + E.d_k]M\frac{e^{\omega\xi}-1}{\omega})$$

Taking $c_2 = L_\eta + Me^{\omega\xi}\exp([K_1 + L_\eta + E.d_k]M\frac{e^{\omega\xi}-1}{\omega})$ *, we get*

$$\begin{aligned} H(\varphi(.,\ \varphi_0),\ 0_{(1,0)}) &= \sup_{0\leq\zeta\leq\xi}\aleph(\varphi(\zeta,\ \varphi_0),\ 0_{(1,0)}) \\ &\leq\ c_2[\aleph(\varphi_0, 0_{(1,0)}) + N_1 + N_3 + N_4 + N_5] \end{aligned}$$

This completes the proof. □

Theorem 4.3. *Suppose that* ϑ, ϖ *are the same as in Thorem 3.1. If*

$$\aleph(\varphi_{n,0}, \varphi_0) \to 0,$$

$$\aleph(g_n(\zeta_1, \zeta_2, \ldots, \zeta_p, \varphi(\cdot)), \varpi(\zeta_1, \zeta_2, \ldots, \zeta_p, \varphi(\cdot))) \to 0,$$

$$\aleph\left(\int_0^\iota \eta_n(\zeta, \iota, \varphi(\iota))\,dt, \int_0^\iota \eta(\zeta, \iota, \varphi(\iota))\,dt\right) \to 0,$$

$$\aleph(I_k(\varphi(\zeta_k^-)), I_k(v(\zeta_k^-))) \to 0, \aleph\left(\int_0^\zeta F_n(\eta, \varphi_n)d\iota, \int_0^\zeta F(\eta, \varphi)d\iota\right) \to 0$$

and

$$\sup_{0\leq\zeta\leq\xi}\aleph(\vartheta_n(\zeta, v), \vartheta(\zeta, v)) \to 0 \text{ as } n \to \infty \text{ for each } v \in L_2,$$

then

$$\sup_{0\leq\zeta\leq\xi}\aleph(\varphi_n(\zeta), \varphi(\zeta)) \to 0 \text{ as } n \to \infty.$$

proof: For any $\zeta \in [0,\ \xi]$ *we have*

$$\begin{aligned} &\aleph(\varphi_n(\zeta),\ \varphi(\zeta)) \\ &\leq \aleph(T(\zeta)\varphi_{n,0},\ T(\zeta)\varphi_0) \\ &\quad + \aleph(T(\zeta)g_n(\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_p,\ \varphi(\cdot)),\ T(\zeta)\varpi(\zeta_1, \zeta_2,\ \ldots, \zeta_p,\ \varphi(\cdot))) \\ &\quad + \aleph\left(\int_0^\zeta \eta_n(\iota, \delta, \varphi_n(\delta))\ d\iota,\ \int_0^\zeta \eta(\iota, \delta, \varphi(\delta))\ , d\iota\right) \\ &\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta_n(\iota,\ \varphi_n(\iota))\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\ \varphi(\iota))\,d\iota\right) \\ &\quad + \aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta_n(\iota, \delta, \varphi_n(\delta))\,d\delta\,d\iota,\ \int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota, \delta, \varphi(\delta))\,d\delta\,d\iota\right) \\ &\quad + \aleph\left(\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-)), \sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right) \end{aligned}$$

