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An Efficient Numerical Technique for Solving the Korteweg-de Vries-Burgers Equation

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ABSTRACT. Nonlinear partial differential equations, such as the Korteweg-de Vries-Burgers equation (KDVB), receive extensive study in a multitude of fields of engineering and physics. This study presents the Variational Homotopy Perturbation Method (VHPM) as a robust numerical technique for approximating solutions to the KDVB equation. The technique integrates the Variational Iteration Method (VIM) with the Homotopy Perturbation Method (HPM), providing an efficient solution without requiring the discretization or linearization of the equation. The efficacy of the proposed scheme is demonstrated through various problems, with the accuracy of the method being assessed using absolute errors in the L_2 and L_∞ error norms. The results indicate that the proposed method is straightforward to implement and provides superior outcomes compared to the existing schemes documented in the literature. This study offers a substantial contribution to the advancement of numerical techniques for solving nonlinear partial differential equations, providing beneficial applications across diverse scientific and engineering fields.

1. Introduction

In many fields, such as physics, engineering, biology, and chemistry, numerous phenomena are modeled by complex equations known as nonlinear partial differential equations. One example of such an equation is known as the Korteweg-de Vries Burgers equation (KDVB) [1], which takes the form:

$$w_t + \alpha w w_x - \beta w_{xx} + \gamma w_{xxx} = 0, \tag{1.1}$$

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where α, β, and γ are real constants. Equation 1.1 explains the behavior of specific types of nonlinear waves. It merges the Korteweg-de Vries (KDV) equation, which explains the propagation of long waves in geophysical systems such as dense oceans, plasma, shallow water, and crystal lattice [2, 3], with the Burgers equation, which describes the behavior of gas-fluid dynamics, heat conduction, traffic flow, and turbulence [4-12]. The parameters β and γ in Equation 1.1 correspond to the damping and dispersion parameters, respectively.

If the parameter β equals zero, then Equation 1.1 becomes:

$$w_t + \alpha w w_x + \gamma w_{xxx} = 0, \tag{1.2}$$

which is known as the Kortweg and De-Vries developed (KDV) equations and KDV equations in 1895, illustrating their crucial role in solitons like waves with slight and limited amplitudes in shallow water.

If the parameter γ equals zero, then Equation 1.1 becomes:

$$w_t + \alpha w w_x - \beta w_{xx} = 0, \tag{1.3}$$

which is known as the Burger equation. The Burgers model of free turbulence is a highly influential fluid dynamics model.

The KDVB equation can serve as a nonlinear wave model in a range of applications, including fluid dynamics within an elastic tube [13], the behavior of liquids with small bubbles [14], and the study of turbulence [15, 16]. The investigation of the traveling wave solution for the KDVB equation has been extensive. Researchers such as Johnson [17], Demiray [18], Antar and Demiray [19] formulated the KDVB equation as the governing evolution equation for viscoelastic tubes or waves propagating in fluid-filled elastic. Their work accounts for the effects of dispersion, dissipation, and nonlinearity.

Numerous investigations in the literature, employing a variety of approaches, have explored deriving solutions for the KDVB Equation 1.1. Over the years, the KDVB equation has been studied numerically by many researchers, including Ali et al. [20], who developed a B-spline finite element scheme for the numerical solution of the KDVB equation. Canosa and Gazdag [21] studied the time evolution and stability of shock solutions of the KDVB equation using numerical methods, employing a scheme named accurate space derivative. They furthermore determined how non-analytic initial data changes into monotonic shocks. Kudryashov [22] analyzed solitary wave solutions of the KDV and KDVB equations, showing that these solutions can be simplified into known forms. Wazzan [23] developed multiple traveling wave solutions for the KDV and KDVB equations using a modified tanh–coth method. Shi et al. [24] proposed a hybrid scheme to solve the KDVB equation, combining the classical constrained interpolation profile (CIP) method and compact methods with third-order Runge-Kutta time discretization, demonstrating its high resolution with a test case. Aydin [25] introduced a novel splitting technique for the approximate solution of the KDVB equation. Kashchenko [26] studied the local dynamics of the KDVB equation with periodic boundary conditions and derived a special nonlinear partial differential equation that serves as a normal form, governing the behavior of solutions in a small neighborhood of equilibrium. Recently, Koroglu [27] proposed new nonstandard finite difference schemes for solving the KDVB equation, providing accurate and efficient numerical simulations compared to standard finite difference schemes. Ahmad et al. [28] presented an improved version of the variational iteration algorithm-II for solving the KDVB equation. Aliyi and Muleta [29] presented a sixth-order compact finite difference method for solving the one-dimensional KDVB equation. Parumasur et al. [30] used the orthogonal collocation on finite elements (OCFE) method with quadratic and cubic B-splines at Gaussian points to solve Burgers', modified Burgers' and KdV–Burgers' equations.

