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Mathematical Modeling and Numerical Simulation of Drug Consumption Dynamics in Burkina Faso

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Abstract. In this paper a mathematical model of drug consumption dynamics is proposed and analyzed. The model is based on the principle of the epidemiological model and takes into account the biological and environmental factors of exposed individuals, treatment and sensitization. The Jacobian determinant method is used to determine the basic reproduction function $\mathcal{R}_0$ of the model. The drug-free equilibrium points and the endemic equilibrium of the model were then identified, and their stabilities were analyzed based on the value of $\mathcal{R}_0$. A sensitivity analysis was performed to assess which parameters have the greatest influence on the dynamics of drug consumption. The numerical simulation was carried out using data from the Burkinabe population in 2020, aged between 11 and 65 years. The numerical results show that sensitization and treatment do not have much effect if the individual evolves in a favorable environment.

1. Introduction

Drugs can be defined as any substance of natural origin or obtained through synthesis that, when absorbed by a living organism, alters one or more of its functions [1]. Since the dawn of civilization, humans have used drugs for healing, pain relief, ritual purposes, and to alter their psyche and behavior [2]. Over the past 20 years, drug use has spread at an unprecedented rate, affecting all regions of the globe. This scourge has led to a concomitant worsening of health and social problems [3]. According to the United Nations Office on Drugs and Crime (UNODC), approximately 284 million people worldwide, aged 15 to 65, the majority of whom are men, are estimated to have used drugs throughout the year 2020. In Africa, the UNODC estimates that the number of drug users among young adults ranges from 22 to 72 million, with a prevalence rate of

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approximately 3.8 % to 12.5 % [1]. In Burkina Faso, about 122 tons of drugs were seized in 2021, compared to 240 tons in 2022 and over 300 tons in 2023. The Burkinabe government has intensified its efforts to combat drug use through initiatives from the National Committee for the Fight Against Drugs (CNLD). However, despite these efforts, the trend appears to indicate an increase in consumption. Most studies on drug use often focus on medical, sociological, and epidemiological aspects; however, in recent decades, mathematical modeling has increasingly been explored as an important tool to understand and mitigate issues related to drug consumption in various countries around the world. In 2007, White and Comiskey established one of the first discrete-event models of opioid dependence based on the principles of mathematical epidemiology [4]. They studied the dynamics using a threshold $\mathcal{R}_0$ and demonstrated that prevention is more effective than treatment. In 2009, the model by White and Comiskey was reconsidered by Mulone and Staughan [5]. They established the stability of the positive equilibrium point of the model using the eigenvalue equation and Poincaré-Bendixson theory. In 2011, Sánchez E, Villanueva Micó RJ, Santonja FJ, and Rubio M proposed a mathematical model to predict cocaine consumption in Spain [6]. They considered cocaine use as a socially transmissible epidemic disease that spreads through peer pressure or social contacts. In 2017, Isaac Mwangi Wangari and Lewi Stone formulated a heroin epidemic model with a saturated treatment function [7]. They based their work on the assumption that heroin use follows a process that can be modeled similarly to infectious diseases. In 2021, M. Chapwanya, J. M. S. Lubuma, H. Lutermann, A. Matusse, F. Nyabadza, and Y. Terefe proposed a mathematical model of cannabis epidemic in a South African province with a nonlinear incidence rate [8]. In June 2023, Moumine, Balatif, and Rachik proposed a SMHTR model that describes population dynamics and analyzes interactions between different classes of drug users [9]. In December 2023, Elbaz and El-Awady formulated a model of soft drug epidemics [10].

In the context of Burkina Faso, to our knowledge, there are no mathematical models on the dynamics of drug consumption. In this work, we present a mathematical model of the dynamics of drug consumption in Burkina Faso where we consider that drug consumption spreads like an epidemic disease, socially transmissible and which spreads through peer pressure or social contacts. We also consider the influence of biological and environmental factors such as births from drug-using parents or drug-using environments as sources of direct exposure to drug consumption. Taking these factors into account will thus make it possible to evaluate their real impact on drug consumption dynamics.

The rest of the work is orginized as follows: Section 2 is devoted to the presentation of the model. Section 3 is devoted to the mathematical analysis of the model in which the existence, uniqueness, and overall stability of the model are established. In Section 4, we make a sensitivity analysis of the model, the numerical simulation on data from the population of Burkina Faso in 2020, aged between 11 and 65 years. We end the work with a conclusion.

2. Model Formulation

In the proposed model, The total population is subdivided into seven different compartments: S, A, E, C_o, C_r, T, and R, whose descriptions are provided in the table 1

Table 1. Table of Model Variable Descriptions

Variable	Description
S	Represents the number of susceptible individuals in the population at a given time t. Here, all individuals are aged 11 years and above.
A	Represents the number of susceptible individuals who are aware of the harmful effects of drug consumption at a given time t
E	Represents the number of exposed individuals at a given time t ; That is, individuals who are in contact with occasional or regular users or those undergoing treatment, as well as individuals born to occasional or regular users or those undergoing treatment; but who do not consume drugs.
C_o	Represents the number of occasional drug users at a given time t; that is, individuals who frequently consume drugs but are not yet dependent.
C_r	Represents the number of regular drug users at a given time t; that is, individuals who are dependent on drug consumption.
T	Represents the number of drug users undergoing treatment at a given time t
R	Represents the number of drug users who have been successfully treated, as well as individuals who have voluntarily stopped using drugs through 'self-healing' at a given time t.
N	Represents the total population at a given time t. $(N = S + E + C_o + C_r + T + R)$.

We consider the following assumptions:

(H 1): The total population is constant within the modeling time period ; therefore, we find:

$$\Lambda = \mu N + \delta_1 C_o + \delta_2 C_r + \delta_3 T$$

(H 2): The population mixes homogeneously; that is, each individual in the population has an equal chance of encountering any other individual.

(H 3): Occasional and regular users are able to stop using drugs either through self-control or by undergoing treatment.

(H 4): Individuals who have successfully undergone detoxification, either through self-control or treatment, become temporarily immune to drug consumption.

(H 5): Individuals undergoing treatment still consume drugs.

(H 6): Occasional, regular, and individuals undergoing treatment can influence others to start using drugs.

(H 7): Drug users undergoing treatment may relapse into occasional or regular drug users due to severe withdrawal side effects or the high cost of treatment.

(H 8): Individuals born to occasional, regular, or treatment-receiving drug users are born exposed to drug consumption.

Under the assumptions below, We establish the following transmission diagram:

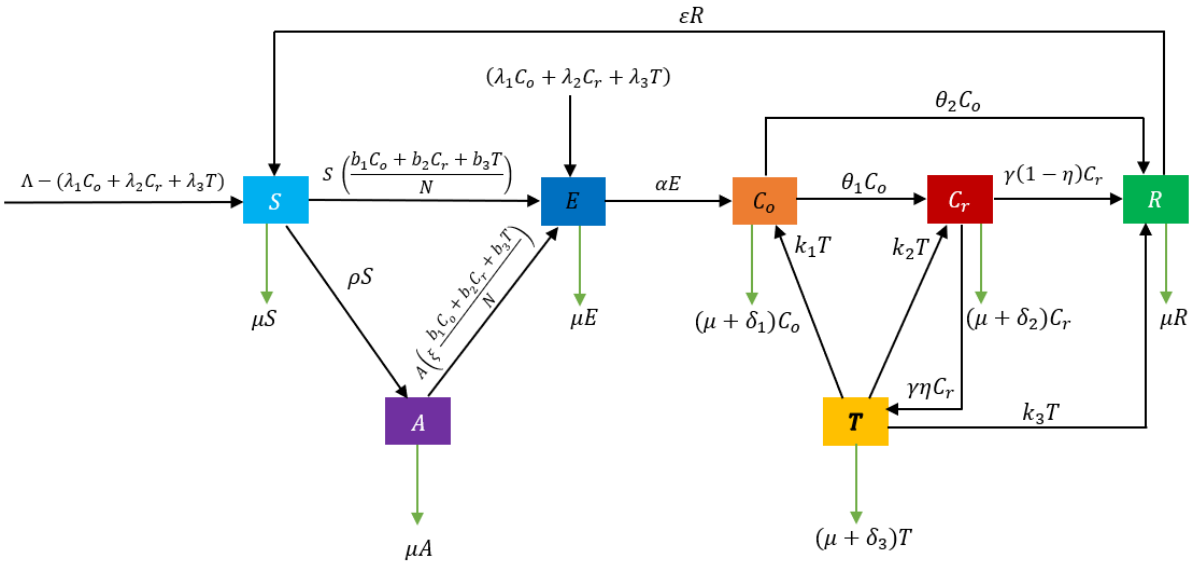


FIGURE 1. Epidemiological Model Diagram of Drug Consumption Dynamics with Awareness in Burkina Faso.

The different parameters of the model are described as follows:

TABLE 2. Table of Parameter Descriptions for the Model

Parameter	Description
Λ	Population recruitment rate (individuals reaching the age of 11 and above during the modeling period).
λ_1	Birth rate of children born to occasional drug users.
λ_2	Birth rate of children born to regular drug users.
λ_3	Birth rate of children born to drug users undergoing treatment.
μ	Natural mortality rate of the general population.
b_1	Contact rate between susceptible individuals and occasional drug users.
b_2	Contact rate between susceptible individuals and regular drug users.
b_3	Contact rate between susceptible individuals and drug users undergoing treatment.
ρ	Media campaign rate to increase population awareness.
ξ	Awareness failure rate.
α	Probability of exposed individuals becoming occasional drug users.
δ_1	Mortality rate related to occasional drug consumption.
δ_2	Mortality rate related to regular drug consumption.
δ_3	Mortality rate related to drug use among individuals undergoing treatment.
ε	Probability of drug users who have successfully undergone detoxification becoming susceptible again.
θ_1	Probability of occasional drug users becoming regular drug users.
θ_2	Probability of occasional drug users who have successfully undergone detoxification through self-control.
k_1	Probability of drug users undergoing treatment who relapse into occasional drug users.
k_2	Probability of drug users undergoing treatment who relapse into regular drug users.
k_3	Probability of drug users undergoing treatment who have successfully undergone detoxification.
γ	Progression rate of the compartment C_r to the compartments T and R.
η	Rate of regular drug users undergoing treatment.

