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Complex Fuzzy Dynamical Graphs and their Applications in Signals Processing

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Abstract. In this paper we introduce the concepts of complex fuzzy dynamic graphs, complex fuzzy diagonal matrices and complex fuzzy Laplacian matrices. We use these graphs and their laplacian matrices as mathematical framework for applications in Sciences, especially signals processing. We define absolute average eigenvalues of the Complex Laplacian matrices and explore the properties of these matrices with their eigenvalues. We develop an algorithm using the absolute eigenvalues of the Laplacian matrices and apply this algorithm to signal and systems. Our study begins by establishing the theoretical foundation of complex fuzzy dynamic graphs, highlighting their role to model within dynamic systems including two dimensional uncertainties. We investigates the complex fuzzy Laplacian matrices obtain from these graphs. Our main focus is on the absolute eigenvalues of these matrices, which hold a vital role into the graph's structural characteristics and behavior. In the context of signals processing, the research demonstrates how these absolute eigenvalues serve as essential matrices for system characterization. This study presents novel methods to analyze signals on complex fuzzy dynamic graphs. These methods are particularly relevant in scenarios where signals are influenced by dynamic and uncertain environments.

1. Introduction

The mathematical models play an invaluable role in the study of complex systems, where uncertainties and dynamic interactions are involved. The study of complex fuzzy dynamic graphs, complex fuzzy sets, complex fuzzy Laplacian matrices and the absolute eigenvalues, provide an elegant platform to represent and analyze the complex relationships within dynamic systems. This amalgamation possess the power of complex fuzzy logic to handle ambiguous and vague information and to illustrate complex interconnections. In this context, the analysis of complex fuzzy Laplacian matrices, particularly emphasizing on their absolute eigenvalues, holds a huge

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significance. Understanding the eigenvalue spectrum provides deep insights into the structure, stability, and behavior of complex systems represented by these graphs. Moreover, the application of these mathematical constructs in the domain of signals processing intensifies their practical role, enabling efficient extraction and analysis of information from real-world data.

The roots of complex fuzzy sets can be traced back to the pioneering work of Lotfi A. Zadeh in the 1960s [8], where he introduced fuzzy sets as a mathematical framework to handle uncertainty and vagueness. Fuzzy sets allow to represent of imprecise information using degrees of membership, providing a way out from the binary nature of classical set theory. Over the years, this concept evolved, incorporating complex numbers to handle both the uncertainty and ambiguity present in various real-world applications [15]. The mixing of complex concepts into fuzzy sets paved the way for the development of complex fuzzy sets, which have found applications in diverse fields including control systems, decision-making, signals and pattern recognition [1–7]. Simultaneously, the study of graphs provide mathematical structures consisting of nodes connected by edges and are used to model relationships between different entities. Traditional graphs, however, do not possess the capability to handle imprecise and uncertain data. With the advent of fuzzy logic, complex fuzzy graphs emerged as an extension of classical graphs, where edges and/or nodes are assigned complex fuzzy values, allowing the representation of complex fuzzy relationships involving two dimensional data, for instance the edge of friendship between two friends involves two dimensions, one representing the strength of friendship and other representing the emotions in friendship. In recent decades, researchers have delved into the synergy between complex numbers and fuzzy logic, leading to the formulation of complex fuzzy graphs [9]- [14]. Complex fuzzy graphs incorporate the nuances of both complex numbers and fuzzy logic, enabling the representation of dynamic, uncertain relationships with a high degree of accuracy and flexibility. These graphs offer a more realistic representation of complex systems where uncertainties, imprecision, and dynamic interactions are prevalent [16]- [18]. This study explores into the complex systems of complex fuzzy dynamic graphs, exploring their theoretical foundations, the characteristics of their Laplacian matrices, and their applications in signals processing. By investigating the absolute average eigenvalues of these matrices, the study aims to unravel the underlying structures of complex systems. Furthermore, the research extends its focus to practical applications, demonstrating how these mathematical constructs can be harnessed to process and interpret signals in real-world scenarios. Through this exploration, a deeper understanding of complex systems and their manifestations in signals will be achieved, paving the way for innovative solutions in fields ranging from telecommunications to biological signal analysis.