In this paper, the KDV-Burgers equation is solved using the Variational Homotopy Perturbation Method (VHPM). The paper is organized as follows: Section 2 provides an analysis of the VHPM. In Section 3, several problems are conducted to assess the accuracy and efficiency of the VHP method, which are then compared with exact solutions and results from other studies in the literature. Section 4 presents our conclusion.

2. Analysis of the method

To illustrate the fundamental concept of the VHPM, we start by considering the following general differential equation:

$$Lw + Nw = f(x), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $f(x)$ represents the forcing term. Moreover, according to the Variational Iteration Method (VIM), a correct function can be formulated as follows:

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\tau)\big(Lw_n(\tau) + N\tilde{w}_n(\tau) - f(\tau)\big)d\tau, \tag{2.2}$$

where λ denotes a Lagrange multiplier [31, 32, 33], which is optimally determined using the Variational Iteration Method (VIM). The subscripts n represent the nth approximation, while $\tilde{w}_n$

is observed as a restricted variation. That is, $\delta\tilde{w}_n = 0$; Equation 2.2 is called a correct function. Subsequently, we proceed to apply the Homotopy Perturbation Method (HPM) [34].

$$\sum_{n=0}^{\infty} \rho^{(n)} w_n = w_0(x) + \rho \int_0^x \lambda(\tau) \left(\sum_{n=0}^{\infty} \rho^{(n)} L\big(w_n(t)\big) + N\left(\sum_{n=0}^{\infty} \rho^{(n)} \tilde{w}_n(\tau) \right) \right) d\tau - \int_0^x \lambda(\tau) f(\tau) d\tau. \tag{2.3}$$

Equation 2.3 defines the VHPM, which is formulated by coupling the variational iteration method (VIM) with the homotopy perturbation method (HPM). Solutions of different orders can be obtained by comparing similar powers of ρ.

3. Numerical experiments and discussion

In this section, we demonstrate the effectiveness of the proposed method by presenting experimental results. We compare the numerical outcomes with existing results from the literature across various parameters and solution domains. The performance of the proposed method is assessed through the computation of error norms L_∞, L_2, and absolute error using the following formulas:

$$L_2 = \sqrt{h \sum_{j=1}^{N} \left| w_j - W_j \right|^2}, \qquad L_\infty = \max_{1 \le j \le N} \left| w_j - W_j \right|, \qquad \text{absolute error} = \left| w_j - W_j \right|,$$

where w_j are the exact solutions and W_j are the approximate solutions at the j-th spatial knot.

Problem 1. Consider the following KDVB equation:

$$w_t + \alpha w w_x - \beta w_{xx} + \gamma w_{xxx} = 0.$$

The exact solution is given by:

$$w(a,t) = A \frac{\beta^2}{\gamma} \left(1 + tanh(\theta) + \frac{1}{2} sech^2 (\theta) \right),$$

where $\theta = \left(\frac{\beta}{10\gamma} \right) \left(x - A \frac{\beta^2}{\gamma} t \right)$, $A = -\frac{6}{25}$.

This problem is solved using the VHPM with $n = 4$, $\alpha = 1$, and different values of β and γ. The results are presented in Table 1 and are compared with the findings reported in [35, 36] in terms of absolute error, which are observed to be superior. Figures 1 and 2 illustrate the mesh between the numerical and exact solutions for $\beta = \gamma = 0.1$, while Figures 3 and 4 illustrate the behavior solution of the KDVB equation at time ($t = 10$) with different values of β and γ, demonstrating an excellent agreement with the exact solution.