- The compartment S: It is generated by a recruitment rate $\Lambda - (\lambda_1 C_o(t) + \lambda_2 C_r(t) + \lambda_3 T(t))$ and is increased due to individuals who have successfully undergone detoxification and become susceptible again at a rate $\varepsilon R(t)$. It is decreased due to contact with occasional, regular, or treatment drug users at a rate $S(t)\left(\frac{b_1 C_o(t)+b_2 C_r(t)+b_3 T(t)}{N}\right)$, by the success of awareness campaigns at a rate $\rho S(t)$, and by the natural mortality rate μ.
- The compartment A: It is increased by the success of awareness campaigns at a rate $\rho S(t)$. It is decreased by the failure of awareness at a rate $A(t)\left(\xi\frac{b_1 C_o(t)+b_2 C_r(t)+b_3 T(t)}{N}\right)$ and by the natural mortality rate μ.
- The compartment E: It is increased by the contact of susceptible individuals with occasional, regular, or treatment-seeking drug users at a rate of $S(t) \times \left(\frac{b_1 C_o(t)+b_2 C_r(t)+b_3 T(t)}{N}\right)$, by the failure of awareness campaigns at a rate of $A(t)\left(\xi\frac{b_1 C_o(t)+b_2 C_r(t)+b_3 T(t)}{N}\right)$, and also by vertical transmission at a rate of $(\lambda_1 C_o(t) + \lambda_2 C_r(t) + \lambda_3 T(t))$. It is decreased when some exposed individuals begin to consume drugs occasionally at a rate of $\alpha E(t)$ and also by the natural mortality rate μ.
- The compartment C_o: It is increased when some exposed individuals begin to consume drugs at a rate of $\alpha E(t)$ and also when some individuals fail their treatment at a rate of $k_1 T(t)$. It is decreased when some occasional consumers become regular consumers at a rate of $\theta_1 C_o(t)$ or when some occasional consumers decide to stop consuming drugs at a rate of $\theta_2 C_o(t)$. It is also decreased by the natural mortality rate μ and the mortality rate related to occasional drug consumption δ_1.
- The compartment C_r: It is increased by the rate $\theta_1 C_o(t)$ of occasional consumers who become regular consumers and by the rate $k_2 T(t)$ of consumers in treatment who relapse. It is decreased when some regular consumers either decide to undergo treatment at a rate of $\gamma\eta C_r(t)$, or stop consuming drugs through self-control at a rate of $\gamma(1-\eta)C_r(t)$, or due to the natural mortality rate μ and the mortality rate related to regular consumption δ_2.
- The compartment T: It is increased by the rate $\gamma\eta C_r(t)$ of regular consumers who decide to undergo treatment. It is decreased when individuals in treatment relapse into occasional consumers at a rate of $k_1 T(t)$ or into regular consumers at a rate of $k_2 T(t)$, or succeed in their detoxification at a rate of $k_3 T(t)$. It is also decreased by the natural mortality rate μ and the mortality rate related to drug use among consumers in treatment.
- The compartment R: It is increased either by the rate $\theta_2 C_o(t)$ of occasional consumers who decide to stop using drugs through self-control, or by the rate $\gamma(1-\eta)C_r(t)$ of regular consumers who also decide to stop using drugs through self-control, or by the rate $k_3 T(t)$ of drug users who have successfully completed their detoxification. It is decreased when certain cured individuals become susceptible again at a rate of $\varepsilon R(t)$ and by the natural mortality rate μ.

We obtain the mathematical model of drug consumption with awareness for the population of Burkina Faso represented by the following system of nonlinear ordinary differential equations:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - (\lambda_1 C_o(t) + \lambda_2 C_r(t) + \lambda_3 T(t)) - S(t)\left(\frac{b_1 C_o(t) + b_2 C_r(t) + b_3 T(t)}{N}\right) - (\mu + \rho) S(t) + \varepsilon R(t) \\ \frac{dA(t)}{dt} = \rho S(t) - A(t)\left(\xi \frac{b_1 C_o(t) + b_2 C_r(t) + b_3 T(t)}{N}\right) - \mu A(t) \\ \frac{dE(t)}{dt} = \left(\frac{b_1 C_o(t) + b_2 C_r(t) + b_3 T(t)}{N}\right)(S(t) + A(t)\xi) + (\lambda_1 C_o(t) + \lambda_2 C_r(t) + \lambda_3 T(t)) - (\alpha + \mu) E(t) \\ \frac{dC_o(t)}{dt} = \alpha E(t) + k_1 T(t) - (\mu + \delta_1 + \theta_1 + \theta_2) C_o(t) \\ \frac{dC_r(t)}{dt} = \theta_1 C_o(t) + k_2 T(t) - (\gamma + \mu + \delta_2) C_r(t) \\ \frac{dT(t)}{dt} = \gamma \eta C_r(t) - (k_1 + k_2 + k_3 + \mu + \delta_3) T(t) \\ \frac{dR(t)}{dt} = \theta_2 C_o(t) + \gamma(1 - \eta) C_r(t) + k_3 T(t) - (\varepsilon + \mu) R(t) \end{cases} \tag{2.1}$$

With the initial conditions $S(0), A(0), E(0), C_o(0), C_r(0), T(0)$, and $R(0)$ all being positive, and $N = S + A + E + C_o + C_r + T + R$ representing the total population.

To study the model, we begin by rewriting it in terms of proportions for simplicity. Thus, we perform the following variable changes as in [12]:

$$s = \frac{S}{N}; a = \frac{A}{N}; e = \frac{E}{N}; c_o = \frac{C_o}{N}; c_r = \frac{C_r}{N}; tr = \frac{T}{N}; r = \frac{R}{N}$$

with

$$0 \le s \le 1; 0 \le a \le 1; 0 \le e \le 1; 0 \le c_o \le 1; 0 \le c_r \le 1; 0 \le tr \le 1; 0 \le r \le 1.$$

For

$$\Lambda = \lambda N$$

We obtain the normalized system below:

$$\begin{cases} \frac{ds(t)}{dt} = \lambda - (\lambda_1 c_o(t) + \lambda_2 c_r(t) + \lambda_3 tr(t)) - s(t)(b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)) - (\mu + \rho) s(t) + \varepsilon r(t) \\ \frac{da(t)}{dt} = \rho s(t) - \xi a(t)\left(b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)\right) - \mu a(t) \\ \frac{de(t)}{dt} = \left(b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)\right)\left(s(t) + \xi a(t)\right) + (\lambda_1 c_o(t) + \lambda_2 c_r(t) + \lambda_3 tr(t)) - (\alpha + \mu) e(t) \\ \frac{dc_o(t)}{dt} = \alpha e(t) + k_1 tr(t) - (\mu + \delta_1 + \theta_1 + \theta_2) c_o(t) \\ \frac{dc_r(t)}{dt} = \theta_1 c_o(t) + k_2 tr(t) - (\gamma + \mu + \delta_2) c_r(t) \\ \frac{dtr(t)}{dt} = \gamma \eta c_r(t) - (k_1 + k_2 + k_3 + \mu + \delta_3) tr(t) \\ \frac{dr(t)}{dt} = \theta_2 c_o(t) + \gamma(1 - \eta) c_r(t) + k_3 tr(t) - (\varepsilon + \mu) r(t) \end{cases} \tag{2.2}$$

3. Mathematical Analysis

3.1. **Positivity, Boundedness, Existence, and Uniqueness of the Solution of the Model.**

Theorem 3.1. *If the initial value* $(s(0), a(0), e(0), c_o(0), c_r(0), tr(0), r(0)) \in \mathbb{R}^7_+$ *then there exists a unique, nonnegative solution to* (2.2) *for all* $t \ge 0$.

Proof. The system (2.2) is described by a system of nonlinear autonomous first-order differential equations. It can be rewritten in the following matrix form:

$$X'(t) = F\left(X(t)\right)$$

Where

$$X(t) = \begin{pmatrix} s(t) & a(t) & e(t) & c_o(t) & c_r(t) & tr(t) & r(t) \end{pmatrix}^t$$

And F is a function of class C^∞ on $\mathbb{R}^7$ with values in $\mathbb{R}^7$ defined by:

$$F\left(X(t)\right) = \begin{pmatrix} \lambda - (\lambda_1 c_o(t) + \lambda_2 c_r(t) + \lambda_3 tr(t)) - s(t)(b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)) - (\mu + \rho)s(t) + \varepsilon r(t) \\ \rho s(t) - \xi a(t)\left(b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)\right) - \mu a(t) \\ \left(b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)\right)\left(s(t) + \xi a(t)\right) + (\lambda_1 c_o(t) + \lambda_2 c_r(t) + \lambda_3 tr(t)) - (\alpha + \mu)e(t) \\ \alpha e(t) + k_1 tr(t) - (\mu + \delta_1 + \theta_1 + \theta_2)c_o(t) \\ \theta_1 c_o(t) + k_2 tr(t) - (\gamma + \mu + \delta_2)c_r(t) \\ \gamma\eta c_r(t) - (k_1 + k_2 + k_3 + \mu + \delta_3)tr(t) \\ \theta_2 c_o(t) + \gamma(1 - \eta)c_r(t) + k_3 tr(t) - (\varepsilon + \mu)r(t) \end{pmatrix}$$

Since F is of class C^1, it is therefore locally Lipschitz continuous on $\mathbb{R}^7$. This leads to the existence and uniqueness of the maximal solution to the Cauchy problem associated with the differential equation of the system (2.2) relative to the condition $(t_0; X(0)) \in \mathbb{R}_+ \times \mathbb{R}^7$.
Furthermore, since F is of class C^∞, we conclude that this solution is also of class C^∞.