In this paper we are emphasizing one practical applications of the proposed methodologies with various aspects. Through extensive simulations and real-world case studies, the effectiveness of the presented approaches in extracting relevant information from complex signals is validated. Additionally, the research investigates the robustness of these methods in the presence of noise and uncertainties, showcasing their applicability in practical, noisy environments. This research contributes not only to the theoretical advancement of complex fuzzy dynamic graphs and their Laplacian matrices but also to the broader field of signals processing. The insights derived from this study have the potential to revolutionize how we understand and analyze complex systems, making it invaluable for researchers and practitioners working at the intersection of graph theory, fuzzy mathematics, and signals processing.

All fuzzy sets deal with only one dimensional data involving certain uncertainties. To deal with two dimensional data in 2002, Ramot et al. [5] introduced a new concept of a complex fuzzy set. Mathematically complex fuzzy sets is denoted and defined as:

$$\mu_S(t) = r_S(x)e^{\iota\omega_S(t)}$$

where $r_S(t)$ amplitude term representing fuzzy sets while $\omega_S(t)$ is a real valued function. Therefore $\mu_S(t)$ is a complex valued function whose range is the unit disc. There is an addition of one more dimension in it as compare to the traditional fuzzy set. The phase term addresses the wave type phenomenon of this set which makes it more applicable than the fuzzy set.

2. Complex Fuzzy Sets and Complex Fuzzy Laplacian Matrices

A modulus of complex fuzzy set is given as

$$|\mu_S(t)| = |r_S(x)e^{\iota\omega_S(t)}| = r_S(x)\sqrt{\cos^2(\omega_S(t)) + \sin^2(\omega_S(t))}$$

where $r_S(t)$ is a fuzzy set and $\omega_S(t)$ is a real valued function.

Let $G = (V, E)$ be a simple (undirected) graph on vertices $1, 2, ..., n$. Then the $n \times n$ matrix, called the complex adjacency matrix $A(G)$ of G (or simply A), is defined by

$$\begin{aligned} A(G) &= [a_{ij}], \\ a_{ij} &= \begin{cases} 1 \text{ if} & \{i, j\} \in E \\ \mu_S(t) \text{ if the edge is not sure} & \\ 0 & \{i, j\} \notin E \end{cases} \end{aligned}$$

Fuzzy Degree

In fuzzy graph theory, the fuzzy degree of a vertex in a fuzzy graph is a measure that indicates the degree of membership of the vertex in the graph. In a traditional (non-fuzzy) graph, the degree of a vertex represents the number of edges incident to that vertex. However, in a fuzzy graph, edges have degrees of membership indicating the strength or intensity of relationships between vertices.

The fuzzy degree of a vertex takes into account not only the number of adjacent vertices but also the fuzzy strengths of those relationships. Here's how you can calculate the fuzzy degree of a vertex in a fuzzy graph:

For a given vertex v, sum up the degrees of membership of all edges incident to v. These are the fuzzy strengths of the relationships connecting v to other vertices in the graph.

$$\text{Fuzzy Degree(v)} = \sum_{\text{edges } e \text{ incident to } v} \text{Degree of Membership}(e)$$

The degree of membership of an edge indicates the strength of the relationship represented by that edge in the fuzzy graph.

We can normalize the fuzzy degree if the degrees of membership of the edges are on a different scale. Normalization ensures that the fuzzy degree is comparable across different vertices and graphs.

$$\text{Normalized Fuzzy Degree}(v) = \frac{\text{Fuzzy degree(v)}}{\text{Maximum possible degree}}$$

Normalization divides the fuzzy degree of a vertex by the maximum possible fuzzy degree in the graph, providing a value between 0 and 1.

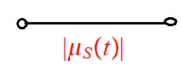
The fuzzy degree of a vertex in a fuzzy graph is a crucial measure, especially when analyzing the centrality or importance of vertices in a network where relationships have varying strengths. It allows for a more nuanced understanding of the connectivity and influence of vertices in fuzzy graphs, where the relationships are not simply present or absent but exist with varying degrees of intensity or fuzziness.