Table 1: Absolute errors of problem 1 with different values of $x,\ t,\ \beta$, and γ.

t	β	γ	x	VHPM	MVIA-II [35]	VIM [36]
100	0.001	0.001	0	5.7712e-07	9.426e-08	9.426-08
			25	2.0295e-10	2.656e-10	2.659e-10
			50	9.3905e-15	1.262e-14	3.000e-13
			75	4.3368e-19	6.505e-19	0.000
			100	5.4210e-20	1.084e-19	2.000e-13
800	0.001	0.001	0	4.6675e-06	6.599e-07	6.603e-07
			25	4.6675e-06	2.071e-09	2.071e-09
			50	6.9294e-14	9.876e-14	2.000e-13
			75	3.0900e-18	4.554e-18	1.000e-13
			100	5.4210e-20	1.084e-19	0.000
100	0.01	0.01	0	5.8473e-05	7.925e-06	7.936e-06
			25	1.8436e-08	2.570e-08	2.570e-08
			50	8.4579e-13	1.227e-12	1.000e-12
			75	3.9031e-17	5.725e-17	1.000e-12
			100	0.000	8.675e-19	2.000e-12
100	0.1	0.1	0	3.3432e-03	3.388e-05	1.268e-03
			25	9.1325e-08	1.854e-06	1.812e-06
			50	1.0225e-11	9.821e-11	8.000e-11
			75	4.7184e-16	4.476e-15	0.000
			100	6.9389e-18	6.939e-15	4.000e-11

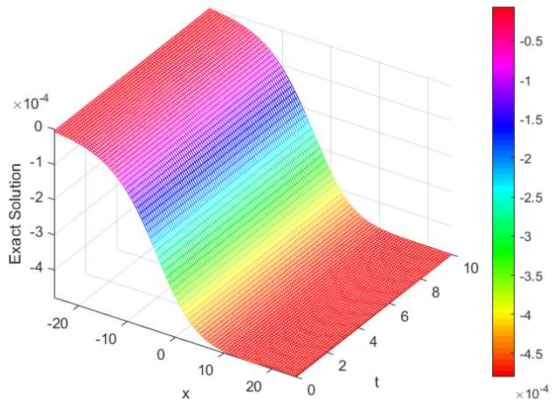

Figure 1: Surface plot of the exact solution to problem 1.

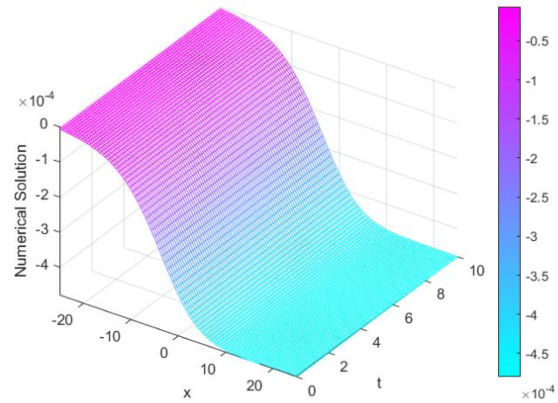

Figure 2: Surface plot of the VHPM solution to problem 1.

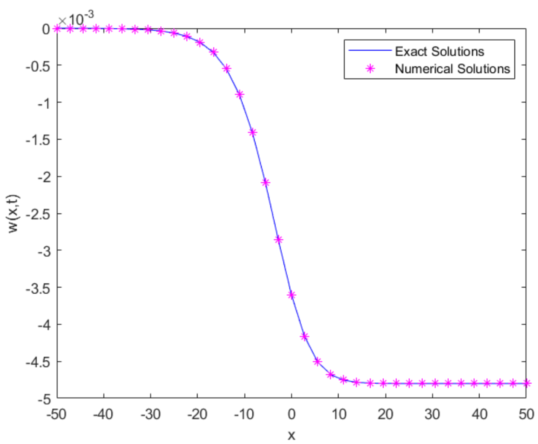

Figure 3: The behavior of numerical and exact solutions of the KDVB equation for problem 1 with $\beta = \gamma = 0.01$ and $t = 10$.

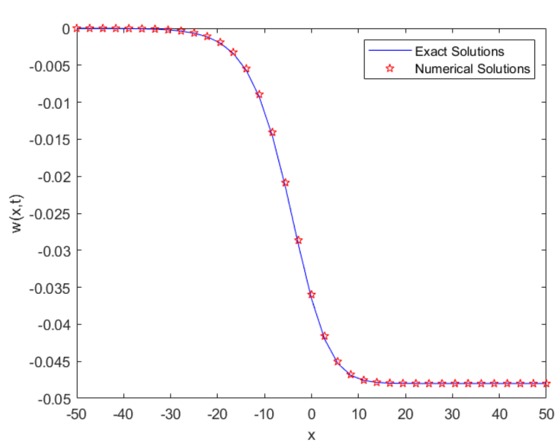

Figure 4: The behavior of numerical and exact solutions of the KDVB equation for problem 1 with $\beta = \gamma = 0.1$ and $t = 10$.

