Int. J. Anal. Appl. (2025), 23:57

International Journal of Analysis and Applications

Fractional Order Delay Differential Equation Constrained by Nonlocal and Weighted Delay Integral Equations

A.M.A. El-Sayed[1], W.G. El-Sayed[1], Kheria M. Msaik[2,*], Hanaa R. Ebead[1]

[1]*Faculty of Science, Alexandria University, Alexandria, Egypt*
[2]*Faculty of Science, Zintan University, Al zintan, Libya*

[*]*Corresponding author:* kheria.msaik@uoz.edu.ly

Abstract. This paper presents theoretical proof of the existence of a unique solution to a constrained problem of the Riemann-Liouville fractional differential equation with time delay functions by utilizing the Schauder fixed point theorem. Moreover, we analyzed the continuous dependence of the solution on the initial conditions and other parameters. Further, we investigate the Hyers-Ulam stability of the problem. We introduce some examples and special cases to illustrate our results.

1. Introduction

Fractional analysis is an area of calculus that is concerned with non traditional orders. It is a relatively new field of study that has gained significant attention from researchers because of its wide-ranging applications in different sectors of science and engineering (see [1, 24, 30, 31]). Fractional calculus applications continue to grow, and it is expected that they will perform an increasingly essential role in the development of new technologies in the future. This has resulted in the development of new mathematical tools and techniques that have been used to solve complex problems in physics, engineering, finance, and other fields.

Studying fractional calculus has yielded the maturation of numerous analytical strategies for solving fractional models, including Riemann, Caputo, and Grunwald-Letnikov's approaches [13, 25, 26]. Fractional differential and integral equations have attracted a lot of research interest due to their significance in various domains. Srivastava et al. [29] analyzed a class of nonlinear boundary value problems of an arbitrary fractional-order differential equation with integral

Received: Nov. 3, 2024.

2020 *Mathematics Subject Classification.* 47H10, 47H10, 46T20, 26A33, 34A08.

Key words and phrases. fractional order delay differential equation; Schauder fixed point theorem; fractional integral equation; nonlocal conditions; Hyers-Ulam stability; continuous dependence.

https://doi.org/10.28924/2291-8639-23-2025-57
ISSN: 2291-8639

boundary conditions and infinite-point. Kucche and Sutarin [22] established the existence and stability results for nonlinear fractional delay differential equations, they also proved the Ulam–Hyers stability of the problem. In [10], the authors discussed the solvability of an implicit hybrid delay nonlinear functional integral equation, they proved the existence of integrable solutions by using technique of measure of noncompactnes, moreover, they examined the uniqueness and the continuous dependence of the solution under suitable assumptions. In [4] El-Sayed et al. studied a constrained problem of the quadratic functional integro-differential equation of arbitrary (fractional) orders, the authors proved the existence and the uniquiness of the solution under suitable assumptions, they also investigated the stability of the problem due to Hyers-Ulam stability and the continuous dependence of the unique solution on some parameters. For further studies, (see [1]- [3], [5]- [9], [11,12,14,16,19,20], [26]- [28], [30,31]).

In this study, we are concerned with the constrained problem of the Riemann-Liouville fractional differential equation,

$$ {}^{R}D^{\alpha}x(t) = f(t, x(\varphi_1(t)),\ t \in (0,1] \tag{1.1} $$

with each one of the nonlocal and weighted delay integral constraints.

$$ I^{1-\alpha}x(t)|_{t=0} = x_0 + \int_0^{1-\tau} g(s, x(\varphi_2(s)))ds,\ \tau \in [0,1]. \tag{1.2} $$

or

$$ t^{1-\alpha}x(t)|_{t=0} = \frac{1}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(s, x(\varphi_2(s)))ds),\ \tau \in [0,1]. \tag{1.3} $$

where ${}^{R}D^{\alpha}$ denoted the fractional derivative of Riemann-Liouville of order $\alpha \in (0,1]$.

In Section 1, we introduce theoretical introduction about the importance of fractional calculus models and some related works, we also analyze the transform of (1.1)-(1.2) or (1.1) and (1.3) into the equivalent integral fractional model (1.4). Moving on to the 2nd section, we establish the existence and uniqueness of the solution by utilizing the fixed point of Schauder's theorem and Kolmogorov's compactness criterion. Section 3 presents the continuous dependence part of the study, we prove the continuous dependence of the unique solution on initial conditions and other parameters. In the 4th section, we study the stability of the problem due to the Ulam–Hyers stability. In Section 5, we close the manuscript by some examples and special cases.

Now, we have the following lemma.

Lemma 1.1. *The solution of the constrained problem (1.1)-(1.2) or (1.1) and (1.3) can be expressed by the fractional order delay integral equation*

$$ x(t) \ = \ \frac{t^{\alpha-1}}{\Gamma(\alpha)}[x_0 \ + \int_0^{1-\tau} g(s, x(\varphi_2(s))ds] + I^{\alpha} f(t, x(\varphi_1(t))). \tag{1.4} $$

Proof. Let $x \in L_1(I)$ be a solution of the constrained problem (1.1)-(1.2) or (1.1) and (1.3), then we have

$$ \frac{d}{dt} I^{1-\alpha}\, x(t) \ = \ f(t, x(\varphi_1(t))). $$

By integration, we get

$$I^{1-\alpha}\, x(t) \;-\; I^{1-\alpha}\, x(t)|_{t=0} \;=\; \int_0^t f(s,x(\varphi_1(s)))ds,$$

$$I^{1-\alpha}\, x(t) \;=\; I^{1-\alpha}\, x(t)|_{t=0} \;+\; \int_0^t f(s,x(\varphi_1(s)))ds$$

and from (1.2), we get

$$I^{1-\alpha}\, x(t) \;=\; x_0 \;+\; \int_0^{1-\tau} g(s,x(\varphi_2(s)))ds \;+\; \int_0^t f(s,x(\varphi_1(s)))ds.$$

