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The Coeffisients of the Spline Minimizing Semi Norm in $K_2(P_3)$

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Abstract. Our goal is to construct an approximation of the unknown function f by Sobolev's method, we construct an approximation form of unknown function by interpolation splines minimizing the semi norm in $K_2(P_3)$ Hilbert space. Explicit formulas for coefficients of the interpolation splines are obtained. The resulting interpolation spline is exact for the hyperbolic functions and constant. In the last section, we obtain several absolute errors graph in interpolating functions with the sixth order algebraic-hyperbolic spline, and we compare absolute errors of cubic spline and algebraic-hyperbolic in interpolating several functions. Numerical results show that the sixth-order spline interpolates the functions with higher accuracy than the cubic spline.

1. Introduction

Nowadays, interpolation plays a crucial role in various fields including mathematics, engineering, computer science, statistics, and more. It involves approximate functions, fill in missing data, smooth noisy data, and create continuous representations from discrete data. There are several types of interpolation methods, each with its own characteristics and suitability for different types of data and applications. Here are some common types: linear Interpolation, polynomial interpolation, Spline Interpolation, Piecewise Interpolation, Inverse Distance Weighting (IDW), Kriging. These are some of the main types of interpolation methods commonly used in various fields. Among of them, splines provide a significant tool for the design of computationally economical curves and surfaces for the construction of various objects like automobiles, ship hulls, airplane

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fuselages and wings, propeller blades, shoe insoles, bottles, etc. It also contributes in the description of geological, physical statistical, and even medical phenomena. Spline methods have proven to be indispensable in a variety of modern industries, including computer vision, robotics, signal and image processing, visualization, textile, graphic design, and even media., Among the methods for constructing splines the approach [1] is known, in which the spline is found as a solution to a boundary value problem for a differential equation with internal boundary conditions at the interpolation nodes. Typically, a finite-difference approach is used to solve such problems, so the method can be called the difference method for constructing splines. For tension splines, a corresponding second-order approximation method was proposed and studied in [2]. According to this method, the conditions for smooth conjugation of the first and second derivatives at interpolation nodes are approximated by three symmetrically located nodes using fixed nodes to the left and right of a given node, the solution values in which require exclusion during the calculation process. In [3], a modification of the method [2] of the same order of approximation was proposed, which differs from the latter by setting internal boundary conditions based on one-sided three-point approximations of derivatives, which does not require the introduction of the fixed nodes.

Moreover, in [4] a medical application based on biomarkers is presented; longitudinal and survival fitting model based on cubic polynomial B-splines sets is presented for modeling the longitudinal markers. The usage of quadratic splines [5], [6], cubic splines [7], piecewise polynomial functions of various degrees [8], rational splines [9], a class of polynomial spline curve with free parameters is established in [10], construction of exponential splines, fourth and m order algebraic trigonometric , is showed in [11], [12], [13] respectively and additional approaches are developed in [14], [15], [16], [17], [18], [26], [27].

Our work is also devoted to the construction of optimal algebraic hyperbolic splines. In the space $K_2(P_3)$. This work consists of 5 sections. In next section, we give a defination of sinxth order algebraic-hyperbolic natural interpolation spline, then take a equations system to construct this spline.

2. Statement of the problem

We consider the problem of recovering an approximation of a function given values in following space:

$$K_2(P_3) := \{ f : \ [0,1] \to R \big| \ f'' \ \textit{is absolutely continuous and} \ f''' \in L_2(0,1)\} \tag{2.1}$$

equipped with the following semi norm:

$$\|f\| := \{ \int_0^1 (f'''(x) - f'(x))^2 dx\}^{\frac{1}{2}}, \tag{2.2}$$

Equality 2.2 gives the semi-norm and $\|f\| = 0$ if and only if $f(x) = d_1 e^x + d_2 e^{-x} + d_3$ and we can rewrite $f(x)$ as $f(x) = d_1 \sinh(x) + d_2 \cosh(x) + d_3$. In this space the inner product is defined as following:

$$\langle f,g\rangle := \int_0^1 (f'''(x) - f'(x))(g'''(x) - g'(x))dx$$

$K_2(P_3)$ is the factorized Hilbert space if we identify functions that differ by a solution of $f'''(x) - f'(x) = 0$. We solve the following interpolation problem:

Problem 1 Find the function $S(x)$ *in* $K_2(P_3)$ which gives minimum to the semi-norm 2.2 and satisfies the interpolation condition

$$S(x_\beta) = f(x_\beta), \qquad x_\beta \in [0,1], \qquad \beta = 0,1,...,N, \tag{2.3}$$

for any $f \in K_2(P_3)$.

This problem is solved in $K_2(P_2)$ space [19].

We give a definition of the interpolation spline function in the space $K_2(P_3)$ following [20] .

Definition 2.1 Let $\Delta : 0 = x_0 < x_1 < \cdots < x_N = 1$ be a mesh on the interval [0,1], then the interpolation spline function with respect to Δ is a function $S(x) \in K_2(P_3)$ and satisfies the following conditions:

(1) $S(x)$ is a linear combination of functions $\sinh x,\ \cosh(x),\ 1,\ x\sinh(x),\ x\cosh(x), x$ on each open mesh interval $(x_\beta,\ x_{\beta+1})$, $\beta = 0,1,...,N-1$;

(2) $S(x)$ is a linear combination of functions $\sinh x,\ \cosh(x),\ 1$ on intervals $(-\infty,\ 0)$ and $(1,\ \infty)$;

(3) $S(x)$ satisfies the following continuity and natural spline conditions :

$$S^\alpha(x_\beta^-) = S^\alpha(x_\beta^+),\ \alpha = 0,1,2,3,4,\ \beta = 1,2,...,N-1,$$
$$S'''(0) - S'(0) = S'''(1) - S'(1) = 0,$$
$$S^{(4)}(0) - S''(0) = S^{(4)}(0) - S''(0) = 0.$$

(4) $S(x)$ satisfies the interpolation conditions.