Problem 2. Consider the following modified KDVB equation:

$$w_t + 2(w^3)_x + \beta w_{xx} - \gamma w_{xxx} = 0.$$

The exact solution is given by:

$$w(x,t) = A\,[1 + tanh(A\,(x - Bt))],$$

where $A = \frac{1}{6}$ and $B = \frac{2}{9}$.

In the domain [-10,10], the solutions to problem 2 are obtained using the VHPM with $n = 5$ and $\beta = \gamma = 1$. Table 2 presents the absolute errors at different levels of time t, in comparison with the findings reported in [37]. Our results show a good agreement in terms of absolute errors with the findings reported in [37]. Moreover, Table 3 presents the absolute errors at various values of t and x, demonstrating a good agreement with the exact solution. Figures 5 and 6 illustrate the mesh between the exact and numerical solutions, while Figures 7 and 8 illustrate the behavior of the solution of the KDVB equation at $t = 1$ and $t = 2$, respectively, showing an excellent agreement with the exact solution.

Table 2: Absolute errors of problem 2 with different values of x and t.

t	VHPM	[37]	VHPM	[37]	VHPM	[37]
	$x = 1.04529$		$x = 3.09017$		$x = 5.00000$	
0.001	1.1478e-10	3.58e-07	2.5203e-10	3.58e-07	2.5012e-10	3.58e-07
0.002	4.5916e-10	9.98e-07	1.0081e-09	1.01e-06	1.0005e-09	1.07e-06
0.006	4.1335e-10	2.76e-06	9.0734e-09	2.83e-06	9.0041e-09	2.80e-06
0.02	4.5970e-08	1.97e-06	1.0083e-07	1.73e-06	1.0004e-07	2.89e-06
0.03	1.0350e-07	1.65e-06	2.2688e-07	4.19e-06	2.2506e-07	1.94e-05
0.04	1.8412e-07	5.16e-06	4.0338e-07	1.37e-05	4.0008e-07	4.97e-05
0.05	2.8787e-07	7.05e-06	6.3033e-07	2.74e-05	6.2509e-07	9.97e-05

Table 3: Absolute errors of problem 2 at $-10 \leq x \leq 10$ and $t = 0.5, 1, 1.5$, and 2.

x	t			
	0.5	**1**	**1.5**	**2**
-10	2.1467e-05	8.6793e-05	1.9740e-05	3.5476e-05
-6	5.5171e-05	2.2197e-04	5.0230e-04	8.9800e-04
-2	4.8763e-05	1.9228e-04	4.2616e-04	7.4573e-04
0	1.0584e-06	8.4675e-06	2.8578e-05	6.7740e-05
2	5.0072e-05	2.0276e-04	4.6152e-04	8.2953e-04
6	5.4513e-05	2.1671e-04	4.8454e-04	8.5589e-04
10	2.1016e-05	8.3187e-05	1.8523e-04	3.2591e-04

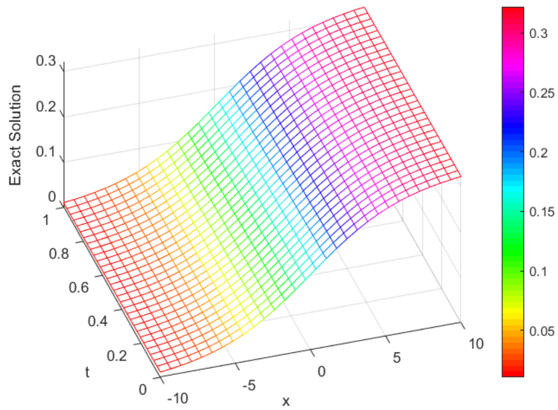

Figure 5: Surface plot of the exact solution to problem 2.

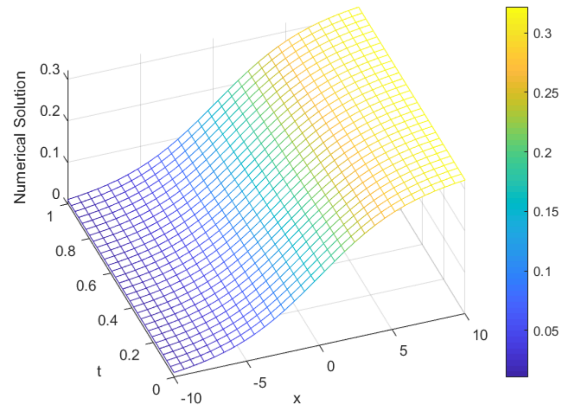

Figure 6: Surface plot of the VHPM solution to problem 2.