Now, from the first equation of the system (2.2), It follows that [9]:

$$\begin{aligned} \frac{ds(t)}{dt} &= \lambda - (\lambda_1 c_o(t) + \lambda_2 c_r(t) + \lambda_3 tr(t)) - s(t)(b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)) - (\mu + \rho)s(t) + \varepsilon r(t) \\ &\geq -s(t)(b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)) - (\mu + \rho)s(t) \end{aligned}$$

Then

$$\frac{ds(t)}{dt} + f(t)s(t) \geq 0 \tag{3.1}$$

Where

$$f(t) = \mu + \rho + b_1 c_o(t) + b_2 c_r(t) + b_3 tr(t)$$

By multiplying both sides of the inequality (3.1) by $\exp\left(\int_0^t f(a)\,da\right)$, we obtain:

$$\exp\left(\int_0^t f(a)\,da\right)\frac{ds(t)}{dt} + f(t)\exp\left(\int_0^t f(a)\,da\right)s(t) \geq 0 \tag{3.2}$$

then

$$\frac{d}{dt}\left(\exp\left(\int_0^t f(a)\,da\right)s(t)\right) \geq 0 \tag{3.3}$$

By integrating the inequality (3.3) from 0 to t, we obtain:

$$s(t) \geq s(0)\exp\left(-\int_0^t f(a)\,da\right)$$

So, the solution $s(t)$ is positive.

To demonstrate the positivity of $a(t)$, $e(t)$, $c_o(t)$, $c_r(t)$, $tr(t)$ and $r(t)$ we will proceed by contradiction as in [9]:
Suppose that at least one of the variables $a(t)$, $e(t)$, $c_o(t)$, $c_r(t)$, $tr(t)$ and $r(t)$ is not positive. Then we are in one of the six (6) following cases:

1. There exists a first time t_1 such that $a(t_1) = 0; \frac{da(t_1)}{dt} < 0; e(t) > 0; c_o(t) > 0; c_r(t) > 0; tr(t) > 0; r(t) > 0$ for all $t \in [0, t_1]$.
2. There exists a first time t_2 such that $e(t_2) = 0; \frac{de(t_2)}{dt} < 0; a(t) > 0; c_o(t) > 0; c_r(t) > 0; tr(t) > 0; r(t) > 0$ for all $t \in [0, t_2]$
3. There exists a first time t_3 such that $c_o(t_3) = 0; \frac{dc_o(t_3)}{dt} < 0; a(t) > 0; e(t) > 0; c_r(t) > 0; tr(t) > 0; r(t) > 0$ for all $t \in [0, t_3]$
4. There exists a first time t_4 such that $c_r(t_4) = 0; \frac{dc_r(t_4)}{dt} < 0; a(t) > 0; e(t) > 0; c_o(t) > 0; tr(t) > 0; r(t) > 0$ for all $t \in [0, t_4]$
5. There exists a first time t_5 such that $tr(t_5) = 0; \frac{dtr(t_5)}{dt} < 0; a(t) > 0; e(t) > 0; c_o(t) > 0; c_r(t) > 0; r(t) > 0$ for all $t \in [0, t_5]$
6. There exists a first time t_6 such that $r(t_6) = 0; \frac{dr(t_6)}{dt} < 0; a(t) > 0; e(t) > 0; c_o(t) > 0; c_r(t) > 0; tr(t) > 0$ for all $t \in [0, t_6]$

In the first case, We have:

$$\frac{da(t_1)}{dt} = \rho s(t_1) > 0$$

which is absurd
In the second case, We have:

$$\frac{de(t_1)}{dt} = (b_1 c_o(t_2) + b_2 c_r(t_2) + b_3 tr(t_2))\,(s(t_2) + \xi a(t_2)) + (\lambda_1 c_o(t_2) + \lambda_2 c_r(t_2) + \lambda_3 tr(t_2)) > 0$$

which is absurd
In the third case, We have:

$$\frac{dc_o(t_3)}{dt} = \alpha e(t_3) + k_1 tr(t_3) > 0$$

which is absurd
In the fourth case, We have:

$$\frac{dc_r(t_4)}{dt} = \theta_1 c_o(t_4) + k_2 tr(t_4) > 0$$

which is absurd
In the fifth case, We have:

$$\frac{dtr(t_5)}{dt} = \gamma\eta c_r(t_5) > 0$$

which is absurd
In the sixth case, We have:

$$\frac{dr(t_6)}{dt} = \theta_2 c_o(t_6) + \gamma(1-\eta) c_r(t_6) + k_3 tr(t_6) > 0$$

which is absurd
Thus $a(t) > 0$; $e(t) > 0$; $c_o(t) > 0$; $c_r(t) > 0$; $tr(t) > 0$ and $r(t) > 0$ for all $t > 0$.

Hence, the solutions $s(t)$; $a(t)$; $e(t)$; $c_o(t)$; $c_r(t)$; $tr(t)$ and $r(t)$ of the system (2.2) are all positive for all $t > 0$ □

Theorem 3.2. *All feasible solutions of the system* (2.2) *are bounded and lie within the following region:*

$$\Omega = \{(s(t), a(t), e(t), c_o(t), c_r(t), tr(t), r(t)) \in \mathbb{R}^7_+ \mid 0 \leq s + a + e + c_o + c_r + tr + r \leq 1\}$$

Proof. Suppose that the initial value $(s(0), a(0), e(0), c_o(0), c_r(0), tr(0), r(0)) \in \mathbb{R}^7_+$. Let the function ϕ be defined as in [12] as:

$$\phi(t) = s(t) + a(t) + e(t) + c_o(t) + c_r(t) + tr(t) + r(t)$$

. By summing the seven equations of the system (2.2), we obtain:

$$\begin{cases} \frac{d\phi(t)}{dt} = \lambda - \mu\phi(t) - (\delta_1 c_o + \delta_2 c_r + \delta_3 tr) \\ \phi(0) = s(0) + a(0) + e(0) + c_o(0) + c_r(0) + tr(0) + r(0) = 1 \end{cases} \tag{3.4}$$

Integrating equation (3.4) over $(0, t)$ for all $0 < t < T$, one can get the following

$$\phi(t) \exp(\mu t) - 1 = \frac{\lambda - (\delta_1 c_o(t) + \delta_2 c_r(t) + \delta_3 tr(t))}{\mu} (\exp(\mu t) - 1)$$

Which implies that

$$\phi(t) = \exp(-\mu t) + \frac{\lambda - (\delta_1 c_o(t) + \delta_2 c_r(t) + \delta_3 tr(t))}{\mu} (1 - \exp(-\mu t))$$

Therefore

$$\phi(t) = \left(1 - \frac{\lambda - (\delta_1 c_o(t) + \delta_2 c_r(t) + \delta_3 tr(t))}{\mu}\right) \exp(-\mu t) + \frac{\lambda - (\delta_1 c_o(t) + \delta_2 c_r(t) + \delta_3 tr(t))}{\mu}$$

When $t \to \infty$ and $\lambda = \mu + \delta_1 c_o(t) + \delta_2 c_r(t) + \delta_3 tr(t)$ we have: $0 \leq n(t) \leq 1$. Thus, all possible solutions of the system (2.2) are within the region Ω. This implies that Ω is a positively invariant region of the system (2.2). Therefore, within the region Ω, we say that the system (2.2) is mathematically and epidemiologically well-posed. □

3.2. **Equilibrium without drugs and the basic reproduction number.**

3.2.1. *Drug-free equilibrium.*

Proposition 3.1. *The system* (2.2) *has a unique drug-free equilibrium point:*

$$X_0 = (\frac{\lambda}{\mu + \rho}; \frac{\rho\lambda}{\mu(\mu + \rho)}; 0; 0; 0; 0; 0)$$

Proof. At the drug-free equilibrium point, we have:

$$\frac{ds}{dt} = \frac{da}{dt} = \frac{de}{dt} = \frac{dc_o}{dt} = \frac{dc_r}{dt} = \frac{dtr}{dt} = \frac{dr}{dt} = 0 \quad \text{et} \quad e = c_o = c_r = tr = 0$$

Then, the system (2.2) becomes:

$$\begin{cases} \lambda - (\mu + \rho)s_0 + \varepsilon r_0 & = 0 \\ \rho s_0 - \mu a_0 & = 0 \\ -(\varepsilon + \mu)r_0 & = 0 \end{cases}$$

So,

$$\begin{cases} s_0 & = \frac{\lambda}{\mu+\rho} \\ a_0 & = \frac{\rho\lambda}{\mu(\mu+\rho)} \\ r_0 & = 0 \end{cases}$$

□

3.2.2. *The basic reproduction number.* In our epidemiological model of drug consumption, $\mathcal{R}_0$ is the average number of total individuals that each occasional or regular drug user, or individual undergoing treatment, will lead to consume drugs. In other words, $\mathcal{R}_0$ is the average number of new drug users resulting from the involvement of an occasional or regular user, or individual in treatment, among susceptible individuals.
To obtain the basic reproduction number for the system (2.2), we will use the Jacobian determinant method as described by B. Seidu *and al.* [13] described in the following aLgorithm [14]:

Algorithm 3.1.
Step 1: Identify the infected compartments of the model.
Step 2: Find the Jacobian $\mathcal{J}$ of the infected subsystem of the model.
Step 3: Evaluate the Jacobian of the infected subsystem at the disease-free equilibrium, ε_0 (i.e.$\mathcal{J}(\varepsilon_0)$)
Step 4: Find the determinant,$|\mathcal{J}(\varepsilon_0)$
Step 5: Express the determinant as $|\mathcal{J}(\varepsilon_0)| = \xi(\frac{B}{D} - 1), \quad \xi \in \mathbb{R}$
Step 6: Find $\mathcal{R}_0$ using $\mathcal{R}_0 = \frac{B}{D}$

where B is the part that contains the transmission factors such as probability of infection, contact rate or infectivity/susceptibility factors, and D is the part containing only transmission and other non-transmission terms.
The infected compartments are $e; c_o; c_r$ and tr ; thus, the infected subsystem is given by:

$$\begin{cases} \frac{de}{dt} & = (b_1c_o + b_2c_r + b_3tr)(s + a\xi) + (\lambda_1c_o + \lambda_2c_r + \lambda_3tr) - (\alpha + \mu)e \\ \frac{dc_o}{dt} & = \alpha e + k_1tr - (\mu + \delta_1 + \theta_1 + \theta_2)c_o \\ \frac{dc_r}{dt} & = \theta_1c_o + k_2tr - (\gamma + \mu + \delta_2)c_r \\ \frac{dtr}{dt} & = \gamma\eta c_r - (k_1 + k_2 + k_3 + \mu + \delta_3)tr \end{cases} \tag{3.5}$$