We get the following theorem based on definition 2.1

Theorem 2.1. *The solution of the problem 1 is a algebraic-hyperbolic spline and it has the following form:*

$$S(x) = \sum_{\gamma=0}^{N} C_\gamma G(x - x_\gamma) + d_1 \sinh(x) + d_2 \cosh(x) + d_3, \tag{2.4}$$

where $G(x)$ is the solution of $G^{(6)}(x) - 2G^{(4)}(x) + G^{(2)}(x) = \delta(x)$ ($\delta(x)$ is Dirac's delta function) differential equation, and has the following form:

$$G(x) = \frac{sign(x)}{4}(x\cosh(x) - 3\sinh(x) + 2x) \tag{2.5}$$

and the coefficients C_γ, $\gamma = 0,1,2,...,N,\ d_1, d_{2,}\ d_3$ of the spline (2.4) are obtained from the following system of N+4 linear equations,

$$\sum_{\gamma=0}^{N} C_\gamma G(x_\beta - x_\gamma) + d_1 \sinh x_\beta + d_2 \cosh x_\beta + d_3 = f(x_\beta),\ \beta = 0,1,...N, \tag{2.6}$$

$$\sum_{\gamma=0}^{N} C_\gamma \sinh(x_\gamma) = 0, \tag{2.7}$$

$$\sum_{\gamma=0}^{N} C_\gamma \cosh(x_\gamma) = 0, \tag{2.8}$$

$$\sum_{\gamma=0}^{N} C_\gamma = 0. \tag{2.9}$$

where $f \in K_2(P_3)$.

Proof: It is clear that the fifth derivative of the function $G(x-x_\gamma) = \frac{sign(x-x_\gamma)}{4}((x-x_\gamma)\cosh(x-x_\gamma) - 3\sinh(x-x_\gamma) + 2(x-x_\gamma))$ has a discontinuity equal to 1 at the points x_γ, $\gamma = 1, 2, ..., N-1$, and from first order to the fourth order derivatives of $G(x-x_\gamma)$ are continuous. Suppose a function $p_\gamma(x)$ coincides with the spline $S(x)$ on the interval $(x_\gamma, x_{\gamma+1})$, i.e., $p_\gamma(x) := p_{\gamma-1}(x) + C_\gamma G(x-x_\gamma),\ x \in (x_\gamma, x_{\gamma+1})$, where C_γ is the jump of the function $S^{(5)}(x)$ at x_γ:

$$C_\gamma = S^{(5)}(x_\gamma^+) - S^{(5)}(x_\gamma^-)$$

Then the spline $S(x)$ can be written in the following form

$$S(x) = \sum_{\gamma=0}^{N} C_\gamma G(x-x_\gamma) + p_{-1}(x), \tag{2.10}$$

where

$$p_{-1}(x) = d_1 \sinh(x) + d_2 \cosh(x) + d_3 \tag{2.11}$$

and $d_1,\ d_2,\ d_3$ are real numbers.
We obtain (2.6) equation from the (2.10), (2.11) and the condition (iv). Furthermore, the function $S(x)$ satisfies the condition (ii) and therefore the function

$$\frac{1}{4}\sum_{\gamma=0}^{N} C_\gamma((x-x_\gamma)\cosh(x-x_\gamma) - 3\sinh(x-x_\gamma) + 2(x-x_\gamma))$$

is a linear combination of the functions $\sinh(x),\ \cosh(x),\ 1$. It leads to the following conditions for C_γ,

$$\sum_{\gamma=0}^{N} C_\gamma \sinh(x_\gamma) = 0,\ \sum_{\gamma=0}^{N} C_\gamma \cosh(x_\gamma) = 0,\ \sum_{\gamma=0}^{N} C_\gamma = 0.$$

in the end , we obtain (2.6)-(2.9) equations system. After all, we have proved theorem 2.1

The rest of the paper is organized as follows. In Sect.3 we give an algorithm for solving the system of equations (2.6)-(2.9) for equally spaced nodes x_β. Using this algorithm the coefficients of the interpolation spline $S(x)$are computed in Sect. 4.

3. An algorithm for computing the coefficients of interpolation splines

Here we use a similar method proposed by S.L.Sobolev [21], [22]for finding the coefficients of optimal quadrature formulas in the space $L_2^{(m)}$. We use mainly the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [22] , [23].

Assume that the nodes x_β are equally spaced, i,e., $x_\beta = h\beta$, $h = {}^1/_N$, $N = 1, 2, ...$and $C_\beta = 0$ when $\beta < 0$ and $\beta > N$. Using convolution, we rewrite equalities (2.6)-(2.9) as follows:

$$G(h\beta) * C_\beta + d_1 \sinh x_\beta + d_2 \cosh x_\beta + d_3 = f(x_\beta),\ \beta = 0, 1, ...N \tag{3.1}$$

$$\sum_{\gamma=0}^{N} C_\gamma \sinh(x_\gamma) = 0 \tag{3.2}$$

$$\sum_{\gamma=0}^{N} C_\gamma \cosh(x_\gamma) = 0, \tag{3.3}$$

$$\sum_{\gamma=0}^{N} C_\gamma = 0, \tag{3.4}$$

where $G(h\beta)$ is a function of discrete argument corresponding to the function G is given in (2.5) . Thus, we have the following problem.

Problem 2 Find the *coefficients* C_β, $\beta = 0, 1, ..., N$ and the *constants* d_1, d_2, d_3 which satisfy the system (3.1)-(3.4).