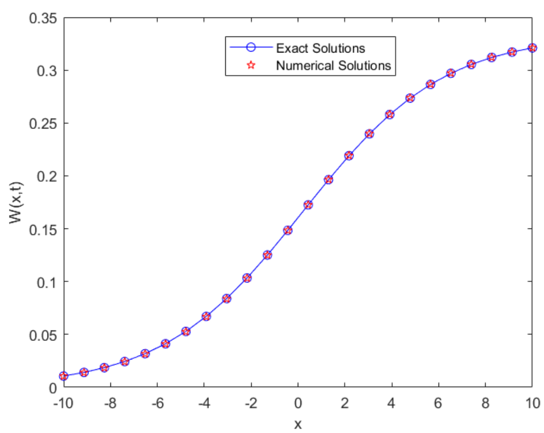

Figure 7: The behavior of numerical and exact solutions of the KDVB equation for problem 2 at $t = 1$.

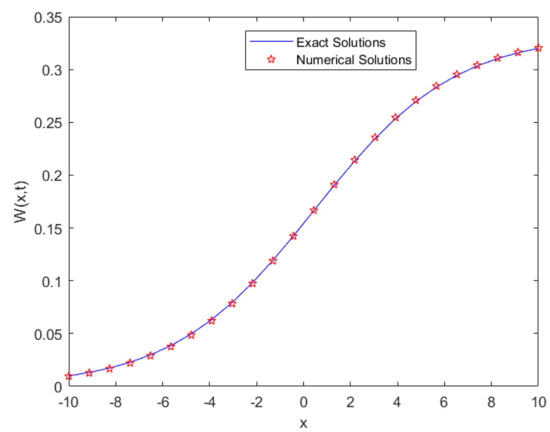

Figure 8: The behavior of numerical and exact solutions of the KDVB equation for problem 2 at $t = 2$.

Problem 3. Consider the following KDVB equation:

$$w_t + (w^2)_x + (w^2)_{xx} - \gamma w_{xxx} = 0.$$

The exact solution is given by:

$$w(x,t) = -\frac{1}{20}\left[1 + tanh\left(\frac{t}{20} + \frac{x}{2}\right)\right].$$

By applying the proposed method with $n = 3$ in the computational domain [-25,25] and setting $\gamma = 0.1$, the problem was successfully solved. Table 4 presents the L_∞ and L_2 norms, indicating results that are superior when compared to those reported in [27]. In Table 5, absolute errors are presented at various values of x and t, illustrating a good match with the exact solution. Moreover, Figures 9 and 10 illustrate the mesh between the exact and numerical solutions. Simultaneously, Figures 11 and 12 illustrate the behavior of the solution for the KDVB equation at $t = 1$ and $t = 10$, respectively, demonstrating an excellent agreement with the exact solution.

Table 4: Error norms of problem 3.

t	VHPM		[27]	
	L_2	L_∞	L_2	L_∞
1	2.9344e-04	6.5616e-05	1.0111e-03	7.0857e-04
10	3.0483e-03	6.8610-04	1.0674e-02	7.4435e-03
25	2.3343e-02	5.6420e-03	2.3922e-02	1.6321e-02

Table 5: Absolute errors of problem 3 at $-25 \leq x \leq 25$ and $t = 0.5, 1, 1.5,$ and 2.

x	t			
	0.5	1	1.5	2
-25	3.4625e-14	1.4592e-13	2.2444e-13	3.0694e-13
-15	1.5676e-09	3.2142e-09	4.9437e-09	6.7606e-09
-5	3.3993e-05	6.9550e-05	1.0677e-04	1.4573e-04
5	3.2292e-05	6.2765e-05	9.1535e-05	1.1871e-04
15	1.4927e-09	2.9141e-09	4.2679e-09	5.5574e-09
25	6.7807e-14	1.3230e-13	1.9376e-13	2.5228e-13

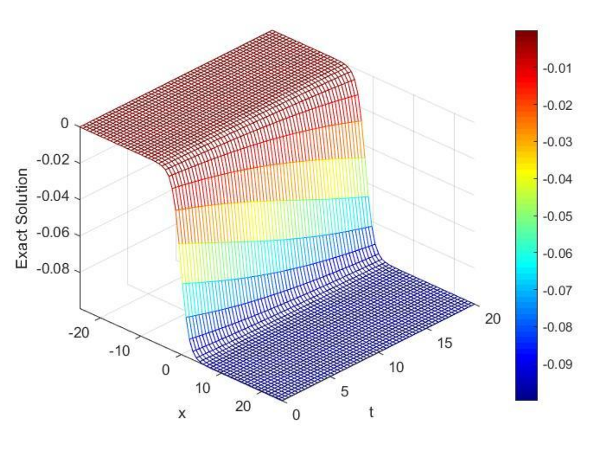

Figure 9: Surface plot of the exact solution to problem 3.