The Jacobian of the right-hand side of the infected subsystem (3.5) evaluated at the drug-free equilibrium is given by:

$$J(X_0) = \begin{pmatrix} -(\alpha+\mu) & \frac{\mu+\rho\xi}{\mu+\rho}b_1+\lambda_1 & \frac{\mu+\rho\xi}{\mu+\rho}b_2+\lambda_2 & \frac{\mu+\rho\xi}{\mu+\rho}b_3+\lambda_3 \\ \alpha & -(\mu+\delta_1+\theta_1+\theta_2) & 0 & k_1 \\ 0 & \theta_1 & -(\gamma+\mu+\delta_2) & k_2 \\ 0 & 0 & \gamma\eta & -(k_1+k_2+k_3+\mu+\delta_3) \end{pmatrix}$$

By setting $q_{33} = \alpha+\mu$, $q_{44} = \mu+\delta_1+\theta_1+\theta_2$, $q_{55} = \gamma+\mu+\delta_2$, $q_{66} = k_1+k_2+k_3+\mu+\delta_3$, $\beta = \mu+\rho\xi$ and $\pi = \mu+\rho$ the matrix J becomes:

$$J(X_0) = \begin{pmatrix} -q_{33} & \frac{\beta}{\pi}b_1+\lambda_1 & \frac{\beta}{\pi}b_2+\lambda_2 & \frac{\beta}{\pi}b_3+\lambda_3 \\ \alpha & -q_{44} & 0 & k_1 \\ 0 & \theta_1 & -q_{55} & k_2 \\ 0 & 0 & \gamma\eta & -q_{66} \end{pmatrix}$$

The determinant of the matrix J evaluated at X_0 gives:

$$\begin{aligned} |J(X_0)| = \frac{1}{\pi}\,(&q_{33}q_{44}q_{55}q_{66}\pi - q_{33}q_{44}\pi\gamma\eta k_2 - q_{33}\pi\gamma\eta k_1\theta_1 - \alpha q_{44}q_{66}\beta b_1 - \alpha q_{55}q_{66}\pi\lambda_1 \\ &-\alpha q_{55}\beta b_2\theta_1 - \alpha q_{66}\pi\lambda_2\theta_1 + \alpha\beta\gamma\eta b_1 k_2 - \alpha\beta\gamma\eta b_3\theta_1 + \alpha\pi\gamma\eta k_2\lambda_1 - \alpha\pi\gamma\eta\lambda_3\theta_1) \end{aligned}$$

This can be written as follows:

$$\begin{aligned} |J(X_0)| = &-(q_{33}q_{44}q_{55}q_{66} - q_{33}q_{44}\gamma\eta k_2 - q_{33}\gamma\eta k_1\theta_1) \\ &\times\left(\frac{\alpha\,(q_{55}q_{66}(\beta b_1+\pi\lambda_1) + q_{66}\theta_1(\beta b_2+\pi\lambda_2) - \gamma\eta k_2(\beta b_1+\pi\lambda_1) + \gamma\eta\theta_1(\beta b_3+\pi\lambda_3))}{q_{33}\pi\,(q_{44}q_{55}q_{66} - q_{44}\gamma\eta k_2 - \gamma\eta k_1\theta_1)} - 1\right) \end{aligned}$$

Here we have

$$\epsilon = -(q_{33}q_{44}q_{55}q_{66} - q_{33}q_{44}\gamma\eta k_2 - q_{33}\gamma\eta k_1\theta_1)$$

and the basic reproduction number $\mathcal{R}_0$ is obtained as follows:

$$\mathcal{R}_0 = \frac{\alpha\,(q_{55}q_{66}(\beta b_1+\pi\lambda_1) + q_{66}\theta_1(\beta b_2+\pi\lambda_2) - \gamma\eta k_2(\beta b_1+\pi\lambda_1) + \gamma\eta\theta_1(\beta b_3+\pi\lambda_3))}{q_{33}\pi\,(q_{44}q_{55}q_{66} - q_{44}\gamma\eta k_2 - \gamma\eta k_1\theta_1)}$$

3.3. **Endemic equilibrium.** Considering that $q_{33} = \alpha+\mu$; $q_{44} = \mu+\delta_1+\theta_1+\theta_2$; $q_{55} = \gamma+\mu+\delta_2$; $q_{66} = k_1+k_2+k_3+\mu+\delta_3$; and letting $q_{77} = \varepsilon+\mu$ the system (2.2) at endemic equilibrium becomes:

$$\begin{cases} \lambda - (\lambda_1 c_o^* + \lambda_2 c_r^* + \lambda_3 t r^*) - s^*(b_1 c_o^* + b_2 c_r^* + b_3 t r^*) - (\mu+\rho)s^* + \varepsilon r^* = 0 \\ \rho s^* - \xi a^*\,(b_1 c_o^* + b_2 c_r^* + b_3 t r^*) - \mu a^* = 0 \\ (b_1 c_o^* + b_2 c_r^* + b_3 t r^*)\,(s^* + \xi a^*) + (\lambda_1 c_o^* + \lambda_2 c_r^* + \lambda_3 t r^*) - q_{33} e^* = 0 \\ \alpha e^* + k_1 t r^* - q_{44} c_o^* = 0 \\ \theta_1 c_o^* + k_2 t r^* - q_{55} c_r^* = 0 \\ \gamma\eta c_r^* - q_{66} t r^* = 0 \\ \theta_2 c_o^* + \gamma(1-\eta) c_r^* + k_3 t r^* - q_{77} r^* = 0 \end{cases} \tag{3.6}$$

Where s^* ; a^* ; e^* ; c_o^* ; c_r^* ; tr^* ; r^* represents the points of endemic equilibrium. By solving the system (3.6), we obtain:

$$\begin{aligned}
tr^* &= \frac{\gamma\eta}{q_{66}}c_r^*;\\
c_0^* &= \frac{q_{55}q_{66}-k_2\gamma\eta}{q_{66}\theta_1}c_r^*;\\
r^* &= \frac{q_{55}q_{66}\theta_2-k_2\theta_2\gamma\eta+q_{66}\theta_1\gamma(1-\eta)+k_3\gamma\eta\theta_1}{q_{66}q_{77}\theta_1}c_r^*;\\
e^* &= \frac{q_{44}q_{55}q_{66}-q_{44}k_2\gamma\eta-k_1\theta_1\gamma\eta}{q_{66}\theta_1\alpha}c_r^*;\\
a^* &= \frac{\rho\theta_1q_{66}\left[\alpha\beta u_1+\pi u_3q_{33}(1-\mathcal{R}_0)\right]}{\alpha\pi u_1(\xi u_1c_r^*+\mu\theta_1q_{66}+\xi\theta_1\rho q_{66})};\\
s^* &= \frac{\left[\alpha\beta u_1+\pi u_3q_{33}(1-\mathcal{R}_0)\right](\xi u_1c_r^*+\mu\theta_1q_{66})}{\alpha\pi u_1(\xi u_1c_r^*+\mu\theta_1q_{66}+\xi\theta_1\rho q_{66})}.
\end{aligned}$$

with c_r^* being the positive solution of the equation $-A(c_r^*)^2+Bc_r+C=0$; where

$$\begin{aligned}
u_1 =& b_1(q_{55}q_{66}-k_2\gamma\eta)+b_2\theta_1q_{66}+b_3\gamma\eta\theta_1\\
u_2 =& \lambda_1(q_{55}q_{66}-k_2\gamma\eta)+\lambda_2\theta_1q_{66}+\lambda_3\gamma\eta\theta_1\\
u_3 =& q_{44}q_{55}q_{66}-q_{44}k_2\gamma\eta-k_1\theta_1\gamma\eta\\
u_4 =& q_{55}q_{66}\theta_2-k_2\theta_2\gamma\eta+q_{66}\theta_1\gamma(1-\eta)+k_3\gamma\eta\theta_1\\
A =& \alpha\pi\xi u_1^2u_2q_{77}+\alpha\beta\xi u_1^3q_{77}+\pi\xi u_1^2u_3q_{33}q_{77}(1-\mathcal{R}_0)\\
B =& \lambda\alpha\pi\xi\theta_1u_1^2q_{66}q_{77}-\mu\alpha\pi\theta_1u_1u_2q_{66}q_{77}-\alpha\pi\xi\theta_1\rho u_1u_2q_{66}-\mu\alpha\beta\theta_1u_1^2q_{66}q_{77}\\
&-\mu\pi\theta_1u_1u_3q_{33}q_{66}q_{77}(1-\mathcal{R}_0)-\alpha\beta\xi(\mu+\rho)\theta_1u_1^2q_{66}q_{77}-\pi\xi(\mu+\rho)\theta_1u_1u_3q_{33}q_{66}q_{77}(1-\mathcal{R}_0)\\
&+\alpha\pi\xi^2u_1^2u_4\\
C =& \lambda\alpha\pi\theta_1^2u_1q_{66}^2q_{77}(\mu+\xi\rho)-\mu\alpha\beta(\mu+\rho)\theta_1^2u_1q_{66}^2q_{77}-\mu\pi(\mu+\rho)\theta_1^2u_3q_{33}q_{66}^2q_{77}(1-\mathcal{R}_0)\\
&+\mu\alpha\pi\xi\theta_1u_1u_4q_{66}+\alpha\pi\xi^2\theta_1\rho u_1u_4q_{66}.
\end{aligned}$$

Thus, for $\mathcal{R}_0>1$, there exists a unique endemic equilibrium $X_1=(s^*;a^*;e^*;c_o^*;c_r^*;tr^*;r^*)$.