Operating by I^{α}, then

$$I\, x(t) \;=\; \frac{t^{\alpha}}{\Gamma(1+\alpha)}(x_0 \;+\; \int_0^{1-\tau} g(s,x(\varphi_2(s)))ds) \;+\; I^{\alpha+1}\, f(t,x(\varphi_1(t))).$$

By differentiation, we get

$$x(t) \;=\; \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 \;+\int_0^{1-\tau} g(s,x(\varphi_2(s)))ds) \;+\; I^{\alpha}\, f(t,x(\varphi_1(t))).$$

Conversely, from (1.4), we have

$$I^{1-\alpha}\, x(t) \;=\; x_0 \;+\int_0^{1-\tau} g(s,x(\varphi_2(s)))ds \;+\; I\, f(t,x(\varphi_1(t)))$$

and

$$\frac{d}{dt}I^{1-\alpha}\, x(t) \;= \frac{d}{dt}\, (x_0 \;+\; \int_0^{1-\tau} g(s,x(\varphi_2(s)))ds) \;+\; \frac{d}{dt}\, I\, f(t,x(\varphi_1(t))),$$

then

$$D^{\alpha}\, x(t) \;=\; f(t,x(\varphi_1(t)))$$

and

$$I^{1-\alpha}x(t)|_{t=0} \;=\; x_0 \;+\int_0^{1-\tau} g(s,x(\varphi_2(s)))ds.$$

Also for the problem (1.1) and (1.3), we have

$$\frac{d}{dt}I^{1-\alpha}\, x(t) \;=\; f(t,x(\varphi_1(t)))$$

Integrating , we get

$$I^{1-\alpha}\, x(t) \;-\; C \;=\; \int_0^t f(s,x(\varphi_1(s)))ds,$$

$$I^{1-\alpha}\, x(t) \;=\; C \;+\; I\, f(t,x(\varphi_1(t))).$$

Operating by I^{α} on both sides, we get

$$I\, x(t) \;=\; \frac{Ct^{\alpha}}{\Gamma(\alpha+1)} \;+\; I^{\alpha+1}\, f(t,x(\varphi_1(t))).$$

Differentiate, then

$$x(t) \;=\; \frac{Ct^{\alpha-1}}{\Gamma(\alpha)} \;+\; I^{\alpha}\, f(t,x(\varphi_1(t))), \tag{1.5}$$

$$t^{1-\alpha}\, x(t) \;=\; \frac{C}{\Gamma(\alpha)} \;+\; t^{1-\alpha}\, I^{\alpha}\, f(t, x(\varphi_1(t))),$$

then

$$t^{1-\alpha}\, x(t)|_{t=0} \;=\; \frac{C}{\Gamma(\alpha)},$$

Thus

$$\frac{1}{\Gamma(\alpha)}(x_0 \;+\; \int_0^{1-\tau} \; g(s, x(\varphi_2(s)))ds) \;=\; \frac{C}{\Gamma(\alpha)}.$$

Substituting in (2.3), we obtain

$$x(t) \;=\; \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 \;+ \int_0^{1-\tau} \; g(s, x(\varphi_2(s)))ds) + \; I^{\alpha}\, f(t, x(\varphi_1(t))).$$

Conversely, let $x \in L_1(I)$ be a solution of (1.4), then we have

$$t^{1-\alpha}\, x(t) = \frac{1}{\Gamma(\alpha)}(x_0 \;+ \int_0^{1-\tau} \; g(s, x(\varphi_2(s)))ds) + \; t^{1-\alpha}\, I^{\alpha}\, f(t, x(\varphi_1(t))),$$

$$t^{1-\alpha}\, x(t)|_{t=0} = \frac{1}{\Gamma(\alpha)}(x_0 \;+ \int_0^{1-\tau} \; g(s, x(\varphi_2(s)))ds)$$

and

$$\frac{d}{dt} I^{1-\alpha}\, x(t) = f(t, x(\varphi_1(t))).$$

2. Existence of solution

Here we study the existence of at least one (and exactly one) integrable solution of the fractional integral equation (1.4). under the following assumptions

(i) $f, g{:}I = [0,1] \times R \to \; R$ are measurable in $t \in I \; \forall \; x \in R$ and continuous in $x \in R \; \forall \; t \in I$ and there exist two integrable functions m_1, m_2 and two constants $a, \; b > 0$, such that

$$\begin{aligned} |f(t,x)| &\;\le\; m_1(t) + a|x|, \qquad (2.1)\\ |g(t,x)| &\;\le\; m_2(t) + b|x| \;\forall\; t \in I,\; x \in R, \end{aligned}$$

where

$$||m_1|| = \int_0^1 |m_1|(t)dt, \; and \; ||m_2|| = \int_0^1 |m_2(t)|dt. \; t \in \; I.$$

(ii) $\varphi_i : I \to \; I; \; i = 1, 2$ is absolutely continuous in $t \in I$ and there exist constant $\gamma > 0$ such that $\varphi_i'(t) \ge \; \gamma$.

Now for the existence of at least one integrable solution of the fractional integral equation (1.4), we have the following theorem

Theorem 2.1. *Let the assumptions $(i) - (ii)$ be satisfied. If*

$$\frac{a+b}{\gamma} \;\;<\;\; 1, \qquad (2.2)$$

then the fractional integral equation (1.4) has at least one integrable solution $x \in \; L_1[0,1]$.