Further we investigate Problem 2 which is equivalent to Problem 1. We introduce the following functions to solve Problem 2

$$v(h\beta) = G(h\beta) * C_\beta, \tag{3.5}$$

$$u(h\beta) = v(h\beta) + d_1 \sinh(h\beta) + d_2 \cosh(h\beta) + d_3. \tag{3.6}$$

In such a statement it is necessary to express the coefficients C_β by the function $u(h\beta)$. For this we have to construct such an operator $D(h\beta)$ which satisfies the equality

$$D(h\beta) * G(h\beta) = \delta(h\beta,$$

where $\delta(h\beta) = \begin{cases} 0, & \beta \neq 0 \\ 1, & \beta = 0 \end{cases}$ is the discrete delta-function. The construction of the discrete analogue $D(h\beta)$ of the differential operator $\frac{d^6}{dx^6} - 2\frac{d^4}{dx^4} + \frac{d^2}{dx^2}$ is given in [24] .
Following [24] we have:

Theorem 3.1. *The discrete analogue of the differential operator* $\frac{d^6}{dx^6} - 2\frac{d^4}{dx^4} + \frac{d^2}{dx^2}$ *has the form*

$$D_3(h\beta) = \frac{2}{p} \begin{cases} \sum\limits_{k=1}^{2} A_k \lambda_k{}^{|\beta|-1}, & |\beta| \geq 0 \\ 1 + \sum\limits_{k=1}^{2} A_k, & |\beta| = 1 \\ C + \sum\limits_{k=1}^{2} \frac{A_k}{\lambda_k}, & |\beta| = 0 \end{cases} \tag{3.7}$$

$$p = h\cosh h - 3\sinh h + 2h, \quad C = -(2+4\cosh h) - \frac{3\sinh 2h - 2h + 6\sinh h - 10h\cosh h}{h\cosh h - 3\sinh h + 2h},$$
$$A_k = \frac{(1-\lambda_k)^2(\lambda_k^2+1-2\lambda_k\cosh h)^2(h\cosh h - 3\sinh h+2h)^2}{\lambda_k P_4{}'(\lambda_k)} \tag{3.8}$$

λ_1, λ_2 are $|\lambda_k| < 1$ zero of the polynomial

$$P_4(\lambda) = (1-\lambda_K)^2[(h\cosh h - 3\sinh)\lambda_k{}^2 + [3\sinh 2h - 2h]\lambda + (h\cosh h - 3\sinh h])$$
$$+2h(\lambda^2{}_k - 2\lambda_k\cosh h + 1)^2$$

We use some property of the discrete analogue $D(h\beta)$. They are shown in the following theorem.

Theorem 3.2. *Discrete analogue $D(h\beta)$ of the differential operator $\frac{d^6}{dx^6} - 2\frac{d^4}{dx^4} + \frac{d^2}{dx^2}$ satisfies the following equalities:*
1)$D(h\beta) * sinh(h\beta) = 0$
2)$D(h\beta) * cosh(h\beta) = 0$
3)$D(h\beta) * (h\beta)sinh(h\beta) = 0$
4)$D(h\beta) * (h\beta)cosh(h\beta) = 0$
5)$D(h\beta) * G(h\beta) = \delta(h\beta)$
6)$D(h\beta) * (h\beta) = 0$
7)$D(h\beta) * 1 = 0.$

This properties was proved in [25]. Then, taking into account (3.6) and Theorem 3.2 for optimal coefficients we have

$$C_\beta = D(h\beta) * u(h\beta). \tag{3.9}$$

Thus, if we find the function $u(h\beta)$ then the coefficients C_β can be obtained from equality (3.9). In order to calculate the convolution 3.9 we need a representation of the function $u(h\beta)$ for all integer values of β. From equality 3.1 we get that $u(h\beta) = \varphi(h\beta)$ when $h\beta \in [0,1]$. Now we need to find a representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C_\beta = 0$ when $h\beta \notin [0,1]$ then $C_\beta = D_m(h\beta) * u(h\beta) = 0,\ h\beta \notin [0,1]$. We calculate now the convolution $v(h\beta) = G(h\beta) * C_\beta$ when $\beta \le 0$ and $\beta \ge N$.

Supposing $\beta \le 0$ and taking into account equalities (2.5), (3.2), (3.3), (3.4), we have

$$v(h\beta) = \sum_{\gamma=-\infty}^{\infty} C_\gamma G(h\beta - h\gamma) = \sum_{\gamma=-\infty}^{\infty} C_\gamma \frac{sign(h\beta-h\gamma)}{4}\{(h\beta-h\gamma)\cosh(h\beta-h\gamma)-$$
$$-3\sinh(h\beta-h\gamma) + 2(h\beta-h\gamma)\} = -\frac{1}{4}\sum_{\gamma=-\infty}^{\infty} C_\gamma\{(h\beta-h\gamma)[\cosh(h\beta)\cosh(h\gamma)-$$
$$-\sinh(h\beta)\sinh(h\gamma)] - 3[\sinh(h\beta)\cosh(h\gamma) - \cosh(h\beta)\sinh(h\gamma)] + 2(h\beta-h\gamma)\}$$
$$= -\frac{1}{4}\sum_{\gamma=-\infty}^{\infty} C_\gamma\{(h\beta)\cosh(h\beta)\cosh(h\gamma) - (h\beta)\sinh(h\beta)\sinh(h\gamma) - (h\gamma)\cosh(h\beta)\cosh(h\gamma)$$
$$+(h\gamma)\sinh(h\beta)\sinh(h\gamma) - 3\sinh(h\beta)\cosh(h\gamma) + 3\cosh(h\beta)\sinh(h\gamma) + 2(h\beta) - 2(h\gamma)\}$$
$$= -\frac{1}{4}\sum_{\gamma=-\infty}^{\infty} C_\gamma\{(h\beta)\cosh(h\beta)\cosh(h\gamma) - (h\beta)\sinh(h\beta)\sinh(h\gamma) - (h\gamma)\cosh(h\beta)\cosh(h\gamma)+$$
$$+(h\gamma)\sinh(h\beta)\sinh(h\gamma)] - 3\sinh(h\beta)\cosh(h\gamma) + 3\cosh(h\beta)\sinh(h\gamma)+$$
$$+2(h\beta) - 2(h\gamma)\} = \cosh(h\beta)\frac{1}{4}\sum_{\gamma=-\infty}^{\infty} C_\gamma(h\gamma)\cosh(h\gamma) - \sinh(h\beta)\frac{1}{4}\sum_{\gamma=-\infty}^{\infty} C_\gamma(h\gamma)\sinh(h\gamma) + \frac{1}{2}\sum_{\gamma=-\infty}^{\infty} C_\gamma(h\gamma).$$