In the context of fuzzy graphs, where edges have degrees of membership indicating the strength of relationships between vertices, the Handshaking Lemma can be adapted to accommodate these fuzzy relationships.

Lemma 2.1. *Let $G(p, q)$ be a complex fuzzy graph and V is the set of fuzzy vertices, then*

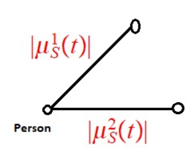
$$\sum_{v \in V} \textit{Fuzzy Degree}(v) = 2 \times \textit{Fuzziness in all edges}$$

Proof. Let us use induction on size of $G = (V, E)$.



$$|\mu_S(t)| + |\mu_S(t)| = 2(|\mu_S(t)|) = 2(\text{Fuzziness in all edges})$$

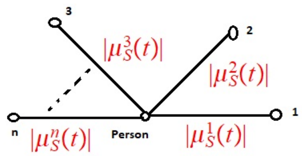
If a person may or may not shakes hand to two people, then



Then total fuzziness in edges

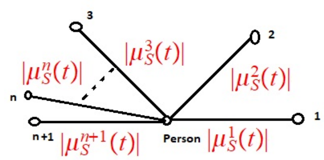
$$|\mu_S^1(t)| + |\mu_S^2(t)| + |\mu_S^1(t)| + |\mu_S^2(t)| = 2(|\mu_S^1(t)| + |\mu_S^2(t)|) = 2(\text{Fuzziness in all edges})$$

Let the result is true for n. A person may or may not shacks hands with n people.



$$
\begin{aligned}
& |\mu_S^1(t)| + |\mu_S^2(t)| + ... + |\mu_S^n(t)| + |\mu_S^1(t)| + |\mu_S^2(t)| + ... + |\mu_S^n(t)| \\
= \; & 2(|\mu_S^1(t)| + |\mu_S^2(t)| + ... + |\mu_S^n(t)|) \\
= \; & 2(\text{Fuzziness in total edges})
\end{aligned}
$$

Add one more vertex and edge that is v shakes hand with n+1 persons (adding one more n+1)



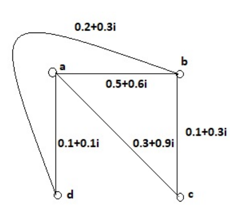
Shaking hand with more person (adding one more edge), mathematically

$$
\begin{aligned}
& |\mu_S^1(t)| + |\mu_S^2(t)| + ... + |\mu_S^n(t)| + |\mu_S^1(t)| + |\mu_S^2(t)| + ... \\
& + |\mu_S^n(t)| \; + |\mu_S^{n+1}(t)| \; + |\mu_S^{n+1}(t)| \\
= \; & 2(|\mu_S^1(t)| + |\mu_S^2(t)| + ... + |\mu_S^n(t)| + |\mu_S^{n+1}(t)|) = 2(\text{Fuzziness in total edges})
\end{aligned}
$$

□

This proof shows that the total strength of relationships (total edge fuzziness) in a fuzzy graph is equal to the sum of the strengths of relationships of all vertices (fuzzy degrees). It adapts the traditional Handshaking Lemma to accommodate the fuzzy nature of relationships in fuzzy graphs.

Example 2.1. *Consider the following complex fuzzy graph*



$$A = \begin{bmatrix} 0 & 0.5+0.6i & 0.3+0.9i & 0.1+0.1i \\ 0.5+0.6i & 0 & 0.1+0.3i & 0.2+0.3i \\ 0.3+0.9i & 0.1+0.3i & 0 & 0 \\ 0.1+0.1i & 0.2+0.3i & 0 & 0 \end{bmatrix}$$

The eigenvalues of this matrix are:

$$0.6466+1.2501i,\ 4.2738\times10^{-2}+0.11438i,\ -0.2196-0.456\,53i,\ -0.46974-0.90799i$$