$$
\begin{aligned}
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi_n)\,d\iota,\int_0^\zeta T(\zeta-\iota)F(\eta,\varphi)\,d\iota\right)\\
&\le \aleph(T(\zeta)\varphi_{n,0},T(\zeta)\varphi_0)\\
&+\aleph(T(\zeta)g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),T(\zeta)\varpi(\zeta_1,\zeta_2,\ \ldots,\zeta_p,\varphi(\cdot)))\\
&+\aleph\left(\int_0^\zeta \eta_n(\iota,\delta,\varphi_n(\delta))\,d\iota,\ \int_0^\zeta \eta(\iota,\delta,\varphi(\delta))\,,d\iota\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta_n(\iota,\varphi_n(\iota))\,d\iota,\int_0^\zeta T(\zeta-\iota)\vartheta_n(\iota,\varphi(\iota))\,d\iota\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)\vartheta_n(\iota,\varphi_n(\iota))\,d\iota,\int_0^\zeta T(\zeta-\iota)\vartheta(\iota,\varphi(\iota))\,d\iota\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta_n(\iota,\delta,\varphi_n(\delta))\,d\delta\,d\iota,\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta_n(\iota,\delta,\varphi_n(\delta))\,d\delta\,d\iota,\int_0^\zeta T(\zeta-\iota)\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\,d\iota\right)\\
&+\aleph\left(\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-)),\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-))\right)\\
&+\aleph\left(\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi_n(\zeta_k^-)),\sum_{0<\zeta_k<z} T(\zeta-\zeta_k)I_k(\varphi(\zeta_k^-))\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi_n)\,d\iota,\int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi)\,d\iota\right)\\
&+\aleph\left(\int_0^\zeta T(\zeta-\iota)F_n(\eta,\varphi_n)\,d\iota,\int_0^\zeta T(\zeta-\iota)F(\eta,\varphi)\,d\iota\right)
\end{aligned}
$$

$$
\begin{aligned}
\aleph(\varphi_n(\zeta),\varphi(\zeta)) \le Me^{\omega\xi}\Big[&\aleph(\varphi_{n,0},\varphi_0)+\aleph(\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi_n(\cdot)),g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))\\
&+\aleph\left(\int_0^\zeta \eta_n(\iota,\delta,\varphi_n(\delta))\,d\iota,\ \int_0^\zeta \eta(\iota,\delta,\varphi(\delta))\,,d\iota\right)\\
&+\aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))\Big]\\
&+\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))\,d\iota\\
&+\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta,\int_0^\iota \eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota\\
&+Me^{\omega\xi}\Big[\int_0^\zeta e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi_n(\iota)),\vartheta_n(\iota,\varphi(\iota)))\,d\iota\\
&+\int_0^\zeta e^{-\omega\iota}\aleph\left(\int_0^\iota \eta_n(\iota,\delta,\varphi_n(\delta))\,d\delta,\int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota
\end{aligned}
$$

$$+ \int_0^{\zeta} e^{-\omega\iota}\aleph\left(F_n(\eta,\varphi_n)\,d\iota, F_n(\eta,\varphi)\,d\iota\right)\Big] + \aleph(I_k(\varphi(\zeta_k^-)), I_k(\upsilon(\zeta_k^-)))\Big]$$

$$\leq Me^{\omega\xi}[\aleph(\varphi_{n,0},\ \varphi_0) + K_2\aleph(\varphi_n(\cdot),\varphi(.))] + L_\eta$$

$$+ \aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))$$

$$+ \int_0^{\zeta} e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))\,d\iota$$

$$+ \int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta_n(\iota,\delta,\varphi(\delta))\,d\delta, \int_0^{\iota}\eta(\iota,\delta,\varphi(\delta))\,d\delta\right)d\iota\Big]$$

$$+ \int_0^{\zeta} e^{-\omega\iota}\aleph\left(F_n(\eta,\varphi_n)\,d\iota, F(\eta,\varphi)\,d\iota\right)$$

$$+ [K_1 + L_\eta]Me^{\omega\xi}\int_0^{\zeta} e^{-\omega\iota}\aleph(\varphi_n(\iota)\varphi_\iota))\,]d\iota + E.d_k$$