Denoting:

$$b_1 = \frac{1}{4}\sum_{\gamma=-\infty}^{\infty} C_\gamma(h\gamma)\sinh(h\gamma),\;\; b_2 = \frac{1}{4}\sum_{\gamma=-\infty}^{\infty} C_\gamma(h\gamma)\cosh(h\gamma),\;\; b_3 = \frac{1}{2}\sum_{\gamma=-\infty}^{\infty} C_\gamma(h\gamma)$$

we get for $\beta \le 0$

$$v(h\beta) = -b_1\sinh(h\beta) + b_2\cosh(h\beta) + b_3$$

And for $\beta \ge N$

$$v(h\beta) = b_1\sinh(h\beta) - b_2\cosh(h\beta) - b_3.$$

Now, setting

$$d_1^- = d_1 - b_1,\; d_2^- = d_2 + b_2,\; d_3^- = d_3 + b_3,$$
$$d_1^+ = d_1 + b_1,\; d_2^+ = d_2 - b_2,\; d_3^+ = d_3 - b_3$$

We formulate the following problem:

Problem 3 Find the solution of the equation

$$D_m(h\beta) * u(h\beta) = 0,\; h\beta \notin [0,1] \tag{3.10}$$

In the form:

$$u(h\beta) = \begin{cases} d_1^- \sinh(h\beta) + d_2^- \cosh(h\beta) + d_3^-, & \beta \le 0, \\ f(h\beta), & 0 \le \beta \le N, \\ d_1^+ \sinh(h\beta) + d_2^+ \cosh(h\beta) + d_3^+, & \beta \ge N, \end{cases} \tag{3.11}$$

where $d_1^-,\ d_2^-,\ d_3^-,\ d_1^+,\ d_2^+,\ d_3^+$ *are unknowns.*

It is clear that

$$d_i = \frac{1}{2}(d_i^+ + d_1^-),\; i = 1.2,3,\; b_1 = \frac{1}{2}(d_1^+ - d_1^-),\; b_2 = \frac{1}{2}(d_2^- - d_2^+),\, b_3 = \frac{1}{2}(d_3^- - d_3^+). \tag{3.12}$$

These unknowns $d_1^-,\ d_2^-,\ d_3^-,\ d_1^+,\ d_2^+,\ d_3^+$ can be found from equation (3.10), using the function $D(h\beta)$. Then the explicit from of the function $u(h\beta)$ and coefficients C_β, d_1, d_2, d_3 can be found. Thus, Problem 3 and respectively Problems 2 and 1 can be solved.

In the next section we realize this algorithm for computing the coefficients $C_\beta, \beta = 0, 1, .., N,\ d_1,\ d_2,\ d_3$ of the interpolation spline (2.4) for any $N = 1, 2...$

4. Computing the coefficients of the interpolation spline

In this section using the algorithm from the previous section we obtain explicit formulae for coefficients of interpolation 2.4 which is the solution of Problem 1.

Theorem 4.1. *Coefficients of interpolation spline 2.4 which minimizes the semi norm 2.2 with equally spaced nodes in the space* $K_2(P_3)$ *have the following form:*

$$C_0 = \frac{2}{p}\{-d_1^- \sinh h + d_2^- \cosh h + d_3^- + Cf(0) + f(h) + \sum_{k=1}^{2}\frac{A_k}{\lambda_k}[M_k + \sum_{\gamma=0}^{N}\lambda_k^\gamma f(h\gamma) + \lambda_k^N N_k]\},$$

$$C_\beta = \frac{2}{p}\{f(h(\beta-1)) + Cf(h\beta) + f(h(\beta+1)) + \sum_{k=1}^{2}\frac{A_k}{\lambda_k}[\lambda_k^\beta M_k + \sum_{\gamma=0}^{N}\lambda_k^{|\beta-\gamma|} f(h\gamma) + \lambda_k^{N-\beta}N_k]\}, \beta = 1, 2, ..., N-1$$

$$C_N = \frac{2}{p}\{d_1^+ \sinh(h+1) + d_2^+ \cosh(h+1) + d_3^+ + f(1-h) + Cf(1) + \sum_{k=1}^{2} \frac{A_k}{\lambda_k}[\lambda_k^N M_k + \sum_{\gamma=0}^{N} \lambda_k^{N-\gamma} f(h\gamma) + N_k]\},$$