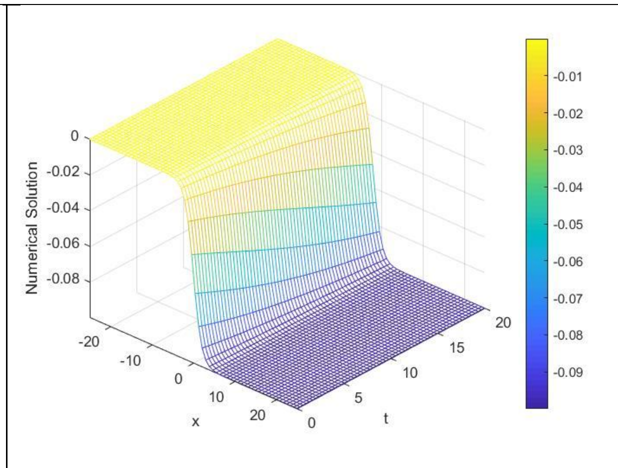

Figure 10: Surface plot of the VHPM solution to problem 3.

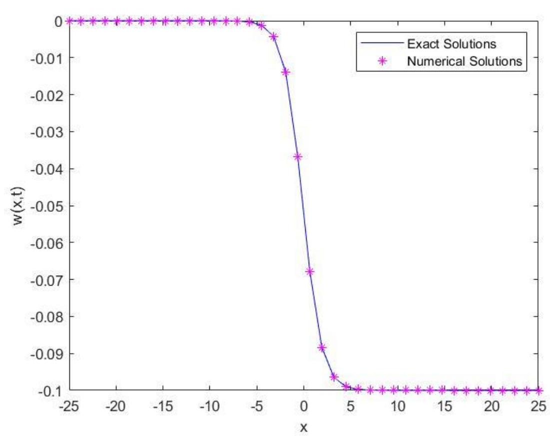

Figure 11: The behavior of numerical and exact solutions of the KDVB equation for problem 3 at $t = 1$.

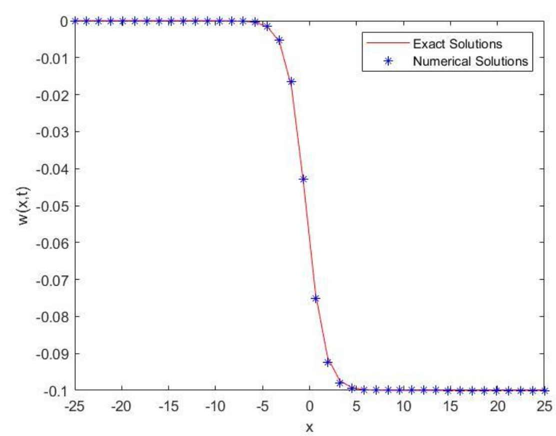

Figure 12: The behavior of numerical and exact solutions of the KDVB equation for problem 3 at $t = 10$.

4. Conclusion

This paper presented a numerical scheme for solving the Korteweg-de Vries-Burgers (KDVB) equation by combining the Variational Iteration Method (VIM) with the Homotopy Perturbation Method (HPM). Through rigorous testing, we have demonstrated the method's accuracy and effectiveness in approximating solutions without the need of discretization or linearization of the equation. The obtained results not only demonstrate excellent agreement with the exact solutions but also outperform existing approaches in the literature, thereby validating their reliability. Our approach offers significant opportunities for solving a broad spectrum of linear and nonlinear time-dependent partial differential equations, thereby rendering this method a valuable tool for researchers in various fields.

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References

[1] C.H. Su, C.S. Gardner, Korteweg-de Vries Equation and Generalizations. III. Derivation of the Korteweg-de Vries Equation and Burgers Equation, J. Math. Phys. 10 (1969), 536–539. https://doi.org/10.1063/1.1664873.