3.4. **Stability of Equilibria.**

3.4.1. *Local and global stability of the drug-free equilibrium.*

Theorem 3.3. *The drug-free equilibrium point X_0 of the system* (2.2) *is locally asymptotically stable if $\mathcal{R}_0<1$ and the following conditions are satisfied:*

$$\begin{gathered}
\sigma_1\sigma_2>\sigma_3\\
(\sigma_1\sigma_2-\sigma_3)\,\sigma_3>(\sigma_1\sigma_4-\sigma_5)\,\sigma_1\\
\left[(\sigma_1\sigma_2-\sigma_3)\,\sigma_3-(\sigma_1\sigma_4-\sigma_5)\,\sigma_1\right](\sigma_1\sigma_4-\sigma_5)>\left[(\sigma_1\sigma_2-\sigma_3)\,\sigma_5-(\sigma_1\sigma_6-\sigma_7)\,\sigma_1\right](\sigma_1\sigma_2-\sigma_3)
\end{gathered}\tag{3.7}$$

$$a_1\left[\left(\sigma_1\sigma_2-\sigma_3\right)\sigma_5-\left(\sigma_1\sigma_6-\sigma_7\right)\sigma_1\right]>a_2\left[\left(\sigma_1\sigma_2-\sigma_3\right)\sigma_3-\left(\sigma_1\sigma_4-\sigma_5\right)\sigma_1\right]$$

$$a_3a_2>q_1^2\left(\sigma_1\sigma_2-\sigma_3\right)\sigma_7$$

Where:

$$a_1=\left[\left(\sigma_1\sigma_2-\sigma_3\right)\sigma_3-\left(\sigma_1\sigma_4-\sigma_5\right)\sigma_1\right]\left(\sigma_1\sigma_4-\sigma_5\right)-\left[\left(\sigma_1\sigma_2-\sigma_3\right)\sigma_5-\left(\sigma_1\sigma_6-\sigma_7\right)\sigma_1\right]\left(\sigma_1\sigma_2-\sigma_3\right)$$

$$a_2=\left[\left(\sigma_1\sigma_2-\sigma_3\right)\sigma_3-\left(\sigma_1\sigma_4-\sigma_5\right)\sigma_1\right]\left(\sigma_1\sigma_6-\sigma_7\right)-\left(\sigma_1\sigma_2-\sigma_3\right)^2\sigma_7$$

$$a_3=a_1\left[\left(\sigma_1\sigma_2-\sigma_3\right)\sigma_5-\left(\sigma_1\sigma_6-\sigma_7\right)\sigma_1\right]-a_2\left[\left(\sigma_1\sigma_2-\sigma_3\right)\sigma_3-\left(\sigma_1\sigma_4-\sigma_5\right)\sigma_1\right]$$

And σ_1, σ_2, σ_3, σ_4, σ_5, σ_6 and σ_7 are defined in 3.9 and 3.10.

Proof. Considering that $q_{33}=\alpha+\mu$; $q_{44}=\mu+\delta_1+\theta_1+\theta_2$; $q_{55}=\gamma+\mu+\delta_2$; $q_{66}=k_1+k_2+k_3+\mu+\delta_3$; $q_{75}=\gamma(1-\eta)$; $q_{77}=\varepsilon+\mu$ and letting $q_{14}=\lambda_1+\frac{b_1\lambda}{\mu+\rho}$; $q_{15}=\lambda_2+\frac{b_2\lambda}{\mu+\rho}$; $q_{16}=\lambda_3+\frac{b_3\lambda}{\mu+\rho}$; $q_{24}=\frac{b_1\xi\rho\lambda}{\mu+\rho}$; $q_{25}=\frac{b_2\xi\rho\lambda}{\mu+\rho}$; $q_{26}=\frac{b_3\xi\rho\lambda}{\mu+\rho}$; $q_{34}=\frac{b_1\lambda(\mu+\xi\rho)}{\mu(\mu+\rho)}+\lambda_1$; $q_{35}=\frac{b_2\lambda(\mu+\xi\rho)}{\mu(\mu+\rho)}+\lambda_2$ and $q_{36}=\frac{b_3\lambda(\mu+\xi\rho)}{\mu(\mu+\rho)}+\lambda_3$ then the Jacobian matrix of the right-hand side of the system (2.2) evaluated at the drug-free equilibrium point X_0 gives:

$$J(X_0)=\begin{pmatrix}-q_{11}&0&0&-q_{14}&-q_{15}&-q_{16}&\varepsilon\\ \rho&-\mu&0&-q_{24}&-q_{25}&-q_{26}&0\\ 0&0&-q_{33}&q_{34}&q_{35}&q_{36}&0\\ 0&0&\alpha&-q_{44}&0&k_1&0\\ 0&0&0&\theta_1&-q_{55}&k_2&0\\ 0&0&0&0&\gamma\eta&-q_{66}&0\\ 0&0&0&\theta_2&q_{75}&k_3&-q_{77}\end{pmatrix}$$

The eigenvalues of the characteristic equation of $J(X_0)$ are the solutions of the equation 3.8

$$\chi^7+\sigma_1\chi^6+\sigma_2\chi^5+\sigma_3\chi^4+\sigma_4\chi^3+\sigma_5\chi^2+\sigma_6\chi+\sigma_7 \tag{3.8}$$

Where:

$$\begin{aligned}\sigma_1=&q_{77}+q_{66}+q_{55}+q_{44}+q_{33}+\mu+q_{11};\\ \sigma_2=&-\eta\gamma k_2-\alpha q_{34}+\mu q_{11}+\mu q_{33}+\mu q_{44}+\mu q_{55}+\mu q_{66}+\mu q_{77}+q_{11}q_{33}+q_{11}q_{44}+q_{11}q_{55}+q_{11}q_{66}+q_{11}q_{77}+q_{33}q_{44}\\ &+q_{33}q_{55}+q_{33}q_{66}+q_{33}q_{77}+q_{44}q_{55}+q_{44}q_{66}+q_{44}q_{77}+q_{55}q_{66}+q_{55}q_{77}+q_{66}q_{77}\\ \sigma_3=&-\eta\gamma\mu k_2-\eta\gamma k_1\theta_1-\eta\gamma k_2q_{11}-\eta\gamma k_2q_{33}-\eta\gamma k_2q_{44}-\eta\gamma k_2q_{77}-\alpha\mu q_{34}-\alpha q_{11}q_{34}-\alpha q_{34}q_{55}-\alpha q_{34}q_{66}-\alpha q_{34}q_{77}\\ &-\alpha q_{35}\theta_1+\mu q_{11}q_{33}+\mu q_{11}q_{44}+\mu q_{11}q_{55}+\mu q_{11}q_{66}+\mu q_{11}q_{77}+\mu q_{33}q_{44}+\mu q_{33}q_{55}+\mu q_{33}q_{66}+\mu q_{33}q_{77}+\mu q_{44}q_{55}\\ &+\mu q_{44}q_{66}+\mu q_{44}q_{77}+\mu q_{55}q_{66}+\mu q_{55}q_{77}+\mu q_{66}q_{77}+q_{11}q_{33}q_{44}+q_{11}q_{33}q_{55}+q_{11}q_{33}q_{66}+q_{11}q_{33}q_{77}+q_{11}q_{44}q_{55}\\ &+q_{11}q_{44}q_{66}+q_{11}q_{44}q_{77}+q_{11}q_{55}q_{66}+q_{11}q_{55}q_{77}+q_{11}q_{66}q_{77}+q_{33}q_{44}q_{55}+q_{33}q_{44}q_{66}+q_{33}q_{44}q_{77}+q_{33}q_{55}q_{66}\\ &+q_{33}q_{55}q_{77}+q_{33}q_{66}q_{77}+q_{44}q_{55}q_{66}+q_{44}q_{55}q_{77}+q_{44}q_{66}q_{77}+q_{55}q_{66}q_{77}\\ \sigma_4=&\alpha\eta\gamma k_2q_{34}-\alpha\eta\gamma q_{36}\theta_1-\eta\gamma\mu k_1\theta_1-\eta\gamma\mu k_2q_{11}-\eta\gamma\mu k_2q_{33}-\eta\gamma\mu k_2q_{44}-\eta\gamma\mu k_2q_{77}-\eta\gamma k_1q_{11}\theta_1-\eta\gamma k_1q_{33}\theta_1\\ &-\eta\gamma k_1q_{77}\theta_1-\eta\gamma k_2q_{11}q_{33}-\eta\gamma k_2q_{11}q_{44}-\eta\gamma k_2q_{11}q_{77}-\eta\gamma k_2q_{33}q_{44}-\eta\gamma k_2q_{33}q_{77}-\eta\gamma k_2q_{44}q_{77}-\alpha\mu q_{11}q_{34}\\ &-\alpha\mu q_{34}q_{55}-\alpha\mu q_{34}q_{66}-\alpha\mu q_{34}q_{77}-\alpha\mu q_{35}\theta_1-\alpha q_{11}q_{34}q_{55}-\alpha q_{11}q_{34}q_{66}-\alpha q_{11}q_{34}q_{77}-\alpha q_{11}q_{35}\theta_1-\alpha q_{34}q_{55}q_{66}\\ &-\alpha q_{34}q_{55}q_{77}-\alpha q_{34}q_{66}q_{77}-\alpha q_{35}q_{66}\theta_1-\alpha q_{35}q_{77}\theta_1+\mu q_{11}q_{33}q_{44}+\mu q_{11}q_{33}q_{55}+\mu q_{11}q_{33}q_{66}+\mu q_{11}q_{33}q_{77}\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&+\mu q_{11}q_{44}q_{55}+\mu q_{11}q_{44}q_{66}+\mu q_{11}q_{44}q_{77}+\mu q_{11}q_{55}q_{66}+\mu q_{11}q_{55}q_{77}+\mu q_{11}q_{66}q_{77}+\mu q_{33}q_{44}q_{55}+\mu q_{33}q_{44}q_{66}\\
&+\mu q_{33}q_{44}q_{77}+\mu q_{33}q_{55}q_{66}+\mu q_{33}q_{55}q_{77}+\mu q_{33}q_{66}q_{77}+\mu q_{44}q_{55}q_{66}+\mu q_{44}q_{55}q_{77}+\mu q_{44}q_{66}q_{77}+\mu q_{55}q_{66}q_{77}\\
&+q_{11}q_{33}q_{44}q_{55}+q_{11}q_{33}q_{44}q_{66}+q_{11}q_{33}q_{44}q_{77}+q_{11}q_{33}q_{55}q_{66}+q_{11}q_{33}q_{55}q_{77}+q_{11}q_{33}q_{66}q_{77}+q_{11}q_{44}q_{55}q_{66}\\
&+q_{11}q_{44}q_{55}q_{77}+q_{11}q_{44}q_{66}q_{77}+q_{11}q_{55}q_{66}q_{77}+q_{33}q_{44}q_{55}q_{66}+q_{33}q_{44}q_{55}q_{77}+q_{33}q_{44}q_{66}q_{77}\\
&+q_{33}q_{55}q_{66}q_{77}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_5 =&\alpha\eta\gamma\mu k_2q_{34}-\alpha\eta\gamma\mu q_{36}\theta_1+\alpha\eta\gamma k_2q_{11}q_{34}+\alpha\eta\gamma k_2q_{34}q_{77}-\alpha\eta\gamma q_{11}q_{36}\theta_1-\alpha\eta\gamma q_{36}q_{77}\theta_1-\eta\gamma\mu k_1q_{11}\theta_1-\eta\gamma\mu k_1q_{33}\theta_1\\
&-\eta\gamma\mu k_1q_{77}\theta_1-\eta\gamma\mu k_2q_{11}q_{33}-\eta\gamma\mu k_2q_{11}q_{44}-\eta\gamma\mu k_2q_{11}q_{77}-\eta\gamma\mu k_2q_{33}q_{44}-\eta\gamma\mu k_2q_{33}q_{77}-\eta\gamma\mu k_2q_{44}q_{77}\\
&-\eta\gamma k_1q_{11}q_{33}\theta_1-\eta\gamma k_1q_{11}q_{77}\theta_1-\eta\gamma k_1q_{33}q_{77}\theta_1-\eta\gamma k_2q_{11}q_{33}q_{44}-\eta\gamma k_2q_{11}q_{33}q_{77}-\eta\gamma k_2q_{11}q_{44}q_{77}-\eta\gamma k_2q_{33}q_{44}q_{77}\\
&-\alpha\mu q_{11}q_{34}q_{55}-\alpha\mu q_{11}q_{34}q_{66}-\alpha\mu q_{11}q_{34}q_{77}-\alpha\mu q_{11}q_{35}\theta_1-\alpha\mu q_{34}q_{55}q_{66}-\alpha\mu q_{34}q_{55}q_{77}-\alpha\mu q_{34}q_{66}q_{77}\\
&-\alpha\mu q_{35}q_{66}\theta_1-\alpha\mu q_{35}q_{77}\theta_1-\alpha q_{11}q_{34}q_{55}q_{66}-\alpha q_{11}q_{34}q_{55}q_{77}-\alpha q_{11}q_{34}q_{66}q_{77}-\alpha q_{11}q_{35}q_{66}\theta_1-\alpha q_{11}q_{35}q_{77}\theta_1\\
&-\alpha q_{34}q_{55}q_{66}q_{77}-\alpha q_{35}q_{66}q_{77}\theta_1+\mu q_{11}q_{33}q_{44}q_{55}+\mu q_{11}q_{33}q_{44}q_{66}+\mu q_{11}q_{33}q_{44}q_{77}+\mu q_{11}q_{33}q_{55}q_{66}+\mu q_{11}q_{33}q_{55}q_{77}\\
&+\mu q_{11}q_{33}q_{66}q_{77}+\mu q_{11}q_{44}q_{55}q_{66}+\mu q_{11}q_{44}q_{55}q_{77}+\mu q_{11}q_{44}q_{66}q_{77}+\mu q_{11}q_{55}q_{66}q_{77}+\mu q_{33}q_{44}q_{55}q_{66}+\mu q_{33}q_{44}q_{55}q_{77}\\
&+\mu q_{33}q_{44}q_{66}q_{77}+\mu q_{33}q_{55}q_{66}q_{77}+\mu q_{44}q_{55}q_{66}q_{77}+q_{11}q_{33}q_{44}q_{55}q_{66}+q_{11}q_{33}q_{44}q_{55}q_{77}+q_{11}q_{33}q_{44}q_{66}q_{77}\\
&+q_{11}q_{33}q_{55}q_{66}q_{77}+q_{11}q_{44}q_{55}q_{66}q_{77}+q_{33}q_{44}q_{55}q_{66}q_{77}\\
\sigma_6 =&\alpha\eta\gamma\mu k_2q_{11}q_{34}+\alpha\eta\gamma\mu k_2q_{34}q_{77}-\alpha\eta\gamma\mu q_{11}q_{36}\theta_1-\alpha\eta\gamma\mu q_{36}q_{77}\theta_1+\alpha\eta\gamma k_2q_{11}q_{34}q_{77}-\alpha\eta\gamma q_{11}q_{36}q_{77}\theta_1-\eta\gamma\mu k_1q_{11}q_{33}\theta_1
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&-\eta\gamma\mu k_1q_{11}q_{77}\theta_1-\eta\gamma\mu k_1q_{33}q_{77}\theta_1-\eta\gamma\mu k_2q_{11}q_{33}q_{44}-\eta\gamma\mu k_2q_{11}q_{33}q_{77}-\eta\gamma\mu k_2q_{11}q_{44}q_{77}-\eta\gamma\mu k_2q_{33}q_{44}q_{77}\\
&-\eta\gamma k_1q_{11}q_{33}q_{77}\theta_1-\eta\gamma k_2q_{11}q_{33}q_{44}q_{77}-\alpha\mu q_{11}q_{34}q_{55}q_{66}-\alpha\mu q_{11}q_{34}q_{55}q_{77}-\alpha\mu q_{11}q_{34}q_{66}q_{77}-\alpha\mu q_{11}q_{35}q_{66}\theta_1\\
&-\alpha\mu q_{11}q_{35}q_{77}\theta_1-\alpha\mu q_{34}q_{55}q_{66}q_{77}-\alpha\mu q_{35}q_{66}q_{77}\theta_1-\alpha q_{11}q_{34}q_{55}q_{66}q_{77}-\alpha q_{11}q_{35}q_{66}q_{77}\theta_1+\mu q_{11}q_{33}q_{44}q_{55}q_{66}\\
&+\mu q_{11}q_{33}q_{44}q_{55}q_{77}+\mu q_{11}q_{33}q_{44}q_{66}q_{77}+\mu q_{11}q_{33}q_{55}q_{66}q_{77}+\mu q_{11}q_{44}q_{55}q_{66}q_{77}+\mu q_{33}q_{44}q_{55}q_{66}q_{77}+q_{11}q_{33}q_{44}q_{55}q_{66}q_{77}
\end{aligned}
$$