$$d_i = \frac{1}{2}(d_i^+ + d_i^-),\ i = 1,2,3$$

where

$$M_k = -d_1^- \frac{\lambda_k \sinh h}{1+\lambda_k{}^2 - 2\lambda_k \cosh h} + d_2^- \frac{\lambda_k(\cosh h - \lambda_k)}{1+\lambda_k{}^2 - 2\lambda_k \cosh h} + d_3^- \frac{\lambda_k}{1-\lambda_k}, \tag{4.1}$$

$$N_k = d_1^+ \frac{\lambda_k(\sinh(h+1) - \lambda_k \sinh 1)}{1+\lambda_k{}^2 - 2\lambda_k \cosh h} + d_2^+ \frac{\lambda_k(\cosh(h+1) - \lambda_k \cosh 1)}{1+\lambda_k{}^2 - 2\lambda_k \cosh h} + d_3^+ \frac{\lambda_k}{1-\lambda_k} \tag{4.2}$$

and p, A_k, C, λ_k *are given in 3.7 and* $d_1^-, d_2^-, d_3^-, d_1^+, d_2^+, d_3^+$ *are defined by 4.3, 4.8.*

Proof. First we find the expression for d_2^- and d_2^+. When $\beta = 0$ and $\beta = N$, from 3.11 we get

$$d_2^- = f(0) - d_3^-,\ d_2^+ = \frac{f(1) - d_1^+ \sinh(1) - d_3^+}{\cosh(1)} \tag{4.3}$$

Now we find other four unknowns $d_1^-, d_3^-, d_1^+, d_3^+$ which can be found from ((3.10)) when $\beta = -1, -2, N+1, N+2$. Taking into account (3.11) and from (3.10) we have:

$$\sum_{\gamma=-\infty}^{-1} D(h\beta - h\gamma)(d_1^- \sinh(h\gamma) + d_2^- \cosh(h\gamma) + d_3^-) + \sum_{\gamma=0}^{N} D(h\beta - h\gamma) f(h\gamma)$$
$$+ \sum_{\gamma=N+1}^{\infty} D(h\beta - h\gamma)(d_1^+ \sinh(h\gamma) + d_2^+ \cosh(h\gamma) + d_3^+) = 0,$$
$$\sum_{\gamma=1}^{\infty} D(h\beta + h\gamma)(-d_1^- \sinh(h\gamma) + d_2^- \cosh(h\gamma) + d_3^-) + \sum_{\gamma=0}^{N} D(h\beta - h\gamma) f(h\gamma)$$
$$+ \sum_{\gamma=1}^{\infty} D(h(N+\gamma-\beta))(d_1^+ \sinh(h\gamma+1) + d_2^+ \cosh(h\gamma+1) + d_3^+) = 0.$$

Now, we use (4.3) and for $\beta = -1, -2, N+1, N+2$ we get the following system of linear equations for $d_1^-, d_3^-, d_1^+, d_3^+$.

$$-d_1^- \sum_{\gamma=1}^{\infty} D_3(h\gamma + h\beta) \sinh(h\gamma) + d_3^- \sum_{\gamma=1}^{\infty} D_3(h\gamma + h\beta)(1 - \cosh(h\gamma))$$
$$+d_1^+ \tfrac{1}{\cosh 1} \sum_{\gamma=1}^{\infty} D_3(h(\gamma + N - \beta)) \sinh(h\gamma) + d_3^+ \sum_{\gamma=1}^{\infty} D_3(h(\gamma + N - \beta))(1 - \tfrac{\cosh(h\gamma+1)}{\cosh 1})$$
$$= -\sum_{\gamma=0}^{N} D_3(h\beta - h\gamma) f(h\gamma) - f(0) \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) \cosh(h\gamma) - \tfrac{f(1)}{\cosh 1} \sum_{\gamma=1}^{\infty} D_3(h(\gamma + N - \beta)) \cosh(h\gamma + 1)$$

and for $\beta = -1, -2, N+1, N+2$ we get the following system of linear equations for $d_1^-, d_3^-, d_1^+, d_3^+$