Proof. Define the set B_r by

$$B_r = \{x \in L_1 : \|x\| \leq r\} \subset L_1.$$

and the operator F by

$$Fx(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(s, x(\varphi_2(s)))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\varphi_1(s))ds,$$

where

$$r = \frac{|x_0| + \|m_1\| + \|m_2\|}{1 - \frac{a+b}{\gamma}}.$$

Now, let $x \in B_r$, then

$$\begin{aligned} |Fx(t)| &= |\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(s, x(\varphi_2(s)))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\varphi_1(s))ds| \\ &\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0| + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{1-\tau} |g(s, x(\varphi_2(s)))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, x(\varphi_1(s)))|ds. \end{aligned}$$

Then

$$\begin{aligned} &\int_0^1 |Fx(t)|dt \\ \leq\ & \int_0^1 [\frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0| + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{1-\tau} |g(s, x(\varphi_2(s)))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, x(\varphi_1(s)))|ds]dt \\ \leq\ & \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0|dt + \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{1-\tau} |g(s, x(\varphi_2(s)))|dsdt + \int_0^1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s, x(\varphi_1(s)))|dsdt \\ \leq\ & \frac{1}{\Gamma(\alpha+1)}|x_0| + \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{1-\tau} |g(s, x(\varphi_2(s)))|dsdt + \int_0^1 |f(s, x(\varphi_1(s)))| \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dtds \\ \leq\ & \frac{1}{\Gamma(\alpha+1)}|x_0| + \frac{1}{\Gamma(\alpha+1)} \int_0^{1-\tau} |g(s, x(\varphi_2(s)))|ds + \frac{1}{\Gamma(\alpha+1)} \int_0^1 |f(s, x(\varphi_1(s)))|ds \\ \leq\ & \frac{1}{\Gamma(\alpha+1)}|x_0| + \frac{1}{\Gamma(\alpha+1)}[\int_0^{1-\tau} |m_2\ (s)|ds + b\int_0^{1-\tau} |x(\varphi_2(s))|ds] + \frac{1}{\Gamma(\alpha+1)}[\int_0^1 |m_1\ (s)|ds \\ &+a\int_0^1 |x(\varphi_1(s))|ds] \\ \leq\ & \frac{1}{\Gamma(\alpha+1)}|x_0| + \frac{1}{\Gamma(\alpha+1)}[\int_0^1 |m_2\ (s)|ds + \frac{b}{\gamma}\int_0^1 |x(\eta)|d\eta] + \frac{1}{\Gamma(\alpha+1)}[\int_0^1 |m_1\ (s)|ds + \frac{a}{\gamma}\int_0^1 |x(\theta)|d\theta] \\ \leq\ & \frac{1}{\Gamma(\alpha+1)}|x_0| + \frac{1}{\Gamma(\alpha+1)}(\|m_2\| + \frac{b}{\gamma}\|x\|_1) + \frac{1}{\Gamma(\alpha+1)}(\|m_1\| + \frac{a}{\gamma}\|x\|_1) \\ \leq\ & |x_0| + \|m_1\| + \|m_2\| + \frac{r(a+b)}{\gamma} = r. \end{aligned}$$

then

$$\|Fx\|_1 \leq r.$$

Then $F : B_r \ \rightarrow \ B_r$, moreover the operator F is uniformly bounded. Now, let $x \in B_r$, then

$$\begin{aligned}
\|(Fx)_h - Fx\|_1 &= \int_0^1 |(Fx)_h \ (t) - (Fx)(t)| dt \\
&= \int_0^1 |\frac{1}{h} \int_t^{t+h} (Fx)(s) ds - (Fx)(t)| dt \\
&= \int_0^1 \frac{1}{h} \int_t^{t+h} |(Fx)(s) - (Fx)(t)| ds dt
\end{aligned}$$

Now, $Fx \in \ L_1[0,1]$ implies

$$\|(Fx)_h - Fx\|_1 \rightarrow \ 0.$$

This means that $Fx(t)_h \rightarrow (Fx)$ uniformly in $L_1(I)$. Thus $\{Fx\}$ is relatively compact [21]. Hence F is compact operator.

Now, let $\{x_n\} \subset B_r$, and $x_n \rightarrow x$, then

$$Fx_n(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(s, x_n(\varphi_2(s)))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_n(\varphi_1(s))ds$$

and

$$\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= \lim_{n\to\infty} \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(s, x_n(\varphi_2(s)))ds) \\
&+ \lim_{n\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_n(\varphi_1(s)))ds.
\end{aligned}$$

Applying Lebesgue dominated convergence Theorem [21], then from our assumptions, we get

$$\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(s, \lim_{n\to\infty} x_n(\varphi_2(s)))ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \lim_{n\to\infty} x_n(\varphi_1(s)))ds \\
&= \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(s, x(\varphi_2(s)))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\varphi_1(s))ds = Fx(t).
\end{aligned}$$

Then $Fx_n(t) \rightarrow Fx(t)$. Which means that the operator F is continuous.

Since all conditions of Schauder fixed point Theorem [21] are satisfied, then the operator F has at least one fixed point $x \in L_1[0,1]$. Consequently there exists at least one solution $x \in L_1[0,1]$ of the problems (1.1)-(1.2) or (1.1) and (1.3).

2.1. **Uniqueness of the solution.** Consider the following assumptions.

$(i)^*$ f, g: $I \times R \rightarrow \ R$ are measurable in $t \in I$ $\forall x \in R$ and satisfy the lipschits conditions with $a, b > 0$,

$$\begin{aligned}
|f(t,x) - f(t,\bar{x})| &\leq a\,|x - \bar{x}| \\
|g(t,x) - g(t,\bar{x})| &\leq b\,|x - \bar{x}|, \ \forall \ t \in I, x \ \in R.
\end{aligned}$$

$(ii)^*$

$$f(t,0) \in L_1[0,1] \text{ and } \|m_1\| = sup_t \int_0^t |f(t,0)|dt,$$

$$g(t,0) \in L_1[0,1] \text{ and } \|m_2\| = sup_t \int_0^t |g(t,0)|dt.$$

Theorem 2.2. *Let the assumption $(i)^*$, $(ii)^*$ and (ii) be satisfied. If*

$$\frac{(a+b)}{\gamma} < 1,$$

then the fractional integral equation (1.4) has a unique solution $x \in L_1[0,1]$.