[2] N. Cheemaa, A.R. Seadawy, T.G. Sugati, D. Baleanu, Study of the Dynamical Nonlinear Modified Korteweg–de Vries Equation Arising in Plasma Physics and Its Analytical Wave Solutions, Results Phys. 19 (2020), 103480. https://doi.org/10.1016/j.rinp.2020.103480.

[3] A. Ankiewicz, M. Bokaeeyan, N. Akhmediev, Shallow-Water Rogue Waves: An Approach Based on Complex Solutions of the Korteweg–de Vries Equation, Phys. Rev. E 99 (2019), 050201. https://doi.org/10.1103/PhysRevE.99.050201.

[4] H. Bateman, Some Recent Researches on The Motion of Fluids, Mon. Weather Rev. 43 (1915), 163–170. https://doi.org/10.1175/1520-0493(1915)43<163:SRROTM>2.0.CO;2.

[5] J.D. Cole, On a Quasi Linear Parabolic Equations Occurring in Aerodynamics, Q. Appl. Math. 9 (1951), 225-236.

[6] C. Yonti Madie, F. Kamga Togue, P. Woafo, Numerical Solution of the Burgers Equation Associated with the Phenomena of Longitudinal Dispersion Depending on Time, Heliyon 8 (2022), e09776. https://doi.org/10.1016/j.heliyon.2022.e09776.

[7] T. Musha, H. Higuchi, Traffic Current Fluctuation and the Burgers Equation, Jpn. J. Appl. Phys. 17 (1978), 811–816. https://doi.org/10.1143/JJAP.17.811.

[8] T. Nagatani, Density Waves in Traffic Flow, Phys. Rev. E 61 (2000), 3564–3570. https://doi.org/10.1103/PhysRevE.61.3564.

[9] N. Su, J.P.C. Watt, K.W. Vincent, M.E. Close, R. Mao, Analysis of Turbulent Flow Patterns of Soil Water under Field Conditions Using Burgers Equation and Porous Suction-Cup Samplers, Soil Res. 42 (2004), 9-16. https://doi.org/10.1071/SR02142.

[10] E. Ben-Naim, S.Y. Chen, G.D. Doolen, S. Redner, Shocklike Dynamics of Inelastic Gases, Physi. Rev. Lett. 83 (1999), 4069–4072. https://doi.org/10.1103/PhysRevLett.83.4069.

[11] S.F. Shandarin, Ya.B. Zeldovich, The Large-Scale Structure of the Universe: Turbulence, Intermittency, Structures in a Self-Gravitating Medium, Rev. Mod. Phys. 61 (1989), 185–220. https://doi.org/10.1103/RevModPhys.61.185.

[12] J. Bec, K. Khanin, Burgers Turbulence, Phys. Rep. 447 (2007), 1–66. https://doi.org/10.1016/j.physrep.2007.04.002.

[13] R.S. Johnson, Nonlinear Waves in Fluid-Filled Elastic Tubes and Related Problems, PhD Thesis, University of London, London, (1969).

[14] L.V. Wijngaarden, One-Dimensional Flow of Liquids Containing Small Gas Bubbles, Annu. Rev. Fluid Mech. 4 (1972), 369–396. https://doi.org/10.1146/annurev.fl.04.010172.002101.

[15] G. Gao, A Theory of Interaction Between Dissipation and Dispersion of Turbulence, Sci. Sin. Ser. A. 28 (1985), 616-627.

[16] S.D. Liu, S.K. Liu, KdV-Burgers Equation Modelling of Turbulence, Sci. Sin. Ser. A. 35 (1992), 576-586.

[17] R.S. Johnson, A Non-Linear Equation Incorporating Damping and Dispersion, J. Fluid Mech. 42 (1970), 49–60. https://doi.org/10.1017/S0022112070001064.

[18] H. Demiray, Nonlinear Waves in a Thick-Walled Viscoelastic Tube Filled with an Inviscid Fluid, Int. J. Eng. Sci. 36 (1998), 345–357. https://doi.org/10.1016/S0020-7225(97)00056-6.

[19] H. Demiray, N. Antar, Nonlinear Waves in an Inviscid Fluid Contained in a Prestressed Viscoelastic Thin Tube, Z. Angew. Math. Phys. 48 (1997), 325-340. https://doi.org/10.1007/s000330050034.

[20] A.H.A. Ali, L.R.T. Gardner, G. Gardner, Numerical Studies of the Korteweg–de Vries–Burgers Equation Using B-Spline Finite Elements. J. Math. Phys. Sci. 27 (1993), 37–53.