$$\begin{aligned}
&-d_1^- \sum_{\gamma=1}^{\infty} D_3(h\gamma - h) \sinh(h\gamma) + d_3^- \sum_{\gamma=1}^{\infty} D_3(h\gamma - h)(1 - \cosh(h\gamma)) \\
&+d_1^+ \tfrac{1}{\cosh 1} \sum_{\gamma=1}^{\infty} D_3(h(\gamma + N + 1)) \sinh(h\gamma) + d_3^+ \sum_{\gamma=1}^{\infty} D_3(h(\gamma + N + 1))(1 - \tfrac{\cosh(h\gamma+1)}{\cosh 1}) \\
&= -\sum_{\gamma=0}^{N} D_3(h\gamma + h) f(h\gamma) - f(0) \sum_{\gamma=1}^{\infty} D_3(h\gamma - h) \cosh(h\gamma) \\
&-\tfrac{f(1)}{\cosh 1} \sum_{\gamma=1}^{\infty} D_3(h(\gamma + N + 1)) \cosh(h\gamma + 1)
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&-d_1^- \sum_{\gamma=1}^{\infty} D_3(h\gamma - 2h)\sinh(h\gamma) + d_3^- \sum_{\gamma=1}^{\infty} D_3(h\gamma - 2h)(1-\cosh(h\gamma)) \\
&+d_1^+ \tfrac{1}{\cosh 1} \sum_{\gamma=1}^{\infty} D_3(h\,(\gamma+N+2))\sinh(h\gamma) + d_3^+ \sum_{\gamma=1}^{\infty} D_3(h\,(\gamma+N+2))(1-\tfrac{\cosh(h\gamma+1)}{\cosh 1}) \\
&= -\sum_{\gamma=0}^{N} D_3(h\gamma+2h)f(h\gamma) - f(0)\sum_{\gamma=1}^{\infty} D_3(h\gamma-2h)\cosh(h\gamma) \\
&-\tfrac{f(1)}{\cosh 1}\sum_{\gamma=1}^{\infty} D_3(h(\gamma+N+2))\cosh(h\gamma+1)
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&-d_1^- \sum_{\gamma=1}^{\infty} D_3(h\,(\gamma+N+1))\sinh(h\gamma) + d_3^- \sum_{\gamma=1}^{\infty} D_3(h(\gamma+N+1))(1-\cosh(h\gamma)) \\
&+d_1^+ \tfrac{1}{\cosh 1} \sum_{\gamma=1}^{\infty} D_3(h\,(\gamma-1))\sinh(h\gamma) + d_3^+ \sum_{\gamma=1}^{\infty} D_3(h\,(\gamma-1))(1-\tfrac{\cosh(h\gamma+1)}{\cosh 1}) \\
&= -\sum_{\gamma=0}^{N} D_3(h\,(N+1-\gamma))f(h\gamma) - f(0)\sum_{\gamma=1}^{\infty} D_3(h\,(\gamma+N+1))\cosh(h\gamma) \\
&-\tfrac{f(1)}{\cosh 1}\sum_{\gamma=1}^{\infty} D_3(h(\gamma-1))\cosh(h\gamma+1)
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&-d_1^- \sum_{\gamma=1}^{\infty} D_3(h\,(\gamma+N+2))\sinh(h\gamma) + d_3^- \sum_{\gamma=1}^{\infty} D_3(h(\gamma+N+2))(1-\cosh(h\gamma)) \\
&+d_1^+ \tfrac{1}{\cosh 1} \sum_{\gamma=1}^{\infty} D_3(h\,(\gamma-2))\sinh(h\gamma) + d_3^+ \sum_{\gamma=1}^{\infty} D_3(h\,(\gamma-2))(1-\tfrac{\cosh(h\gamma+1)}{\cosh 1}) \\
&= -\sum_{\gamma=0}^{N} D_3(h\,(N+2-\gamma))f(h\gamma) - f(0)\sum_{\gamma=1}^{\infty} D_3(h\,(\gamma+N+2))\cosh(h\gamma) \\
&-\tfrac{f(1)}{\cosh 1}\sum_{\gamma=1}^{\infty} D_3(h(\gamma-2))\cosh(h\gamma+1)
\end{aligned} \tag{4.7}$$

Since $|\lambda_k| < 1,\ k = 1.2$, the series in the previous system of equations are convergent.

Using (4.3) and taking into account 3.7, after some calculations, from (4.4)-(4.7) we obtain the following equations system:

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \times \begin{pmatrix} d_1^- \\ d_3^- \\ d_1^+ \\ d_1^+ \end{pmatrix} = \begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{matrix} \tag{4.8}$$

where

$$B_{11} = \frac{2}{p}[C\sinh(h) + \sinh(2h) + \sum_{k=1}^{2}\frac{A_k}{\lambda_k^2}L_1],$$

$$B_{21} = \frac{2}{p}[(1+\sum_{k=1}^{2}A_k - \sum_{k=1}^{2}\frac{A_k}{\lambda_k^2})\sinh(h) + C\sinh(2h) + \sinh(3h) + \sum_{k=1}^{2}\frac{A_k}{\lambda_k^3}L_1]$$

$$B_{12} = \frac{2}{p}[C(1-\cosh(h)) + 1 - \cosh(2h) + \sum_{k=1}^{2}\frac{A_k}{\lambda_k^2}L_5 - \sum_{k=1}^{2}\frac{A_k}{\lambda_k^2}L_2)],$$

$$B_{22}=\frac{2}{p}[(1+\sum_{k=1}^{2}A_k-\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2})(1-\cosh(h))+C(1-\cosh(2h))+1-\cosh(3h)$$

$$+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^3}L_5-\sum_{k=1}^{2}\frac{A_k}{\lambda_k^3}L_2]$$

$$B_{i3}=\frac{2}{p\ \cosh 1}\sum_{k=1}^{2}A_k{\lambda_k}^{N+i-1}L_1,\ i=1,2$$

$$B_{i4}=\frac{2}{p}[\sum_{k=1}^{2}A_k{\lambda_k}^{N+i-1}L_5-\frac{1}{\cosh 1}\sum_{k=1}^{2}A_k{\lambda_k}^{N+i-1}L_4],\qquad i=1,2$$

$$B_{(i+2)1}=\frac{2}{p}\sum_{k=1}^{2}A_k{\lambda_k}^{N+i-1}L_1,\ i=1,2$$

$$B_{(i+2)2}=\frac{2}{p}[\sum_{k=1}^{2}A_k{\lambda_k}^{N+i-1}L_5-\frac{1}{\cosh 1}\sum_{k=1}^{2}A_k{\lambda_k}^{N+i-1}L_2],,\ i=1,2$$

$$B_{33}=\frac{2}{p\ \cosh 1}[C\sinh(h)+\sinh(2h)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2}L_1],$$

$$B_{43}=\frac{2}{p\ \cosh 1}[(1+\sum_{k=1}^{2}A_k-\sum_{k=1}^{2}\frac{A_k}{\lambda_k^3})\sinh(h)+C\sinh(2h)+\sinh(3h)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^3}L_1],$$

$$B_{34}=\frac{2}{p}[C+1+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2}L_5-\frac{1}{\cosh 1}\{(C\cosh(h+1)+$$

$$+\cosh(2h+1)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2}L_4\}]$$

$$B_{44}=\frac{2}{p}[(1+\sum_{k=1}^{2}A_k-\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2})+C+1+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^3}L_5-\frac{1}{\cosh 1}\{(1+\sum_{k=1}^{2}A_k-\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2})\ \cosh(h+1)$$