Proof. Let $x_1,\ x_2$ be two solution of fractional integral equation (1.4)

$$\begin{aligned}
|x_2(t) - x_1(t)| &= |\frac{t^{\alpha-1}}{\Gamma(\alpha)}[x_0 + \int_0^{1-\tau} g(s, x_2(\varphi_2(s)))ds] + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_2(\varphi_1(s)))ds \\
&- \frac{t^{\alpha-1}}{\Gamma(\alpha)}[x_0 + \int_0^{1-\tau} g(s, x_1(\varphi_2(s)))ds] - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x_1(\varphi_1(s)))ds| \\
&\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{1-\tau} |g(s, x_2(\varphi_2(s)) - g(s, x_1(\varphi_2(s))|ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x_2(\varphi_1(s))) - f(s, x_1(\varphi_1(s)))\Big| ds \\
&\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\tau-1} b|x_2(\varphi_2(s)) - x_1(\varphi_2(s))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a|x_2(\varphi_1(s)) - x_1(\varphi_1(s))|ds,
\end{aligned}$$

then

$$\begin{aligned}
&\int_0^1 |x_2(t) - x_1(t)|dt \\
&\leq \int_0^1 (\frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\tau-1} b|x_2(\varphi_2(s)) - x_1(\varphi_2(s))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a|x_2(\varphi_1(s)) - x_1(\varphi_1(s))|ds)dt, \\
&\leq \int_0^1 (\frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\tau-1} b|x_2(\varphi_2(s)) - x_1(\varphi_2(s))|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a|x_2(\varphi_1(s)) - x_1(\varphi_1(s))|ds)dt, \\
&\leq \int_0^1 (\frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \int_0^{\tau-1} b|x_2(\varphi_2(s)) - x_1(\varphi_2(s))|ds + \int_0^1 \int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a|x_2(\varphi_1(s)) - x_1(\varphi_1(s))|dtds, \\
&\leq \frac{1}{\Gamma(\alpha+1)} \int_0^{\tau-1} b|x_2(\varphi_2(s)) - x_1(\varphi_2(s))|ds + \int_0^1 \frac{1}{\Gamma(\alpha+1)} a|x_2(\varphi_1(s)) - x_1(\varphi_1(s))|ds, \\
&\leq \frac{1}{\Gamma(\alpha+1)}\, b \int_0^{1-\tau} |x_2(\varphi_2(s)) - x_1(\varphi_2(s))|ds + \frac{1}{\Gamma(\alpha+1)}\, a \int_0^1 |x_2(\varphi_1(s)) - x_1(\varphi_1(s))|ds \\
&\leq \frac{1}{\Gamma(\alpha+1)} \frac{b}{\gamma} \int_0^1 |x_2(\eta) - x_1(\eta)|d\eta + \frac{1}{\Gamma(\alpha+1)} \frac{a}{\gamma} \int_0^1 |x_2(\theta) - x_1(\theta)|d\theta \\
&\leq \frac{1}{\Gamma(\alpha+1)} (\frac{b}{\gamma} \|x_2 - x_1\|_1 + \frac{a}{\gamma} \|x_2 - x_1\|_1)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha+1)}\|x_2 - x_1\|_1(\frac{b}{\gamma} + \frac{a}{\gamma}) \\
&\leq \frac{a+b}{\gamma}\|x_2 - x_1\|_1.
\end{aligned}
$$

Then

$$
\|x_2 - x_1\|_1(1 - \frac{a+b}{\gamma}) \quad \leq \quad 0,
$$

$$
\|x_2 - x_1\|_1 \quad \leq \quad 0.
$$

Which implies that $x_1 = x_2$, then the solution of the fractional integral equation (1.4) is unique integrable solution. Consequently the solution $x \in L_1[0,1]$ of the problems (1.1)-(1.2) or (1.1) and (1.3) is unique.

3. Continuous dependence

Theorem 3.1. *Let the assumptions of Theorem 2.2 be satisfied then the unique solution $x \in L_1(I)$ depends continuously on x_0, f, g in the sense that $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon)$ such that*

$$
\max\{|x_0 - x_0^*|,\ |f(t,x) - f^*(t,x)|,\ |g(t,x) - g^*(t,x)|\} < \delta,
$$

implies

$$
\|x - x^*\|_1 < \epsilon,
$$

where x^* be the solution of

$$
x^*(t) \ = \ \frac{t^{\alpha-1}}{\Gamma(\alpha)}[x_0^* + \int_0^{1-\tau} g^*(s, x^*(\varphi_2(s)))ds] + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s, x^*(\varphi_1(s)))ds.
$$

Proof.

$$
\begin{aligned}
& |x(t) - x^*(t)| \\
= \ & |\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(s, x((\varphi_2(s)))ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\varphi_1(s)))ds \\
- \ & \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0^* + \int_0^{1-\tau} g^*(s, x^*(\varphi_2(s))ds) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f^*(s, x^*(\varphi_1(s))ds| \\
\leq \ & \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0 - x_0^*| + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{1-\tau} |g(s, x(\varphi_2(s))) - g^*(s, x^*((\varphi_2(s)))|ds \\
+ \ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(\varphi_1(s))) - f^*(s, x^*(\varphi_1(s)))|ds \\
\leq \ & \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0 - x_0^*| + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{1-\tau} |g(s, x((\varphi_2(s))) + g(s, x^*((\varphi_2(s))) - g(s, x^*((\varphi_2(s))) - g^*(s, x^*((\varphi_2(s)))|ds \\
+ \ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(\varphi_1(s))) + f(s, x^*(\varphi_1(s))) - f(s, x^*(\varphi_1(s))) - f^*(s, x^*(\varphi_1(s)))|ds \\
\leq \ & \frac{t^{\alpha-1}}{\Gamma(\alpha)}|x_0 - x_0^*| + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{1-\tau} [|g(s, x((\varphi_2(s))) - g(s, x^*(\varphi_2(s)))| + |g(s, x^*(\varphi_2(s))) - g^*(s, x^*(\varphi_2(s)))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[|f(s,x(\varphi_1(s))) - f(s,x^*(\varphi_1(s)))| + |f(s,x^*(\varphi_1(s))) - f^*(s,x^*(\varphi_1(s)))|]ds \\
&\leq \frac{\delta\, t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{1-\tau}[b|x(\varphi_2 s) - x^*(\varphi_2(s))| + \delta]ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[a|x(\varphi_1(s)) - x^*(\varphi_1(s))| + \delta|]ds,
\end{aligned}
$$