$$
\sigma_7 =q_{77}\mu q_{11}\left(\alpha\eta\gamma k_2q_{34}-\alpha\eta\gamma q_{36}\theta_1-\eta\gamma k_1q_{33}\theta_1-\eta\gamma k_2q_{33}q_{44}-\alpha q_{34}q_{55}q_{66}-\alpha q_{35}q_{66}\theta_1+q_{33}q_{44}q_{55}q_{66}\right)
$$

we have $\sigma_1 > 0$; Thus, the equilibrium point without drugs of the system (2.2) is locally asymptotically stable if and only if the Routh-Hurwitz criterion is satisfied, that is, the conditions 3.7 are satisfied [15]. □

Theorem 3.4. *Drug-free equilibrium X_0 of the system* (2.2) *is globally asymptotically stable if $\mathcal{R}_0 \leq 1$.*

Proof. As in [10], we introduce the following Lyapunov function:

$$
\mathcal{V} = s - s_0 - s_0\ln(\frac{s}{s_0}) + a - a_0 - a_0\ln(\frac{a}{a_0}) + e + c_o + c_r + tr + r
$$

Then the derivative of $\mathcal{V}$ gives:

$$
\begin{aligned}
\dot{\mathcal{V}} &= \dot{s} + \dot{a} + \dot{e} + \dot{c_o} + \dot{c_r} + \dot{tr} + \dot{r} - \frac{s_0}{s}\dot{s} - \frac{a_0}{a}\dot{a}\\
&= \lambda - \mu(s + a + e + c_o + c_r + tr + r) - (\delta_1c_o + \delta_2c_r + \delta_3tr) - \frac{s_0}{s}\left[\lambda - (\lambda_1c_o + \lambda_2c_r + \lambda_3tr)\right.\\
&\left.-s\left(b_1c_o + b_2c_r + b_3tr\right) - (\mu + \rho)s + \varepsilon r\right] - \frac{a_0}{a}\left[\rho s - \xi a\left(b_1c_o + b_2c_r + b_3tr\right) - \mu a\right]
\end{aligned}
$$

For $\mathcal{R}_0 \leq 1$, and $\lambda = (\mu + \rho)s_0$, we have:

$$\begin{aligned}\dot{\mathcal{V}} =&\mu s_0(2 - \frac{s}{s_0} - \frac{s_0}{s}) + \rho s_0(2 - \frac{s_0}{s} - \frac{a_0 s}{a s_0}) + \mu a_0(1 - \frac{a}{a_0}) - \mu e - \mu r - \varepsilon r\frac{s_0}{s} + c_o(s_0 + \xi a_0)\left(b_1 - \frac{\mu}{s_0 + \xi a_0}\right)\\ &+ c_r(s_0 + \xi a_0)\left(b_2 - \frac{\mu}{s_0 + \xi a_0}\right) + tr(s_0 + \xi a_0)\left(b_3 - \frac{\mu}{s_0 + \xi a_0}\right) + \frac{c_o s_0}{s}\left(\lambda_1 - \frac{\delta_1 s}{s_0}\right) + \frac{c_r s_0}{s}\left(\lambda_2 - \frac{\delta_2 s}{s_0}\right)\\ &+ \frac{trs_0}{s}\left(\lambda_3 - \frac{\delta_3 s}{s_0}\right)\end{aligned}$$

Since the arithmetic mean is greater than or equal to the geometric mean, then $\dot{\mathcal{V}} < 0$ and $\dot{\mathcal{V}} = 0$ only if $s = s_0$; $a = a_0$; $e = e_0$; $c_o = c_o^0$; $c_r = c_r^0$; $tr = tr_0$ and $r = r_0$.