$$+C\cosh(2h+1)+\cosh(3h+1)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^3}L_4\}]$$

$$T_1=-[f(0)\cdot(C\cosh(h)+\cosh(2h)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2}L_2)+(f(0)+\sum_{k=1}^{2}A_k\sum_{\gamma=0}^{N}\lambda_k^{\gamma}f(h\gamma))+$$

$$+\frac{f(1)}{\cosh(1)}\cdot\sum_{k=1}^{2}A_k\lambda_k^{N}L_4];$$

$$T_2=-[f(0)\cdot((1+\sum_{k=1}^{2}A_k-\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2})\cosh(h)+C\cosh(2h)+\cosh(3h)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^3}L_2)+$$

$$+\sum_{k=1}^{2}A_k\sum_{\gamma=0}^{N}\lambda_k^{\gamma+1}f(h\gamma)+\frac{f(1)}{\cosh(1)}\cdot\sum_{k=1}^{2}A_k\lambda_k^{N+1}L_4];$$

$$T_3 = -[f(0)\sum_{k=1}^{2} A_k\lambda_k^N L_2 + \sum_{k=1}^{2} A_k \sum_{\gamma=0}^{N} \lambda_k^{N-\gamma} f(h\gamma) + f(1)+$$

$$+\frac{f(1)}{\cosh(1)}(C\cosh(h+1)+\cosh(2h+1)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2}L_4)];$$

$$T_4 = -[f(0)\sum_{k=1}^{2} A_k\lambda_k^{N+1} L_2 + \sum_{k=1}^{2} A_k \sum_{\gamma=0}^{N} \lambda_k^{N+1-\gamma} f(h\gamma)+$$

$$+\frac{f(1)}{\cosh(1)}((1+\sum_{k=1}^{2}A_k-\sum_{k=1}^{2}\frac{A_k}{\lambda_k^2})\cosh(h+1)+C\cosh(2h+1)+\cosh(3h+1)+\sum_{k=1}^{2}\frac{A_k}{\lambda_k^3}L_4)].$$

Where $L_i, i=\overline{1,5}$ are given follow:

1) $L_1 = \sum\limits_{\gamma=1}^{\infty} {\lambda_k}^{\gamma}\sinh(h\gamma) = \frac{\lambda_k \sinh h}{1+{\lambda_k}^2-2\lambda_k\cosh h}$;

2)$L_2 = \sum\limits_{\gamma=1}^{\infty} {\lambda_k}^{\gamma}\cosh(h\gamma) = \frac{\lambda_k(\cosh h-\lambda_k)}{1+{\lambda_k}^2-2\lambda_k\cosh h}$;

3) $\sum\limits_{\gamma=1}^{\infty} {\lambda_k}^{\gamma}\sinh(h\gamma+1) = \frac{\lambda_k(\sinh(h+1)-\lambda_k\sinh 1)}{1+{\lambda_k}^2-2\lambda_k\cosh h}$;

4) $\sum\limits_{\gamma=1}^{\infty} {\lambda_k}^{\gamma}\cosh(h\gamma+1) = \frac{\lambda_k(\cosh(h+1)-\lambda_k\cosh 1)}{1+{\lambda_k}^2-2\lambda_k\cosh h}$

5) $\sum\limits_{\gamma=1}^{\infty} {\lambda_k}^{\gamma} = \frac{\lambda_k}{1-\lambda_k}$;

Combaining (4.8) and (4.3) we obtain $d_1^-,\ d_2^-,\ d_3^-,\ d_1^+,\ d_2^+,\ d_3^+$. Then we obtain d_1, d_2, d_3 which are given in the statement of theorem 4.1.

Now, We calculate the coefficients C_β, $\beta = 0, 1, 2..., N$. Taking into account (2.11) from (2.9) for C_β we get

$$C_\beta = D_3(h\beta) * u(h\beta) = \sum_{\gamma=1}^{\infty} D(h\beta-h\gamma)u(h\gamma) = \sum_{\gamma=1}^{\infty} D(h\beta+h\gamma)(-d_1^-\sinh(h\gamma)+d_2^-\cosh(h\gamma)+d_3^-)$$
$$+\sum_{\gamma=0}^{N} D(h\beta-h\gamma)f(h\gamma)+\sum_{\gamma=1}^{\infty} D(h(N+\gamma-\beta))(d_1^+\sinh(h\gamma+1)+d_2^+\cosh(h\gamma+1)+d_3^+).$$

From which, using (3.7) and taking into account notations (4.1), (4.2) , when $\beta = 0, 1, ..., N$, for C_β we obtain the expression given in (4.1).

5. Numerical results

In this section, using (4.1), we obtain several absolute errors graph in interpolating functions with the sixth order algebraic-hyperbolic spline, and we compare absolute errors of cubic spline and algebraic-hyperbolic in interpolating several functions. We denote the sixth order algebraic-hyperbolic natural spline as $S_6(x)$ and the cubic spline as $S_3(x)$.

Applying $S_6(x)$ with N=5,10, using (4.1) for the functions x^2 , $\cos(2x)$, $e^x + x$ we obtain absolute errors. The graphs of the corresponding absolute values of errors are presented in Figure 1 and Figure 2.