then

$$
\begin{aligned}
&\int_0^1 |x(t) - x^*(t)|dt \\
&\leq \int_0^1 (\frac{\delta_1 t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{1-\tau}[b|x(\varphi_2 s) - x^*(\varphi_2(s))| + \delta]ds \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[a|x(\varphi_1(s)) - x^*(\varphi_1(s))| + \delta|]ds)dt, \\
&\leq \frac{\delta}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\int_0^{1-\tau}[b|x(\varphi_2 s) - x^*(\varphi_2(s))| + \delta]ds \\
&+ \int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(a|x(\varphi_1(s)) - x^*(\varphi_1(s))| + \delta_3|)dtds, \\
&\leq \frac{\delta}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\int_0^{1-\tau}[b|x(\varphi_2 s) - x^*(\varphi_2(s))| + \delta]ds \\
&+ \int_0^1\int_s^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(a|x(\varphi_1(s)) - x^*(\varphi_1(s))| + \delta|)dtds, \\
&\leq \frac{\delta}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\ [b\int_0^1 |x(\varphi_2(s)) - x^*((\varphi_2(s)))|ds + \delta] \\
&+ \frac{1}{\Gamma(\alpha+1)}\ (a\ \int_0^1 |x(\varphi_1(s)) - x^*(\varphi_1(s))|ds + \delta) \\
&\leq \frac{\delta}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\ [\frac{b}{\gamma}\int_0^1 |x(\eta) - x^*(\eta)|d\eta + \delta] \\
&+ \frac{1}{\Gamma(\alpha+1)}\ [\frac{a}{\gamma}\int_0^1 |x(\theta) - x^*(\theta)|d\theta + \delta].
\end{aligned}
$$

Now

$$
\begin{aligned}
||x - x^*||_1 &\leq \frac{\delta}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\ (\frac{b}{\gamma}||x - x^*||_1 + \delta) \\
&+ \frac{1}{\Gamma(\alpha+1)}\ (\frac{a}{\gamma}||x - x^*||_1 + \delta) \\
&\leq \frac{1}{\Gamma(\alpha+1)}[3\delta + \frac{b}{\gamma}||x - x^*||_1 + \frac{a}{\gamma}||x - x^*||_1] \\
&\leq 3\delta + \frac{a+b}{\gamma}||x - x^*||_1.
\end{aligned}
$$

Thus

$$\|x - x^*\|_1 \leq \frac{3\delta}{1 - \frac{a+b}{\gamma}} = \epsilon.$$

□

4. Hyers-Ulam stability [17]

Now, from the equivalence between the problems (1.1)-(1.2) or (1.1) and (1.3) and the integral equation (1.4) we can study the Hyers-Ulam stability of problem (1.1)-(1.2) or (1.1) and (1.3) as follows:

Definition 4.1.

Let the solution $x \in L_1(I)$ of (1.1)-(1.2) or (1.1) and (1.3) be exist, then constrained problem (1.1)-(1.2) or (1.1) and (1.3) is Hyers-Ulam stable if $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon)$ such that for any $\delta-$approximate solution x_s satisfies,

$$|\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(\theta, x_s(\varphi_2(\theta)))d\theta) + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f(\theta, x_s(\varphi_1(\theta)))d\theta - x_s(t)| < \delta, \tag{4.1}$$

then $\|x - x_s\|_1 < \epsilon$.

Theorem 4.1. *Let the assumptions of Theorem 2.2 be satisfied, then constrained problem (1.1)-(1.2) or (1.1) and (1.3) is Hyers-Ulam stable.*

Proof. From (4.1), we have

$$-\delta < \frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(\theta, x_s(\varphi_2(\theta)))d\theta) + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f(\theta, x_s(\varphi_1(\theta)))d\theta - x_s(t) < \delta.$$

Now

$$\begin{aligned}
& |x(t) - x_s(t)| \\
= & |\frac{t^{\alpha-1}}{\Gamma(\alpha)}(x_0 + \int_0^{1-\tau} g(\theta, x(\varphi_2(\theta)))d\theta) + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f(\theta, x(\varphi_1(\theta)))d\theta - x_s(t)| \\
\leq & \delta + |\frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{1-\tau} g(\theta, x(\varphi_2(\theta)))d\theta + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f(\theta, x(\varphi_1(\theta)))d\theta \\
- & \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{1-\tau} g(\theta, x_s(\varphi_2(\theta)))d\theta - \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} f(\theta, x_s(\varphi_1(t)))d\theta| \\
\leq & \delta + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{1-\tau} |g(\theta, x(\varphi_2(\theta))) - g(\theta, x_s(\varphi_2(\theta)))|d\theta \\
+ & \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} |f(\theta, x(\varphi_1(\theta))) - f(\theta, x_s(\varphi_1(\theta)))|d\theta \\
\leq & \delta + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 b\,|x(\varphi_2(\theta)) - x_s(\varphi_2(\theta))|d\theta + \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)} a|x(\varphi_1(\theta)) - x_s(\varphi_1(\theta))|d\theta,
\end{aligned}$$