The complex energy of the this graph is the sum of eigenvalues

The multiplicative inverse of the matrix with trace 0 and rank 4 is.

$$\begin{bmatrix} 0 & 0 & 0.348\,31-1.1573i & -0.561\,80+0.89888i \\ 0 & 0 & -0.04494\,4+0.47191i & 1.685\,4-2.6966i \\ 0.348\,31-1.1573i & -449.4\,4+0.47191i & -0.270\,82-0.85297i & -0.450\,83+3.3573i \\ -0.561\,80+0.89888i & 1.685\,4-2.6966i & -0.450\,83+3.357\,i & 2.158\,8-4.8024i \end{bmatrix}$$

The singular values are: $[1.447\,5, 1.087\,0, 0.478\,73, 0.118\,16]$

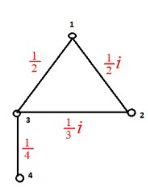
In robotics and sensor networks, graphs model connectivity and relationships between different sensors or robotic agents. Singular values can be applied to optimize sensor placements, network connectivity, and localization algorithms.

$$\begin{bmatrix} 0 & 0.5+0.6i & 0.3+0.9i & 0.1+0.1i \\ 0.5+0.6i & 0 & 0.1+0.3i & 0.2+0.3i \\ 0.3+0.9i & 0.1+0.3i & 0 & 0 \\ 0.1+0.1i & 0.2+0.3i & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0.5+0.6i & 0.3+0.9i & 0.1+0.1i \\ 0.5+0.6i & 0 & 0.1+0.3i & 0.2+0.3i \\ 0.3+0.9i & 0.1+0.3i & 0 & 0 \\ 0.1+0.1i & 0.2+0.3i & 0 & 0 \end{bmatrix}$$

The trace usual product of matrices is $-1.92+2.68i$ with transpose

$$\begin{bmatrix} -0.83+1.16i & -0.25+0.23i & -0.13+0.21i & -0.08+0.27i \\ -0.25+0.23i & -0.24+0.78i & -0.39+0.63i & -0.01+0.11i \\ -0.13+0.21i & -0.39+0.63i & -0.8+0.6i & -0.13+0.21i \\ -0.08+0.27i & -0.01+0.11i & -0.13+0.21i & -0.05+0.14i \end{bmatrix}$$

Example 2.2. *Consider the following complex fuzzy graph*



The complex fuzzy adjacency of this graph

$$A_{CFAM}=\begin{bmatrix} 0 & \frac{1}{2}i & \frac{1}{2} & 0 \\ \frac{1}{2}i & 0 & \frac{1}{3}i & 0 \\ \frac{1}{2} & \frac{1}{3}i & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 \end{bmatrix}$$

The eigenvalues:

$$\frac{1}{6\sqrt[6]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}}\sqrt{9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}}+$$

$$\frac{1}{6\sqrt[6]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}\sqrt[4]{9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}}}$$

$$\sqrt{\begin{aligned}&\frac{3839}{20\,736}\sqrt{9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}}\\&-\frac{1}{2}\sqrt{6}\sqrt{\frac{173}{13\,824}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{1492\,992}}\\&-\frac{7}{12}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}\sqrt{9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}}\\&-9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}\sqrt{9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}}\end{aligned}},$$

$$\frac{1}{6\sqrt[6]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}}\sqrt{9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}}$$

$$-\frac{1}{6\sqrt[6]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}\sqrt[4]{9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}}}$$

$$\sqrt{\begin{array}{l}\frac{3839}{20736}\sqrt{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}\\ -\frac{1}{2}\sqrt{6}\sqrt{\frac{173}{13824}\sqrt{3}\sqrt{1433}+\frac{1201735}{1492992}}\\ -\frac{7}{12}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}\sqrt{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}\\ -9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}\sqrt{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}\end{array}},$$

$$\frac{1}{6\sqrt[6]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}\sqrt[4]{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}}$$

$$\sqrt{\begin{array}{l}\frac{3839}{20736}\sqrt{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}+\\ \frac{1}{2}\sqrt{6}\sqrt{\frac{173}{13824}\sqrt{3}\sqrt{1433}+\frac{1201735}{1492992}}-\frac{7}{12}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}\\ \sqrt{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}\\ -9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}\sqrt{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}\end{array}}-$$