Hence drug-free equilibrium X_0 is globally asymptotically stable according to LaSalle's invariance principle [15]. □

3.4.2. *Global stability of endemic equilibrium.*

Theorem 3.5. *If $\mathcal{R}_0 > 1$, then the endemic equilibrium point X_1 of the system* (2.2) *is globally asymptotically stable.*

Proof. Regarding the global stability of X_1, let us consider the following Lyapunov function:

$$\begin{aligned}\mathcal{V} = &s - s^* - s^* \ln\frac{s}{s^*} + a - a^* - a^* \ln\frac{a}{a^*} + e - e^* - e^* \ln\frac{e}{e^*} + c_o - c_o^* - c_o^* \ln\frac{c_o}{c_o^*} + c_r - c_r^* - c_r^* \ln\frac{c_r}{c_r^*}\\ &+ tr - tr^* - tr^* \ln\frac{tr}{tr^*} + r - r^* - r^* \ln\frac{r}{r^*}\end{aligned}$$

Then the derivative of $\mathcal{V}$ gives:

$$\dot{\mathcal{V}} = \dot{s} + \dot{a} + \dot{e} + \dot{c_o} + \dot{c_r} + \dot{tr} + \dot{r} - \frac{s^*}{s}\dot{s} - \frac{a^*}{a}\dot{a} - \frac{e^*}{e}\dot{e} - \frac{c_o^*}{c_o}\dot{c_o} - \frac{c_r^*}{c_r}\dot{c_r} - \frac{tr^*}{tr}\dot{tr} - \frac{r^*}{r}\dot{r}$$

For $\mathcal{R}_0 > 1$, and $\lambda = \mu(s^* + a^* + e^* + c_o^* + c_r^* + tr^* + r^*) + \delta c_o^* + \delta_2 c_r^* + \delta_3 tr^*$, we have:

$$\begin{aligned}\dot{\mathcal{V}} =&\mu s^*\left(2 - \frac{s}{s^*} - \frac{s^*}{s}\right) + \rho s^*\left(1 - \frac{a^* s}{a s^*}\right) + \mu a^*\left(2 - \frac{a}{a^*} - \frac{s^*}{s}\right) + \mu e^*\left(2 - \frac{e}{e^*} - \frac{s^*}{s}\right) + \mu c_o^*\left(2 - \frac{c_o}{c_o^*} - \frac{s^*}{s}\right)\\ &+ \mu c_r^*\left(2 - \frac{c_r}{c_r^*} - \frac{s^*}{s}\right) + \mu tr^*\left(2 - \frac{tr}{tr^*} - \frac{s^*}{s}\right) + \mu r^*\left(2 - \frac{r}{r^*} - \frac{s^*}{s}\right) + \delta_1 c_o^*\left(2 - \frac{c_o}{c_o^*} - \frac{s^*}{s}\right)\\ &+ \delta_2 c_r^*\left(2 - \frac{c_r}{c_r^*} - \frac{s^*}{s}\right) + \delta_3 tr^*\left(2 - \frac{tr}{tr^*} - \frac{s^*}{s}\right) + s^*\left(\frac{b_1 c_o + b_2 c_r + b_3 tr}{N}\right)\left(1 - \frac{e^* s}{e s^*}\right)\\ &+ \xi a^*\left(\frac{b_1 c_o + b_2 c_r + b_3 tr}{N}\right)\left(1 - \frac{e^* a}{e a^*}\right) + \frac{s^*}{s}(\lambda_1 c_o + \lambda_2 c_r + \lambda_3 tr)\left(1 - \frac{e^* s}{e s^*}\right) + \alpha e^*\left(1 - \frac{c_o^* e}{c_o e^*}\right)\\ &+ \theta_1 c_o^*\left(1 - \frac{c_r^* c_o}{c_r c_o^*}\right) + \theta_2 c_o^*\left(1 - \frac{r^* c_o}{r c_o^*}\right) + \gamma\eta c_r^*\left(1 - \frac{tr^* c_r}{tr c_r^*}\right) + \gamma(1-\eta)c_r^*\left(1 - \frac{r^* c_r}{r c_r^*}\right) + k_1 tr^*\left(1 - \frac{c_o^* tr}{c_o tr^*}\right)\\ &+ k_2 tr^*\left(1 - \frac{c_r^* tr}{c_r tr^*}\right) + k_3 tr^*\left(1 - \frac{r^* tr}{r tr^*}\right) + \varepsilon r^*\left(1 - \frac{s^* r}{s r^*}\right).\end{aligned}$$

Since the arithmetic mean is greater than or equal to the geometric mean, then $\dot{\mathcal{V}} < 0$ and $\dot{\mathcal{V}} = 0$ only if $s = s^*$; $a = a^*$; $e = e^*$; $c_o = c_o^*$; $c_r = c_r^*$; $tr = tr^*$; $r = r^*$.

Hence endemic equilibrium X_1 is globally asymptotically stable according to LaSalle's invariance principle ([16], [17]). □

4. Sensitivity Analysis and Numerical Simulation

A sensitivity analysis of the model (2.2) is conducted to determine the relative importance of the model parameters on the propagation of drug consumption. This analysis is crucial for identifying parameters that have a significant impact on $\mathcal{R}_0$ and should be targeted by intervention strategies. The parameter with a higher sensitivity index is more influential than one with a lower sensitivity index. The sign of the sensitivity indices of $\mathcal{R}_0$ concerning the parameters indicates the positive or negative impact of these parameters. Here, we calculate the sensitivity index for each parameter included in the expression of $\mathcal{R}_0$ except for the parameters $\mu, \delta_1, \delta_2, \delta_3$ which cannot be targeted in intervention strategies. The standard equation for the sensitivity index of a parameter Φ of $\mathcal{R}_0$ is given by as in ([16], [18], [19]):

$$\chi_{\Phi}^{\mathcal{R}_0} = \frac{\Phi}{\mathcal{R}_0} \times \frac{\partial \mathcal{R}_0}{\partial \Phi}$$

Given the complexity of the expression for $\mathcal{R}_0$, we utilized numerical differentiation. Thus, the numerical values of the sensitivity indices are provided in the table 4.

The values of the parameters are given in the table below 3.

Parameter	Value	Reference
λ_1	0.03	Assumed
λ_2	0.08	Assumed
λ_3	0.002	Assumed
μ	0.009	[11]
b_1	0.03	[11]
b_2	0.02	[11]
b_3	0.001	[11]
ρ	0.03	Assumed
ξ	0.05	Assumed
α	0.07	Assumed
δ_1	0.01636	[6]
δ_2	0.059	[11]
δ_3	0.01636	[6]
ε	0.008	[11]
θ_1	0.05	Assumed
θ_2	0.02	[11]
k_1	0.2	[7]
k_2	0.15	[7]
k_3	0.1	[7]
γ	0.008	Assumed
η	0.9999	Assumed

Table 3. Table of values of the model parameters

Sensitivity Index	Value
$\chi_{\lambda_1}^{\mathcal{R}_0}$	0.317222
$\chi_{\lambda_2}^{\mathcal{R}_0}$	0.559462
$\chi_{\lambda_3}^{\mathcal{R}_0}$	0.000224
$\chi_{b_1}^{\mathcal{R}_0}$	0.085406
$\chi_{b_2}^{\mathcal{R}_0}$	0.037656
$\chi_{b_3}^{\mathcal{R}_0}$	0.000030
$\chi_{\alpha}^{\mathcal{R}_0}$	0.113924
$\chi_{\theta_1}^{\mathcal{R}_0}$	0.089412
$\chi_{\theta_2}^{\mathcal{R}_0}$	-0.216949
$\chi_{k_1}^{\mathcal{R}_0}$	0.008167
$\chi_{k_2}^{\mathcal{R}_0}$	0.006920
$\chi_{k_3}^{\mathcal{R}_0}$	-0.013123
$\chi_{\gamma}^{\mathcal{R}_0}$	-0.028859
$\chi_{\eta}^{\mathcal{R}_0}$	0.054709
$\chi_{\rho}^{\mathcal{R}_0}$	-0.077102
$\chi_{\xi}^{\mathcal{R}_0}$	0.017585

TABLE 4. Sensitivity Indices of the Parameters of the Second Model

Figure 2 shows the graphical representation of the sensitivity indices.

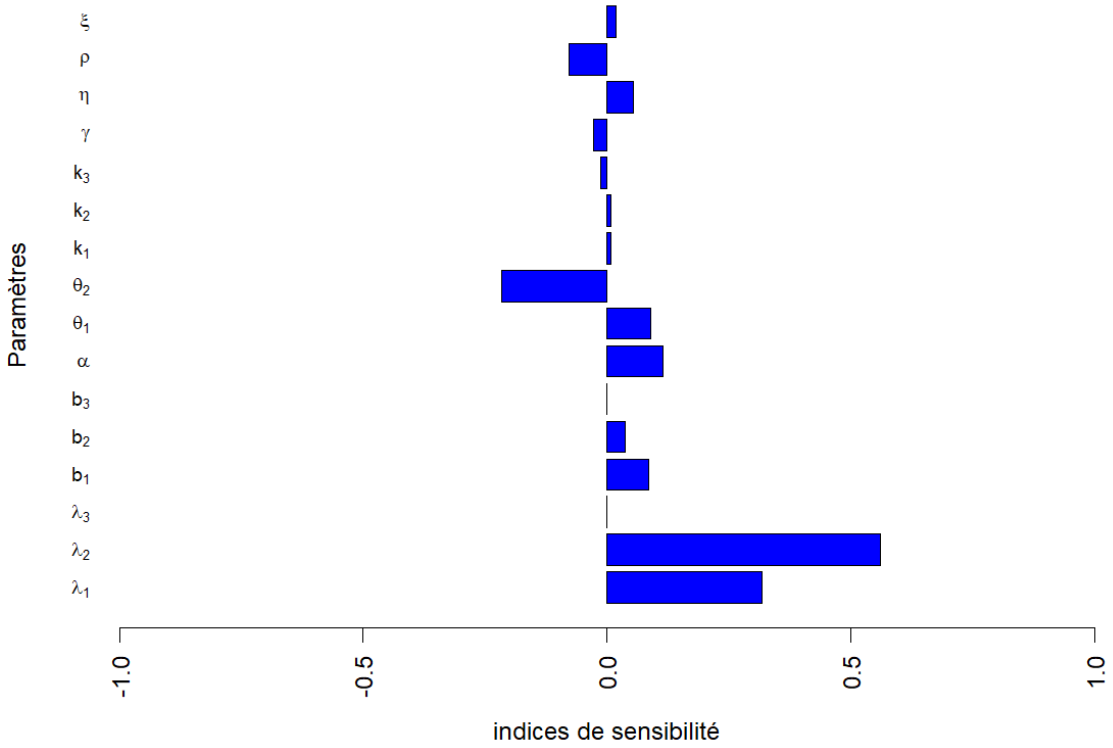
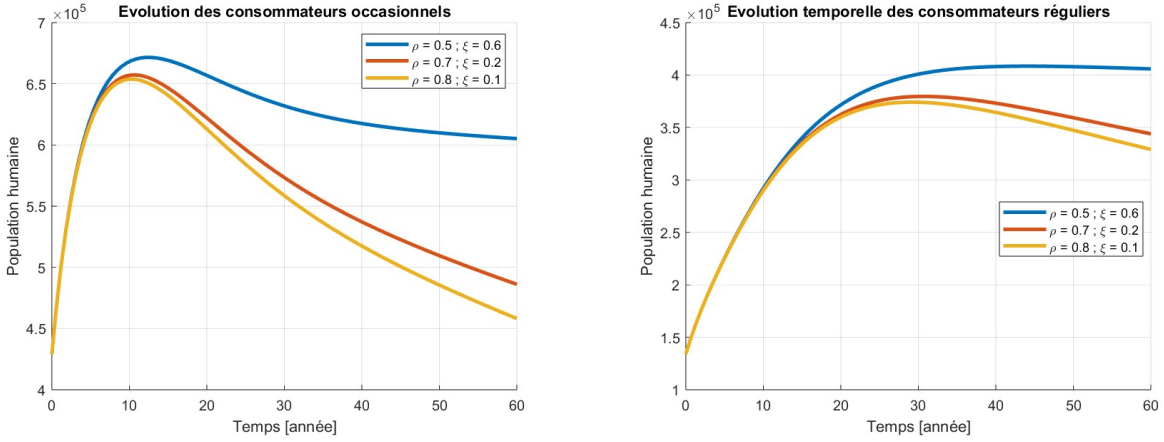