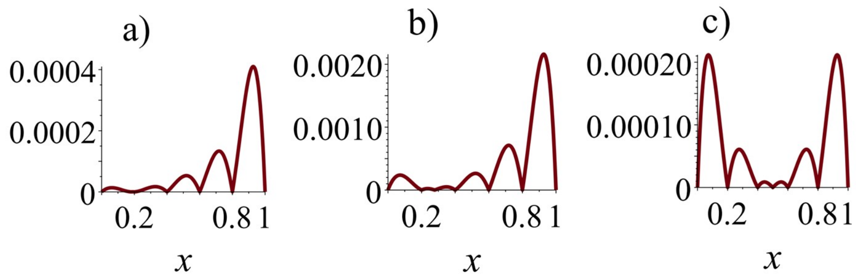

FIGURE 1. Graphs show absolute errors when sixth order algebraic-hyperbolic spline approximate x^2 , $\cos(2x)$, $e^x + x$ functions at N=5: a) $\left|x^2 - S_6(x)\right|$ b)$\left|\cos(2x) - S_6(x)\right|$ c) $\left|e^x + x - S_6(x)\right|$.

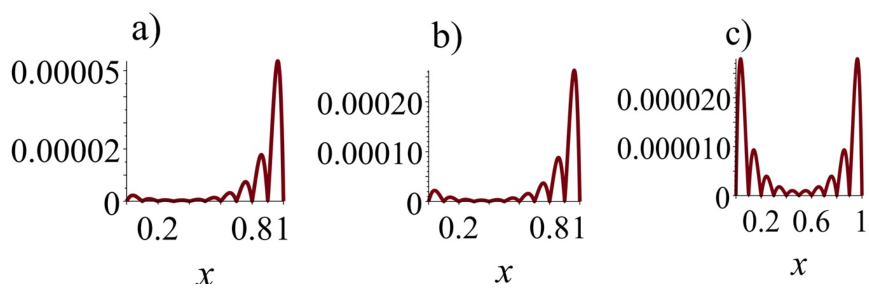

FIGURE 2. Graphs show absolute errors when sixth order algebraic-hyperbolic spline approximate x^2 , $\cos(2x)$, $e^x + x$ functions at N=10: a) $\left|x^2 - S_6(x)\right|$ b)$\left|\cos(2x) - S_6(x)\right|$ c) $\left|e^x + x - S_6(x)\right|$.

Now, we compare the graphs of the absolute errors of interpolating the functions $sin(x)$, $\frac{x^2}{2-x}$, e^x with sixth order algebraic-hyperbolic natural spline and cubic spline, where we get N=10.

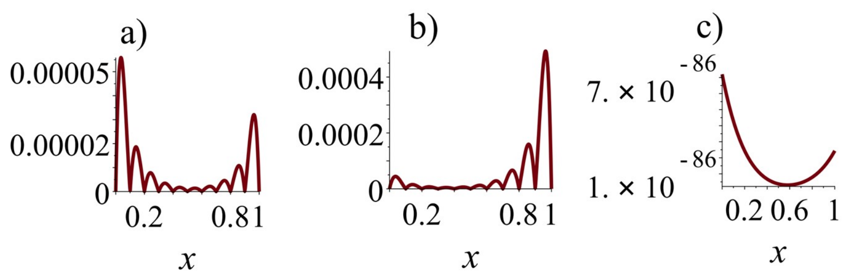

FIGURE 3. Graphs show absolute errors when sixth order algebraic-hyperbolic spline approximate $sin(x)$, $\frac{x^2}{2-x}$, e^x functions at N=10 a) $\left|sin(x) - S_6(x)\right|$ b) $\left|\frac{x^2}{2-x} - S_6(x)\right|$ c) $\left|e^x - S_6(x)\right|$.

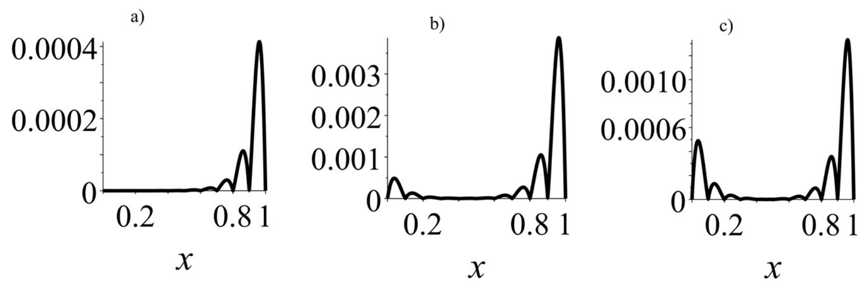

Figure 4. This Graphs show absolute errors when cubic spline approximate $sin(x)$, $\frac{x^2}{2-x}$, e^x functions at N=10 a) $\left|sin(x) - S_3(x)\right|$ b) $\left|\frac{x^2}{2-x} - S_3(x)\right|$ c) $\left|e^x - S_3(x)\right|$

Figure 1,2 and 3 show that it is possible to interpolate functions belonging tovarious classes with high accuracy using an algebraic hyperbolic spline. Figure 3 and Figure 4 show that the sixth order algebraic-hyperbolic spline gives several times better results than the cubic spline in interpolating functions.

6. Conclusion

In this work, we constructed an sixth order algebraic-hyperbolic interpolation, natural spline . To solve this problem, we used the Sobolev method and obtain a spline function for the approximate calculation of the unknown function. We first presented the interpolation spline function under which conditions gives a minimum to the norm in a certain Hilbert space. To find the coefficients of this spline, we created a system of equations based on certain conditions. We used Sobolev method and gave the algorithem to solve equations system. When we found the coefficients of the sixth order algebraic-hyperbolic interpolation natural siline, we obtain the exact expression of this spline. The absolute error in approximating functions with the sixth orderalgebraic-hyperbolic natural spline has been seen in the example of several functions and the absolute errors in approximating fununctions with the kubic spline and the spline we built are compared. The results show that, the spline we built approximates functions better than the cubic spline. We know, in many areas the kubic spline is widely used, This means that, we can take better results through the spline we have built.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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