thus

$$\begin{aligned}
&\int_0^1 |x(t)-x_s(t)|dt\\
\leq\ & \int_0^1 (\delta\ + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 b\,|x(\varphi_2(\theta))-x_s(\varphi_2(\theta))|d\theta\\
+\ & \int_0^t \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}(a|x(\varphi_1(\theta))-x_s(\varphi_1(\theta))|)d\theta)dt\\
\leq\ & \int_0^1 \delta\,dt+\int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)}dt\int_0^1 b\,|x(\varphi_2(\theta))-x_s(\varphi_2(\theta))|d\theta\\
+\ & \int_0^1\int_\theta^1 \frac{(t-\theta)^{\alpha-1}}{\Gamma(\alpha)}dt(a|x(\varphi_1(\theta))-x_s(\varphi_1(\theta))|d\theta)\\
\leq\ & \delta+\frac{1}{\Gamma(\alpha+1)}\int_0^1 b\,|x(\varphi_2(\theta))-x_s(\varphi_2(\theta))|d\theta+\int_0^1\frac{1}{\Gamma(\alpha+1)}a|x(\varphi_1(\theta))-x_s(\varphi_1(\theta))|d\theta\\
\leq\ & \delta+\frac{1}{\Gamma(\alpha+1)}b\int_0^{1-\tau} |x(\varphi_2(\theta))-x_s(\varphi_2(\theta))|d\theta+\frac{1}{\Gamma(\alpha+1)}a\int_0^1 |x(\varphi_1(\theta))-x_s(\varphi_1(\theta))|d\theta\\
\leq\ & \delta+\frac{1}{\Gamma(\alpha+1)}\ b\int_0^{1-\tau} |x(\varphi_2(\theta))-x_s(\varphi_2(\theta))|d\theta+\frac{1}{\Gamma(\alpha+1)}\ a\int_0^1 |x(\varphi_1(\theta))-x_s(\varphi_2(\theta))|d\theta\\
\leq\ & \delta+\frac{1}{\Gamma(\alpha+1)}\ \frac{b}{\gamma}\int_0^1 |x(\eta)-x_s(\eta)|d\eta+\frac{1}{\Gamma(\alpha+1)}\ \frac{a}{\gamma}\int_0^1 |x(\eta)-x_s(\eta)|d\eta\\
\leq\ & \delta+\frac{1}{\Gamma(\alpha+1)}(\frac{b}{\gamma}\|x-x_s\|_1)+\frac{1}{\Gamma(\alpha+1)}(\frac{a}{\gamma}\|x-x_s\|_1)\\
\leq\ & \delta+\|x-x_s\|_1(\frac{a+b}{\gamma}).
\end{aligned}$$

Thus

$$\|x-x_s\|_1 \quad\leq\quad \frac{\delta}{1-\frac{a+b}{\gamma}}=\epsilon.$$

5. Examples

Example 1.

Consider the following fractional order differential equation

$$^{R}D^{\frac{1}{2}}x(t)=\frac{e^{-t}sint}{1+10t}+\frac{1}{12}x(t^2),\ a.e.\ t\in\ (0,1] \tag{5.1}$$

subject to the nonlocal and weighted integral constraints

$$I^{\frac{1}{2}}x(t)|_{t=0}\ =\ \frac{1}{9}\ +\int_0^{1-\tau}(\frac{sins}{7}+\frac{1}{9}x(s^2))ds. \tag{5.2}$$

or

$$t^{\frac{1}{2}}x(t)|_{t=0}\ =\ \frac{1}{\Gamma(\frac{1}{2})}(\frac{1}{9}+\int_0^{1-\tau}(\frac{sins}{7}+\frac{1}{9}x(s^2))ds). \tag{5.3}$$

This problem can be expressed by the fractional order integral equation

$$x(t) \ = \ \frac{t^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})}(\frac{1}{9} + \int_0^{1-\tau} (\frac{sins}{7} + \frac{1}{9}x(s^2))ds) + I^{\frac{1}{2}} \, (\frac{e^{-t}sint}{1+10t} + \frac{1}{12}x(t^2)). \tag{5.4}$$

Set

$$f(t, x(\varphi_1(t)) = \frac{e^{-t}\sin\ t}{1+10t} + \frac{1}{12}x(t^2),$$

then

$$|f(t,x)| \leq \frac{1}{1+10t} + \frac{1}{12}|x|.$$

Also

$$h(t, x(\varphi_2(t)) = \frac{sins}{7} + \frac{1}{9}x(s^2),$$

then

$$|h(t,x)| \leq \ \frac{1}{7} + \frac{1}{9}|x|,$$

where

$$m_1(t) = \frac{1}{1+10t} \ and \ \|m_1\| = \int_0^1 \frac{1}{1+10t}ds = \frac{\ln(11)}{10},$$

$$m_2(t) = \frac{1}{7} \ and \ \|m_2\| = \int_0^1 \frac{1}{7}\,ds = \frac{1}{7}.$$

Now, we have $\alpha = \frac{1}{2}$, $x_0 = \frac{1}{9}$, $a = \frac{1}{12}$, $b = \frac{1}{9}$, $\gamma = 1$, $and\ r = 0.61294069396$.

then

$$\frac{a+b}{\gamma} = 0.19444444444444 < 1.$$

Now all the conditions of Theorem 2.1 are satisfied, then the problem (5.1)-(5.2) or (5.1) and (5.3) has at least one solution $x \in \ L_1[0,1]$. Moreover,

$$\begin{aligned} |f(t,x) - f(t,\bar{x})| \ &\leq \ \frac{1}{12}\,|x-\bar{x}|, \\ |h(t,x) - h(t,\bar{x})| \ &\leq \ \frac{1}{9}\,|x-\bar{x}|, \ \forall\ t \in I,\ x \in R. \end{aligned} \tag{5.5}$$

Then the solution of problem (5.1)-(5.2) or (5.1) and (5.3) is unique.

Example 2.

Consider the following fractional order differential equation

$${}^{R}D^{\frac{1}{4}}x(t) = \frac{e^{-t}sint}{4-t} + \frac{1}{8}(\frac{x(\frac{1}{2}t)e^{-x^2(\frac{1}{2}t)}}{1+\sin^2 x(\frac{1}{2}t)}),\ a.e.\ t \in \ (0,1] \tag{5.6}$$

subject to the nonlocal and weighted integral constraints

$$I^{\frac{3}{4}}x(t)|_{t=0} \ = \ \frac{1}{4} \ + \int_0^{1-\tau} (\frac{sins}{2-s} + \frac{1}{6}(\frac{x^2(\frac{1}{2}s)\sin^2 x(\frac{1}{2}s)}{1+x^2(\frac{1}{2}s)}))ds. \tag{5.7}$$

or

$$t^{\frac{3}{4}}x(t)|_{t=0} \;=\; \frac{1}{\Gamma(\frac{1}{4})}(\frac{1}{4}+\int_0^1(\frac{sins}{2-s}+\frac{1}{6}(\frac{x^2(\frac{1}{2}s)\sin^2 x(\frac{1}{2}s)}{1+x^2(\frac{1}{2}s)})ds). \tag{5.8}$$