FIGURE 2. Sensitivity analysis of the model parameters

Figure 2 clearly illustrates the impact of each parameter on $\mathcal{R}_0$. We observe that $\mathcal{R}_0$ is most positively sensitive to changes in the parameters λ_1, λ_2 and α. An increase in the value of any of these parameters will result in a proportional increase in $\mathcal{R}_0$ (similarly, a decrease in the value of any of these parameters will lead to an equivalent decrease in $\mathcal{R}_0$). The parameter θ_2 has an inversely proportional relationship with $\mathcal{R}_0$. An increase in the value of θ_2 will result in a decrease in $\mathcal{R}_0$ and while a decrease in the value of θ_2 will lead to an increase in $\mathcal{R}_0$. However, reducing the rate of exposed individuals becoming occasional consumers α does not contribute positively. It is therefore preferable to focus efforts on reducing the birth rate of individuals born to occasional or regular consumer parents (λ_1 and λ_2) and on increasing the success rate of detoxification through self-control for occasional consumers θ_2. Since $\mathcal{R}_0$ is more sensitive to variations in λ_1 and λ_2 than to θ_2, it is wise to concentrate on λ_1 and λ_2, the birth rates from consumer parents, to control drug abuse within the population.

4.1. **Numerical Simulation.** In this section, we perform some numerical simulations to illustrate the theoretical results obtained in the previous sections. To do this, we used MATLAB software with the fourth-order Runge-Kutta method. We present the impact of awareness based on the birth rates from drug-using parents on the total population aged 11 to 65 years in Burkina Faso in 2020, which was estimated to be $N = 13407908$ inhabitants $(INSD2023)$. We assume the following initial conditions: $S = 10458168; A = 938554; E = 1340791; C_o = 429053; C_r = 134079; T = 93855; R = 13408$.

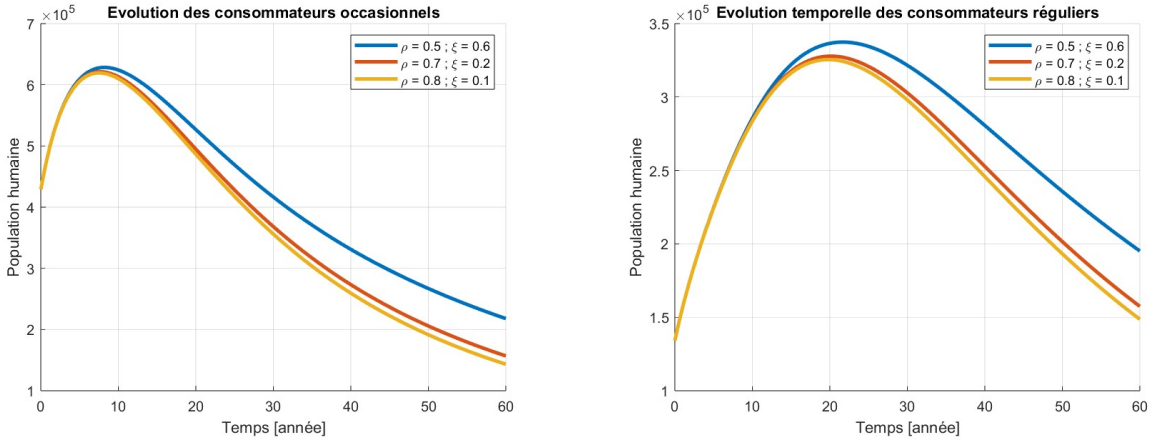


(A) Evolution of Occasional Drug Users

(B) Evolution of Regular Drug Users

FIGURE 3. Temporal Evolution of Consumers for $\lambda_1 = 0.03$, $\lambda_2 = 0.08$ and $\lambda_3 = 0.002$

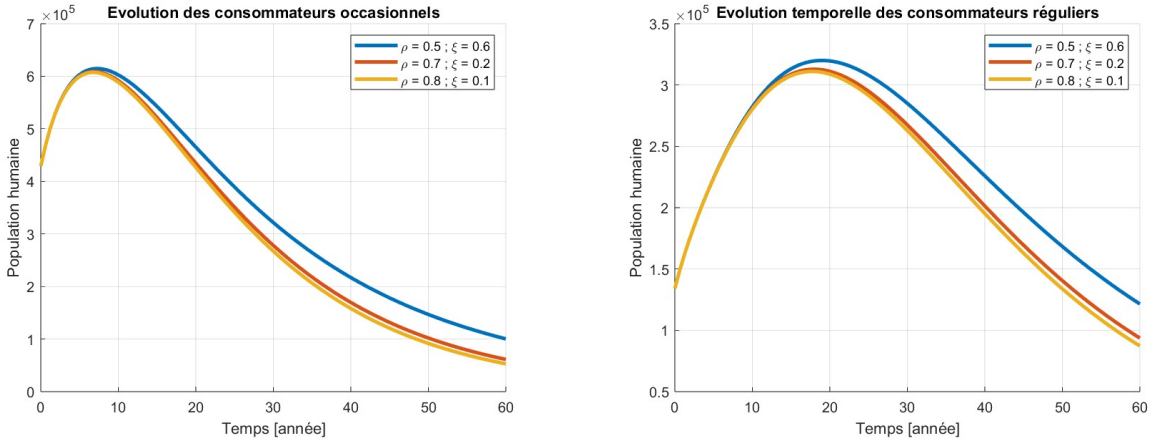
Figure 3 above presents the temporal evolution of occasional consumers 3a and regular consumers 3b. Here, we have varied the success and failure rates of the awareness campaign. We observe a decrease in consumption when the success rate is high and the failure rate is low, as also indicated by the values of $\mathcal{R}_0$: for $\rho = 0.5$, $\xi = 0.6$, $\mathcal{R}_0 = 1.0494$; for $\rho = 0.7$, $\xi = 0.2$, $\mathcal{R}_0 = 0.8844$ and for $\rho = 0.8$, $\xi = 0.1$, $\mathcal{R}_0 = 0.8428$



(A) Evolution of Occasional Drug Users (B) Evolution of Regular Drug Users

FIGURE 4. Temporal Evolution of Consumers for $\lambda_1 = 0.01$, $\lambda_2 = 0.03$ et $\lambda_3 = 0.001$

Figure 4 above presents the temporal evolution of occasional consumers 4a and regular consumers 4b. Here, we have reduced the birth rates of individuals from drug-consuming parents and varied the success and failure rates of the awareness campaign. We observe a significant decrease in consumption, as also indicated by the values of $\mathcal{R}_0$: for $\rho = 0.5$, $\xi = 0.6$, $\mathcal{R}_0 = 0.5392$; for $\rho = 0.7$, $\xi = 0.2$, $\mathcal{R}_0 = 0.3743$ and for $\rho = 0.8$, $\xi = 0.1$, $\mathcal{R}_0 = 0.3326$



(A) Evolution of Occasional Drug Users (B) Evolution of Regular Drug Users

FIGURE 5. Temporal Evolution of Consumers for $\lambda_1 = 0.001$, $\lambda_2 = 0.002$ et $\lambda_3 = 0.0008$

Figure 5 above presents the temporal evolution of occasional consumers 5a and regular consumers 5b. Here, we have further reduced the birth rates of individuals from drug-consuming parents and varied the success and failure rates of the awareness campaign. We observe a very

significant decrease in consumption, as also indicated by the values of $\mathcal{R}_0$: for $\rho = 0.5$, $\xi = 0.6$, $\mathcal{R}_0 = 0.2747$; for $\rho = 0.7$, $\xi = 0.2$, $\mathcal{R}_0 = 0.1097$ and for $\rho = 0.8$, $\xi = 0.1$, $\mathcal{R}_0 = 0.0681$.
Thus, these numerical simulations demonstrate that awareness has no significant effect on drug dynamics if the children are in a favorable environment.

5. Conclusion

In this article, we proposed a deterministic model of drug consumption dynamics with awareness. This allowed us to understand the dynamics of consumption and evaluate the impacts of awareness campaigns on the population. The analytical results show that the model is mathematically significant and defined within the positive region Ω. We established the conditions for the existence of equilibrium states of the model and found that at the drug-free equilibrium point, the model is stable if $\mathcal{R}_0 \leq 1$ and at the endemic equilibrium point, the model is stable if $\mathcal{R}_0 > 1$. The sensitivity analysis and numerical simulation have allowed us to understand that the most influential parameters are those related to biological and genetic dispositions. This indicates that the environment in which an individual evolves plays a crucial role in the propagation of drug use. Awareness efforts have minimal impact on drug dynamics if an individual evolves in a favorable environment. Hence, there is a pressing need to educate parents about the harmful effects of their consumption on their children.

In the future, we plan to use a stochastic model of drug consumption dynamics in Burkina Faso for a better understanding of the epidemic.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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