[21] J. Canosa, J. Gazdag, The Korteweg-de Vries-Burgers Equation, J. Comput. Phys. 23 (1977), 393–403. https://doi.org/10.1016/0021-9991(77)90070-5.

[22] N.A. Kudryashov, On "New Travelling Wave Solutions" of the KdV and the KdV–Burgers Equations, Commun. Nonlinear Sci. Numer. Simul. 14 (2009), 1891–1900. https://doi.org/10.1016/j.cnsns.2008.09.020.

[23] L. Wazzan, A Modified Tanh–Coth Method for Solving the KdV and the KdV–Burgers' Equations, Commun. Nonlinear Sci. Numer. Simul. 14 (2009), 443–450. https://doi.org/10.1016/j.cnsns.2007.06.011.

[24] Y. Shi, B. Xu, Y. Guo, Numerical Solution of Korteweg-de Vries-Burgers Equation by the Compact-Type CIP Method, Adv. Diff. Equ. 2015 (2015), 353. https://doi.org/10.1186/s13662-015-0682-5.

[25] A. Aydin, An Unconventional Splitting for Korteweg de Vries-Burgers Equation, Eur. J. Pure Appl. Math. 8 (2015), 50-63.

[26] S.A. Kashchenko, Normal Form for the KdV–Burgers Equation, Dokl. Math. 93 (2016), 331–333. https://doi.org/10.1134/S1064562416030170.

[27] C. Koroglu, Exact and Nonstandard Finite Difference Schemes for the Generalized KdV–Burgers Equation, Adv. Diff. Equ. 2020 (2020), 134. https://doi.org/10.1186/s13662-020-02584-2.

[28] H. Ahmad, A.R. Seadawy, T.A. Khan, Numerical Solution of Korteweg–de Vries-Burgers Equation by the Modified Variational Iteration Algorithm-II Arising in Shallow Water Waves, Phys. Scripta 95 (2020), 045210. https://doi.org/10.1088/1402-4896/ab6070.

[29] K. Koroche, H. Chemeda, Sixth-Order Compact Finite Difference Method for Solving KDV-Burger Equation in the Application of Wave Propagations, Iran. J. Numer. Anal. Optim. 12 (2022), 277-300. https://doi.org/10.22067/ijnao.2021.72366.1058.

[30] N. Parumasur, R.A. Adetona, P. Singh, Efficient Solution of Burgers', Modified Burgers' and KdV–Burgers' Equations Using B-Spline Approximation Functions, Mathematics 11 (2023), 1847. https://doi.org/10.3390/math11081847.

[31] J.-H. He, Some Asymptotic Methods for Strongly Nonlinear Equations, Int. J. Mod. Phys. B 20 (2006), 1141–1199. https://doi.org/10.1142/S0217979206033796.

[32] J.-H. He, Variational Iteration Method – a Kind of Non-Linear Analytical Technique: Some Examples, Int. J. Non-Linear Mech. 34 (1999), 699–708. https://doi.org/10.1016/S0020-7462(98)00048-1.

[33] M. Inokuti, H. Sekine, T. Mura, General Use of the Lagrange Multiplier in Nonlinear Mathematical Physics, in: Variational Methods in the Mechanics of Solids, Elsevier, 1980: pp. 156–162. https://doi.org/10.1016/B978-0-08-024728-1.50027-6.

[34] J.-H. He, Homotopy Perturbation Technique, Comput. Methods Appl. Mech. Eng. 178 (1999), 257–262. https://doi.org/10.1016/S0045-7825(99)00018-3.

[35] H. Ahmad, A.R. Seadawy, T.A. Khan, Numerical Solution of Korteweg–de Vries-Burgers Equation by the Modified Variational Iteration Algorithm-II Arising in Shallow Water Waves, Phys. Scripta 95 (2020), 045210. https://doi.org/10.1088/1402-4896/ab6070.

[36] A.A. Soliman, A Numerical Simulation and Explicit Solutions of KdV-Burgers' and Lax's Seventh-Order KdV Equations, Chaos Solitons Fractals 29 (2006), 294–302. https://doi.org/10.1016/j.chaos.2005.08.054.

[37] A.H. Khater, R.S. Temsah, M.M. Hassan, A Chebyshev Spectral Collocation Method for Solving Burgers'-Type Equations, J. Comput. Appl. Math. 222 (2008), 333–350. https://doi.org/10.1016/j.cam.2007.11.007.