$$\frac{1}{6\sqrt[6]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}}\sqrt{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}},-$$

$$\frac{1}{6\sqrt[6]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}}\sqrt{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}-$$

$$\frac{1}{6\sqrt[6]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}\sqrt[4]{9\left(\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}\right)^{\frac{2}{3}}-\frac{7}{24}\sqrt[3]{\frac{173}{746496}\sqrt{3}\sqrt{1433}+\frac{1201735}{80621568}}-\frac{3839}{20736}}}$$

$$\sqrt{\begin{array}{c}\frac{3839}{20\,736}\sqrt{\begin{array}{c}9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}\\ -\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}\\ +\frac{1}{2}\sqrt{6}\sqrt{\frac{173}{13\,824}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{1492\,992}}\end{array}}\\ -\frac{7}{12}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}\sqrt{\begin{array}{c}9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}\\ -\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}\end{array}}\\ -9\left(\begin{array}{c}\frac{173}{746\,496}\sqrt{3}\sqrt{1433}\\ +\frac{1201\,735}{80\,621\,568}\end{array}\right)^{\frac{2}{3}}\sqrt{\begin{array}{c}9\left(\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}\right)^{\frac{2}{3}}\\ -\frac{7}{24}\sqrt[3]{\frac{173}{746\,496}\sqrt{3}\sqrt{1433}+\frac{1201\,735}{80\,621\,568}}-\frac{3839}{20\,736}\end{array}}\end{array}}$$

This example has its own importance while we are dealing real life phenomenon. It is sometimes quite difficult to apply the energy of the adjacency matrix. Thus it is worth mentioning here that some times the energy of the adjacency matrix of a complex fuzzy graph is so complicated and is quite hard to use it for applications in other branches of science, especially in signals. Therefore we need to find the complex Laplacian energy, especially for applications point of view if we are dealing with models carrying two dimensional ill-defined dynamic phenomenon.

In order to find the complex fuzzy Laplacian energy, first we need to find the complex fuzzy diagonal matrix.

Example 2.3. *The complex fuzzy degree matrix of the above complex fuzzy graph is*

$$D_{CFDM}=\begin{bmatrix}0.5+0.5i & 0 & 0 & 0\\ 0 & 0.833i & 0 & 0\\ 0 & 0 & 1.16i & 0\\ 0 & 0 & 0 & 0.25\end{bmatrix}$$

Now the complex fuzzy Laplacian matrix is

$$\begin{aligned}L_{CFLM} &= \begin{bmatrix}0.5+0.5i & 0 & 0 & 0\\ 0 & 0.833i & 0 & 0\\ 0 & 0 & 1.16i & 0\\ 0 & 0 & 0 & 0.25\end{bmatrix}-\begin{bmatrix}0 & \frac{1}{2}i & \frac{1}{2} & 0\\ \frac{1}{2}i & 0 & \frac{1}{3}i & 0\\ \frac{1}{2} & \frac{1}{3}i & 0 & \frac{1}{4}\\ 0 & 0 & \frac{1}{4} & 0\end{bmatrix}\\ &= \begin{bmatrix}0.5+0.5i & -\frac{1}{2}i & -\frac{1}{2} & 0\\ -\frac{1}{2}i & 0.833i & -\frac{1}{3}i & 0\\ -\frac{1}{2} & -\frac{1}{3}i & 1.16i & -\frac{1}{4}\\ 0 & 0 & -\frac{1}{4} & 0.25\end{bmatrix}\end{aligned}$$

The eigenvalues of complex fuzzy Laplacian matrix are

$$0.692\,26+0.73777i, 0.270\,77+1.891\,5\times10^{-2}i, -3.316\,3\times10^{-2}+0.316\,73i, -0.179\,87+1.4196i$$