From Gronwall's inequality, we get

$$\aleph(\varphi_n(\zeta),\varphi(\zeta))$$

$$\leq Me^{\omega\xi}\Big[\aleph(\varphi_{n,0},\varphi_0) + K_2\aleph(\varphi_n(\cdot),\varphi(\cdot)) + \aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))$$

$$+ \int_0^{\zeta} e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))\,d\iota$$

$$+ \int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta_n(\delta,\iota,\varphi(\delta))\,d\delta, \int_0^{\iota}\eta(\delta,\iota,\varphi(\delta))\,d\delta\right)d\iota\Big]$$

$$+ \int_0^{\zeta} e^{-\omega\iota}\aleph(F_n(\eta,\varphi)), F(\eta,\varphi)d\iota\Big]\exp\left((K_1+L_\eta)Me^{\omega\xi}\frac{1-e^{-\omega\zeta}}{\omega}\right) + E.d_k$$

That is,

$$(1 - K_2Me^{\omega\xi}\exp\left((K_1+L_\eta)M\frac{e^{\omega\xi}-1}{\omega}\right))\sup_{0\leq\zeta\leq\xi}\aleph(\varphi_n(\zeta),\varphi(\zeta))$$

$$\leq Me^{\omega\xi}\Big[\aleph(\varphi_{n,0},\varphi_0) + \aleph(g_n(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)),\varpi(\zeta_1,\zeta_2,\ldots,\zeta_p,\varphi(\cdot)))$$

$$+ \sup_{0\leq\zeta\leq\xi}\int_0^{\zeta} e^{-\omega\iota}\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))\,d\iota$$

$$+ \sup_{0\leq\zeta\leq\xi}\int_0^{\zeta} e^{-\omega\iota}\aleph\left(\int_0^{\iota}\eta_n(\delta,\iota,\varphi(\delta))\,d\delta, \int_0^{\iota}\eta(\delta,\iota,\varphi(\delta))\,d\delta\right)d\iota\Big]$$

$$+ \sup_{0\leq\zeta\leq\xi}\int_0^{\zeta} e^{-\omega\iota}\aleph\left((F_n(\eta,\varphi)), F(\eta,\varphi)\right)d\iota\Big]\exp\left((K_1+L_\eta)M\frac{1-e^{-\omega\zeta}}{\omega}\right) + E.d_k(5)$$

And,

$$\aleph(\vartheta_n(\iota,\varphi(\iota)),\vartheta(\iota,\varphi(\iota)))$$

$$\leq \aleph\left(\int_0^\iota \eta_n(\iota,\delta,\varphi(\delta))\,d\delta, \int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta\right)$$
$$+\aleph\left(\int_0^\iota \eta_n(\iota,\delta,0_{(1,0)})\,d\delta, \int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta\right)$$
$$+\aleph\left(\int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta, \int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta\right)$$
$$+\aleph\left(\int_0^\iota F_n(\eta,0_{(1,0)})\,d\iota, \int_0^\iota F(\eta,0_{(1,0)})\,d\iota\right)$$
$$\leq 2L_\eta\aleph(\varphi(\iota),0_{(1,0)}) + \sup_{0\leq\zeta\leq\xi}\aleph\left(\int_0^\iota \eta_n(\iota,\delta,0_{(1,0)})\,d\delta, \int_0^\iota \eta(\iota,\delta,0_{(1,0)})\,d\delta\right)$$
$$+\sup_{0\leq\zeta\leq\xi}\aleph\left(\int_0^\iota F_n(\eta,0_{(1,0)})\,d\iota, \int_0^\iota F(\eta,0_{(1,0)})\,d\iota\right)$$
$$\leq 2L_\eta C_2\left(\aleph(\varphi_0,0_{(1,0)}) + N_1 + N_3 + N_4 + N_5\right) + 1$$

As soon as n is sufficiently large, utilizing condition 2 of Theorem 3.1. Consequently, by utilizing the dominated convergence theorem in (5), we derive the theorem's conclusion. □

5. Conclusion

This paper examines the controllability of IF integro-differential equations with nonlocal conditions. By using IF semigroups and functional analysis, we identify conditions that allow control of system states to achieve desired results. Our findings show that intuitionistic fuzzy control functions can effectively handle uncertainties, ensuring the system can be controlled. This study advances theoretical knowledge and suggests practical uses in various fields. Future research might focus on advanced control methods and numerical techniques to improve IF control systems in complex situations [41–43]. This work sets the stage for further investigation into controllability in intuitionistic fuzzy systems.