This problem can be expressed by the fractional order integral equation

$$x(t) \;=\; \frac{t^{\frac{-3}{4}}}{\Gamma(\frac{1}{4})}(\frac{1}{4}+\int_0^{1-\tau}(\frac{sins}{2-s}+\frac{1}{6}(\frac{x^2(\frac{1}{2}s)\sin^2 x(\frac{1}{2}s)}{1+x^2(\frac{1}{2}s)}))ds)+I^{\frac{1}{4}}\,(\frac{e^{-t}sint}{4-t}+\frac{1}{8}(\frac{x(\frac{1}{2}t)e^{-x^2(\frac{1}{2}t)}}{1+\sin^2 x(\frac{1}{2}t)})). \tag{5.9}$$

Set

$$f(t,x(\varphi_1(t))=\frac{e^{-t}sint}{4-t}+\frac{1}{8}(\frac{x(\frac{1}{2}t)e^{-x^2(\frac{1}{2}t)}}{1+\sin^2 x(\frac{1}{2}t)}),$$

then

$$|f(t,x)|\leq\frac{1}{4-t}+\frac{1}{8}|x|.$$

Also

$$h(t,x(\varphi_2(t))=\frac{sint}{2-t}+\frac{1}{6}(\frac{x^2(\frac{1}{2}t)\sin^2 x(\frac{1}{2}t)}{1+x^2(\frac{1}{2}t)}),$$

then

$$|h(t,x)|\leq\;\frac{1}{2-t}+\frac{1}{6}|x|,$$

where

$$m_1(t)=\frac{1}{4-t}\;and\;\|m_1\|=\int_0^1\frac{1}{4-s}ds=\ln(4)-\ln(3),$$

$$m_2(t)=\frac{1}{2-t}\;and\;\|m_2\|=\int_0^1\frac{1}{2-s}\,ds=\ln(2).$$

Now, we have $\alpha=\frac{1}{4},\ x_0=\frac{1}{4},\ a=\frac{1}{8},\ b=\frac{1}{6},\ \gamma=\frac{1}{2},\ and\ r=2.953990207228.$

then

$$\frac{a+b}{\gamma}=0.588333333333<1.$$

Now all the conditions of Theorem 2.1 are satisfied, then the problem (5.6)-(5.7) or (5.6) and (5.8) has at least one solution $x\in\ L_1[0,1]$. Moreover,

$$\begin{aligned}|f(t,x)-f(t,\bar{x})| &\leq \frac{1}{8}\,|x-\bar{x}|,\\ |h(t,x)-h(t,\bar{x})| &\leq \frac{1}{6}\,|x-\bar{x}|,\ \forall\ t\in I,\ x\in R.\end{aligned} \tag{5.10}$$

Then the solution of problem (5.6)-(5.7) or (5.6) and (5.8) is unique.

6. Special cases

Corollary 6.1. *Let the assumption of Theorem 2.2 be satisfied. If*

(1) $\tau = 0$, *then the problem*

$$^{R}D^{\alpha}x(t) = f(t, x(\varphi_1(t)), \ t \in (0,1]$$

$$I^{1-\alpha}x(t)|_{t=0} \ = \ x_0 \ + \int_0^1 \ g(s, x(\varphi_2(s)))ds$$

or

$$t^{1-\alpha}x(t)|_{t=0} \ = \ \frac{1}{\Gamma(\alpha)}(x_0 + \int_0^1 \ g(s, x(\varphi_2(s)))ds)$$

has a unique solution.

(2) *If* $\tau = 1$, *then the problem*

$$^{R}D^{\alpha}x(t) = f(t, x(\varphi_1(t)), \ t \in (0,1]$$

$$I^{1-\alpha}x(t)|_{t=0} = x_0$$

or

$$t^{1-\alpha}x(t)|_{t=0} = \frac{1}{\Gamma(\alpha)}x_0$$

has unique solution.

7. conclusion

Fractional order derivatives, which expand the concept of classical derivatives to non-integer orders, can pose various theoretical and practical issues. Establishing the existence and uniqueness of solutions to fractional differential equations includes numerous theoretical frameworks and approaches. Stability analysis is an extensive and varied field with deep theoretical roots and numerous applications in engineering, economics, biology, physics, and other disciplines. Hyers-Ulam stability evaluates the problem's resilience to interruptions while another concept in stability theory is Continuous dependency which examines how even minor parameter changes affect the problem's unique solution. In this manuscript, we are concerned with the study of the solvability of a delay differential equation of a fractional integral equation under two constraints. We analyzed the Hyers-Ulam stability and the continuous dependence of the solution on the initial conditions and other parameters. Finally, we provided some special cases and examples.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S.M. Al-Issa, A.M.A. El-Sayed, H.H.G. Hashem, An Outlook on Hybrid Fractional Modeling of a Heat Controller with Multi-Valued Feedback Control, Fractal Fract. 7 (2023), 759. https://doi.org/10.3390/fractalfract7100759.

[2] D. Baleanu, S. Etemad, Sh. Rezapour, On a Fractional Hybrid Integro-Differential Equation with Mixed Hybrid Integral Boundary Value Conditions by Using Three Operators, Alexandria Eng. J. 59 (2020), 3019–3027. https://doi.org/10.1016/j.aej.2020.04.053.

[3] B. Ahmad, J.J. Nieto, Existence Results for a Coupled System of Nonlinear Fractional Differential Equations with Three-Point Boundary Conditions, Comput. Math. Appl. 58 (2009), 1838–1843. https://doi.org/10.1016/j.camwa.2009.07.091.

[4] A.M.A. El-Sayed, A.A.A. Alhamali, E.M.A. Hamdallah, H.R. Ebead, Qualitative Aspects of a Fractional-Order Integro-Differential Equation with a Quadratic Functional Integro-Differential Constraint, Fractal Fract. 7 (2023), 835. https://doi.org/10.3390/fractalfract7120835.

[5] A.M.A. El-Sayed, Fractional-Order Diffusion-Wave Equation, Int. J. Theor. Phys. 35 (1996), 311–322. https://doi.org/10.1007/BF02083817.