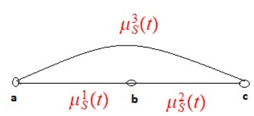
The absolute average eigenvalue of the complex fuzzy Laplacian energy is give as:

$$
\begin{aligned}
|\lambda|_{AAEV} &= |0.692\,26 + 0.73777i + 0.270\,77 + 189.15i - 331.63 + 0.316\,73i \\
&\quad + -0.179\,87 + 1.4196i| \div 4 \\
&= \frac{|-330.84684 + 191.6241i|}{4} = \frac{382.334184}{4} = 95.583546
\end{aligned}
$$

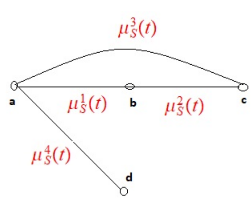
The average eigenvalue of the complex fuzzy Laplacian matrix is directly related to the connectivity of graphs. A high average eigenvalue shows that the graph is well-connected with vertices are strongly interrelated which indicates strongly two dimensional relations. In the context of a complex fuzzy graph, this suggests strong and complex relationships between individuals (vertices). A large average eigenvalue in signals that signals can efficiently circulate across the vertices (nodes), allowing for efficient processing and extraction of meaningful information from the complex signals represented by the complex fuzzy graphs.

Practical Example of Complex Fuzzy Dynamic Graphs.

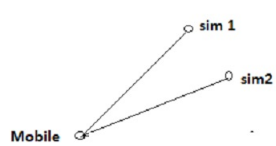
Example 2.4. *Consider the following three friends a, b and c are connected through social network and the edges shows the strength and emotions of their friendship, then the complex fuzzy graph is given as under*



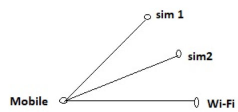
A new person (vertex) say d joining the network and add the person (vertex) a, then



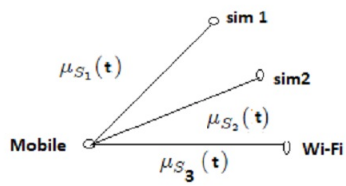
Example 2.5. *Consider the simple example of three signals received on mobile.*



Then appearance of the Wi-Fi signals on mobile, Wi-Fi is the addition of a new vertex and edge between mobile and Wi-Fi is the addition of a new edge.



A mobile (a digital receiver) received three signals of sim 1, sim 2 and Wi-Fi. Once can easily observe that at different times and different places the mobile will show some times full signals, half and some times less than half signals for each signals. Thus there is uncertainties of received signals. In order to represent his situation we draw the following graph



Now the degree of the vertices will be in fractions. Thus we get the following degree matrix.

$$D_{CFDM} = \begin{pmatrix} \frac{1}{2}+i & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3}+0.2i & 0 \\ 0 & 0 & 0 & \frac{4}{3}+0.3i \end{pmatrix}$$

Where $\mu_{S_i}(t)$ *for* $1 \le i \le 3$, *represents the complex fuzzy uncertainties of all three signals.*

Now the complex adjacency matrix of above simple graph is considered as

$$A_{CFAM} = A_{CFAM}(G) = \begin{pmatrix} 0 & 0.5+0.1i & 0.6+0.7i & 0.4+0.2i \\ 0.5+0.1i & 0 & 0 & 0 \\ 0.6+0.7i & 0 & 0 & 0 \\ 0.4+0.2i & 0 & 0 & 0 \end{pmatrix}$$

The the Laplacian matrix is

$$L_{CFLM} = D_{CFDM} - A_{CFAM} = \begin{pmatrix} \frac{1}{2}+i & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3}+0.2i & 0 \\ 0 & 0 & 0 & \frac{4}{3}+0.3i \end{pmatrix} - \begin{pmatrix} 0 & 0.5+0.1i & 0.6+0.7i & 0.4+0.2i \\ 0.5+0.1i & 0 & 0 & 0 \\ 0.6+0.7i & 0 & 0 & 0 \\ 0.4+0.2i & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+i & -0.5-0.1i & -0.6-0.7i & -0.4-0.2i \\ -0.5-0.1i & \frac{1}{2} & 0 & 0 \\ -0.6-0.7i & 0 & \frac{1}{3}+0.2i & 0 \\ -0.4-0.2i & 0 & 0 & \frac{4}{3}+0.3i \end{pmatrix}$$

Universal complex adjacency matrix. The complement of a complex adjacency matrix $A = [\mu_S(t)]_{n\times n}$ is defined as an $n \times n$ matrix whose all entries are $1+1i$.