Acknowledgment: This research is funded partially by Zarqa University-Jordan.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R. Saadeh, O. Ala'yed, A. Qazza, Analytical Solution of Coupled Hirota–Satsuma and KdV Equations, Fractal and Fractional 6 (2022), 694. https://doi.org/10.3390/fractalfract6120694.

[2] A.B.M. Alzahrani, R. Saadeh, M.A. Abdoon, M. Elbadri, M. Berir, A. Qazza, Effective Methods for Numerical Analysis of the Simplest Chaotic Circuit Model with Atangana–Baleanu Caputo Fractional Derivative, J. Eng. Math. 144 (2024), 9. https://doi.org/10.1007/s10665-023-10319-x.

[3] A. Hazaymeh, R. Saadeh, R. Hatamleh, M.W. Alomari, A. Qazza, A Perturbed Milne's Quadrature Rule for n-Times Differentiable Functions with L^p-Error Estimates, Axioms 12 (2023), 803. https://doi.org/10.3390/axioms12090803.

[4] S. Melliani, M. Elomari, L.S. Chadli, R. Ettoussi, Intuitionistic Fuzzy Metric Space, Notes IFS 21 (2015), 43–53.

[5] S.A. Altaie, N. Anakira, A. Jameel, O. Ababneh, A. Qazza, A.K. Alomari, Homotopy Analysis Method Analytical Scheme for Developing a Solution to Partial Differential Equations in Fuzzy Environment, Fractal Fract. 6 (2022), 419. https://doi.org/10.3390/fractalfract6080419.

[6] O. Kaleva, Fuzzy Differential Equations, Fuzzy Sets Syst. 24 (1987), 301–317. https://doi.org/10.1016/0165-0114(87)90029-7.

[7] J.U. Jeong, Fuzzy Differential Equations With Nonlocal Condition, J. Appl. Math. Comput. 17 (2005), 509–517.

[8] Y. Feng, Fuzzy Stochastic Differential Systems, Fuzzy Sets Syst. 115 (2000), 351–363. https://doi.org/10.1016/S0165-0114(98)00389-3.

[9] E. Szmidt, J. Kacprzyk, Distances between Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 114 (2000), 505–518. https://doi.org/10.1016/S0165-0114(98)00244-9.

[10] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2010. https://doi.org/10.1007/978-0-387-70914-7.

[11] L.S. Chadli, S. Melliani, On Distance Between Intuitionistic Fuzzy Sets, Notes IFS 8 (2002), 34–36.

[12] P. Grzegorzewski, Distances between Intuitionistic Fuzzy Sets and/or Interval-Valued Fuzzy Sets Based on the Hausdorff Metric, Fuzzy Sets Syst. 148 (2004), 319–328. https://doi.org/10.1016/j.fss.2003.08.005.

[13] M.L. Puri, D.A. Ralescu, Fuzzy Random Variables, J. Math. Anal. Appl. 114 (1986), 409–422. https://doi.org/10.1016/0022-247X(86)90093-4.

[14] R. Saadati, J.H. Park, On the Intuitionistic Fuzzy Topological Spaces, Chaos Solitons Fractals 27 (2006), 331–344. https://doi.org/10.1016/j.chaos.2005.03.019.

[15] T. Gunasekar, F. P. Samuel, and M. M. Arjunan, Existence Results for Impulsive Neutral Functional Integrodifferential Equation With Infinite Delay, J. Nonlinear Sci. Appl. 6 (2013), 234–243.