[6] A.M.A. El-Sayed, E.O. Bin-Taher, Positive Nondecreasing Solutions for a Multi-Term Fractional-Order Functional Differential Equation with Integral Conditions, Electron. J. Differ. Equ. 2011 (2011), 166.

[7] A. M. A. El-Sayed and E. O. Bin-Taher, a multi-term fractional-order differential equation with nonlocal condition, Egypt.-Chin. J. Comput. Appl. Math. 1 (2012), 54-60.

[8] A.M.A. El-Sayed, M.M.S. Ba-Ali, E.M.A. Hamdallah, An Investigation of a Nonlinear Delay Functional Equation with a Quadratic Functional Integral Constraint, Mathematics 11 (2023), 4475. https://doi.org/10.3390/math11214475.

[9] A.M.A. El-Sayed, F.M. Gaafar, Fractional Calculus and Some Intermediate Physical Processes, Appl. Math. Comput. 144 (2003), 117–126. https://doi.org/10.1016/S0096-3003(02)00396-X.

[10] A.M.A. El-Sayed, H.H.G. Hashem, S.M. Al-Issa, An Implicit Hybrid Delay Functional Integral Equation: Existence of Integrable Solutions and Continuous Dependence, Mathematics 9 (2021), 3234. https://doi.org/10.3390/math9243234.

[11] A.M.A. El-Sayed, E.M.A. Hamdallah, M.M.S. Ba-Ali, Qualitative Study for a Delay Quadratic Functional Integro-Differential Equation of Arbitrary (Fractional) Orders, Symmetry 14 (2022), 784. https://doi.org/10.3390/sym14040784.

[12] A.M.A. El-Sayed, S.Z. Rida, A.A.M. Arafa, On the Solutions of the Generalized Reaction-Diffusion Model for Bacterial Colony, Acta Appl. Math. 110 (2010), 1501–1511. https://doi.org/10.1007/s10440-009-9523-4.

[13] I. Podlubny, A.M.A. El-Sayed, On Two Definitions of Fractional Calculus, Slovak Academy of Science, Institute of Experimental Phys., Bratislava, (1996).

[14] K.M. Furati, N. Tatar, Power-Type Estimates for a Nonlinear Fractional Differential Equation, Nonlinear Anal.: Theory Methods Appl. 62 (2005), 1025–1036. https://doi.org/10.1016/j.na.2005.04.010.

[15] F.M. Gaafar, Continuous and Integrable Solutions of a Nonlinear Cauchy Problem of Fractional Order with Nonlocal Conditions, J. Egypt. Math. Soc. 22 (2014), 341–347. https://doi.org/10.1016/j.joems.2013.12.008.

[16] G.M. N'Guérékata, A Cauchy Problem for Some Fractional Abstract Differential Equation with Non Local Conditions, Nonlinear Anal.: Theory Methods Appl. 70 (2009), 1873–1876. https://doi.org/10.1016/j.na.2008.02.087.

[17] D.H. Hyers, On the Stability of the Linear Functional Equation, Proc. Natl. Acad. Sci. 27 (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222.

[18] Y. Jalilian, M. Ghasemi, On the Solutions of a Nonlinear Fractional Integro-Differential Equation of Pantograph Type, Mediterranean J. Math. 14 (2017), 194. https://doi.org/10.1007/s00009-017-0993-8.

[19] T. Jankowski, Fractional Equations of Volterra Type Involving a Riemann–Liouville Derivative, Appl. Math. Lett. 26 (2013), 344–350. https://doi.org/10.1016/j.aml.2012.10.002.

[20] K.M. Furati, N. Tatar, Power-Type Estimates for a Nonlinear Fractional Differential Equation, Nonlinear Anal.: Theory Methods Appl. 62 (2005), 1025–1036. https://doi.org/10.1016/j.na.2005.04.010.

[21] A.N. Kolmogorov, S.V. Fomin, Itroductory Real Analysis, Dover Publications, 1975.

[22] K.D. Kucche, S.T. Sutar, On Existence and Stability Results for Nonlinear Fractional Delay Differential Equations, Bol. Soc. Parana. Mat. 36 (2018), 55–75. https://doi.org/10.5269/bspm.v36i4.33603.

[23] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, World Scientific, 2010.

[24] R. Metzler, J. Klafter, The Random Walk's Guide to Anomalous Diffusion: A Fractional Dynamics Approach, Phys. Rep. 339 (2000), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3.

[25] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley, New York, 1993.

[26] S. Samko, O.L. Marichev, Fractional Integral and Derivatives, Gordon and Breach Science Publisher, 1993.

[27] T. Sandev, R. Metzler, Ž. Tomovski, Fractional Diffusion Equation with a Generalized Riemann–Liouville Time Fractional Derivative, J. Phys. A: Math. Theor. 44 (2011), 255203. https://doi.org/10.1088/1751-8113/44/25/255203.

[28] A.A. Shaikh, S. Qureshi, Comparative Analysis of Riemann-Liouville, Caputo-Fabrizio, and Atangana-Baleanu Integrals, J. Appl. Math. Comput. Mech. 21 (2022), 91–101. https://doi.org/10.17512/jamcm.2022.1.08.

[29] H.M. Srivastava, A.M.A. El-Sayed, F.M. Gaafar, A Class of Nonlinear Boundary Value Problems for an Arbitrary Fractional-Order Differential Equation with the Riemann-Stieltjes Functional Integral and Infinite-Point Boundary Conditions, Symmetry 10 (2018), 508. https://doi.org/10.3390/sym10100508.

[30] H. Xu, Analytical Approximations for a Population Growth Model with Fractional Order, Commun. Nonlinear Sci. Numer. Simul. 14 (2009), 1978–1983. https://doi.org/10.1016/j.cnsns.2008.07.006.

[31] K. Zhao, Stability of a Nonlinear Langevin System of ML-Type Fractional Derivative Affected by Time-Varying Delays and Differential Feedback Control, Fractal Fract. 6 (2022), 725. https://doi.org/10.3390/fractalfract6120725.