[16] T. Gunasekar, F.P. Samuel, M.M. Arjunan, Existence of Solutions for Impulsive Partial Neutral Functional Evolution Integrodifferential Inclusions With Infinite Delay, Int. J. Pure Appl. Math. 85 (2013), 939–954. https://doi.org/10.12732/ijpam.v85i5.12.

[17] S. Manimaran, Existence of Solutions for Neutral Functional Integrodifferential Evolution Equations with Non Local Conditions, Indian J. Sci. Technol. 8 (2015), 358–363. https://doi.org/10.17485/ijst/2015/v8i4/60133.

[18] T. Gunasekar, F.P. Samuel, M.M. Arjunan, Controllability of Impulsive Partial Neutral Functional Integro-Differential Systems With Infinite Delay, Int. J. Appl. Eng. Res. 8 (2013), 1103–1115.

[19] M.L. Suresh, T. Gunasekar, F.P. Samuel, Existence Results for Nonlocal Impulsive Neutral Functional Integro-Differential Equations, Int. J. Pure Appl. Math. 116 (2017), 337–345.

[20] F. Acharya, V. Kushawaha, J. Panchal, D. Chalishajar, Controllability of Fuzzy Solutions for Neutral Impulsive Functional Differential Equations with Nonlocal Conditions, Axioms 10 (2021), 84. https://doi.org/10.3390/axioms10020084.

[21] N. S, N. M, Existence and Controllability Result for the Nonlinear First Order Fuzzy Neutral Integrodifferential Equations with Nonlocal Conditions, Int. J. Fuzzy Logic Syst. 3 (2013), 39–53. https://doi.org/10.5121/ijfls.2013.3304.

[22] M. Nagarajan, K. Karthik, Controllability Results for Nonlinear Impulsive Functional Neutral Integrodifferential Equations in n-Dimensional Fuzzy Vector Space, Appl. Appl. Math.: Int. J. 17 (2022), 17–30.

[23] T. Gunasekar, M. Angayarkanni, K.R. Salini, Existence and Controllability Results for Impulsive Neutral Mixed-Type Functional Integrodifferential Systems With Infinite Delay, J. Adv. Res. Dyn. Control Syst. 10 (2018), 449–458.

[24] T. Gunasekar, K.A. Venkatesan, R. Shanmugapriya, K. Nithyanandhan, Results for Damped Second-Order Neutral Integral Equation With Impulses and Infinite Delay, Adv. Math.: Sci. J. 8 (2019), 683–691.

[25] T. Gunasekar, M. Angayarkanni, S. Yasotha, Controllability Results for Impulsive Neutral Stochastic Functional Integrodifferential Inclusions With Infinite Delay, Int. J. Pure Appl. Math. 116 (2017), 311–326.

[26] T. Gunasekar, M.M. Arjunan, F.P. Samuel, Controllability of Impulsive Partial Neutral Functional Integrodifferential Systems With Infinite Delay, Int. J. Appl. Eng. Res. 8 (2013), 1103–1115.

[27] A.U.K. Niazi, N. Iqbal, R. Shah, F. Wannalookkhee, K. Nonlaopon, Controllability for Fuzzy Fractional Evolution Equations in Credibility Space, Fractal Fract. 5 (2021), 112. https://doi.org/10.3390/fractalfract5030112.

[28] K. Abuasbeh, R. Shafqat, A. Alsinai, M. Awadalla, Analysis of Controllability of Fractional Functional Random Integroevolution Equations with Delay, Symmetry 15 (2023), 290. https://doi.org/10.3390/sym15020290.

[29] A. Georgieva, Double Fuzzy Sumudu Transform to Solve Partial Volterra Fuzzy Integro-Differential Equations, Mathematics 8 (2020), 692. https://doi.org/10.3390/math8050692.

[30] P.B. Dhandapani, J. Thippan, C. Martin-Barreiro, V. Leiva, C. Chesneau, Numerical Solutions of a Differential System Considering a Pure Hybrid Fuzzy Neutral Delay Theory, Electronics 11 (2022), 1478. https://doi.org/10.3390/electronics11091478.

[31] T. Gunasekar, J. Thiravidarani, M. Mahdal, P. Raghavendran, A. Venkatesan, M. Elangovan, Study of Non-Linear Impulsive Neutral Fuzzy Delay Differential Equations with Non-Local Conditions, Mathematics 11 (2023), 3734. https://doi.org/10.3390/math11173734.

[32] F.I. Chou, T.H. Huang, P.Y. Yang, et al. Controllability of Fractional-Order Particle Swarm Optimizer and Its Application in the Classification of Heart Disease, Appl. Sci. 11 (2021), 11517. https://doi.org/10.3390/app112311517.

[33] H.M. Ahmed, M.M. El-Borai, H.M. El-Owaidy, A.S. Ghanem, Existence Solution and Controllability of Sobolev Type Delay Nonlinear Fractional Integro-Differential System, Mathematics 7 (2019), 79. https://doi.org/10.3390/math7010079.

[34] D.N. Chalishajar, A. Kumar, Total Controllability of the Second Order Semi-Linear Differential Equation with Infinite Delay and Non-Instantaneous Impulses, Math. Comput. Appl. 23 (2018), 32. https://doi.org/10.3390/mca23030032.

[35] M.D. Johansyah, A.K. Supriatna, E. Rusyaman, J. Saputra, The Existence and Uniqueness of Riccati Fractional Differential Equation Solution and Its Approximation Applied to an Economic Growth Model, Mathematics 10 (2022), 3029. https://doi.org/10.3390/math10173029.

[36] B. Telli, M.S. Souid, J. Alzabut, H. Khan, Existence and Uniqueness Theorems for a Variable-Order Fractional Differential Equation with Delay, Axioms 12 (2023), 339. https://doi.org/10.3390/axioms12040339.

[37] O. Tunç, C. Tunç, J.-C. Yao, Global Existence and Uniqueness of Solutions of Integral Equations with Multiple Variable Delays and Integro Differential Equations: Progressive Contractions, Mathematics 12 (2024), 171. https://doi.org/10.3390/math12020171.

[38] B. Ahmad, Y. Alruwaily, A. Alsaedi, S. K. Ntouyas, Existence and Stability Results for a Fractional Order Differential Equation with Non-Conjugate Riemann-Stieltjes Integro-Multipoint Boundary Conditions, Mathematics 7 (2019), 249. https://doi.org/10.3390/math7030249.

[39] A. Mohammed Djaouti, Z.A. Khan, M.I. Liaqat, A. Al-Quran, Existence, Uniqueness, and Averaging Principle of Fractional Neutral Stochastic Differential Equations in the Lp Space with the Framework of the Ψ-Caputo Derivative, Mathematics 12 (2024), 1037. https://doi.org/10.3390/math12071037.

[40] O. Algahtani, Ulam Stability of Fractional Hybrid Sequential Integro-Differential Equations with Existence and Uniqueness Theory, Symmetry 14 (2022), 2438. https://doi.org/10.3390/sym14112438.

[41] A. Hazaymeh, A. Qazza, R. Hatamleh, M.W. Alomari, R. Saadeh, On Further Refinements of Numerical Radius Inequalities, Axioms 12 (2023), 807. https://doi.org/10.3390/axioms12090807.

[42] M. Abu-Ghuwaleh, R. Saadeh, A. Qazza, A Novel Approach in Solving Improper Integrals, Axioms 11 (2022), 572. https://doi.org/10.3390/axioms11100572.

[43] T. Qawasmeh, A. Qazza, R. Hatamleh, M.W. Alomari, R. Saadeh, Further Accurate Numerical Radius Inequalities, Axioms 12 (2023), 801. https://doi.org/10.3390/axioms12080801.