Example 2.6.

$$[\mu_S(t)]^{UCAM}_{m\times n} = \begin{pmatrix} 1+1i & 1+1i & 1+1i & 1+1i \\ 1+1i & 1+1i & 1+1i & 1+1i \\ 1+1i & 1+1i & 1+1i & 1+1i \\ 1+1i & 1+1i & 1+1i & 1+1i \end{pmatrix}.$$

Zero complex adjacency matrix. A complex adjacency matrix $A = [\mu_S(t)]\, n$ is called zero matrix if each entry of the matrix is 0, and is denoted by $[0]_{ZCAM}$.

Example 2.7.

$$[0]_{ZCFAM} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Complement of complex adjacency matrix. If $A(G) = [\mu_S(t)]_{n\times n}$ is a complex adjacency matrix, then complement of $A(G)$ is $A^c(G) = [\mu_S(t)]^c_{n\times n}$ and defined as:

$$[\mu_S(t)]^c = [\mu_S(t)]^{UCAM}_{m\times n} - [\mu_S(t)].$$

Example 2.8. *Consider $A_{CFAM}(G)$ be a complex adjacency matrix such that*

$$A_{CFAM} = A(G) = \begin{pmatrix} 0.2+0.3i & 0.3+0.2i \\ 0.1+i & 0.9 \end{pmatrix}_{2\times 2}$$

Then complement of this matrix is given as:

$$A^C_{CFAM}(G) = \begin{pmatrix} 0.8+0.7i & 0.7+0.8i \\ 0.9 & 0.1+i \end{pmatrix}_{2\times 2}.$$

Union of complex adjacency Matrices. Let $A(G) = [\mu_S(t)]$ and $B(G) = [\chi_S(t)]$ be complex fuzzy adjacency matrices. Then Union of $A(G)$ and $B(G)$ denoted by $A(G) \cup B(G)$ and defined as

$$C(G) = [\max\{\mathrm{Re}\{\mu_S(t), \chi_S(t)\}\} + \max\{imaginary\{\mu_S(t), \chi_S(t)\}\}], \text{ for all } t.$$

Intersection of complex adjacency Matrices. Let $A(G) = [\mu_S(t)]$ and $B(G) = [|\chi_S(t)|]$ be complex adjacency matrices. Then intersection of $A(G)$ and $B(G)$ denoted by $A(G) \cap B(G)$ and defined as

$$C(G) = [\min\{\mathrm{real}(\mu_S(t)), (\chi_S(t))\} + \min\{imaginary(\mu_S(t)), (\chi_S(t))], \text{ for all } t.$$

We introduce the following results with rather simple proofs.

Lemma 2.2. *Let* $[\mu_S(t)]$ *be a* $n \times n$ *complex adjacency matrix then*

(i) $([\mu_S(t)]^c)^c = [\mu_S(t)]$

(ii) $[0]^c_{ZCFAM} = [\mu_S(t)]^{UCAM}_{m\times n}$ *and* $\left([\mu_S(t)]^{UCAM}_{m\times n}\right)^c = [0]_{ZCFAM}$

Lemma 2.3. *If* $[\mu_S(t)]$, $[\chi_S(t)]$ *and* $[\gamma_S(t)]$ *are* $n \times n$ *complex adjacency matrices then*

(i) $[\mu_S(t)] \cup [\chi_S(t)] = [\chi_S(t)] \cup [\mu_S(t)]$

(ii) $[\mu_S(t)] \cap [\chi_S(t)] = [\chi_S(t)] \cap [\mu_S(t)]$

(iii) $([\mu_S(t)] \cup [\chi_S(t)]) \cup [\gamma_S(t)] = [\mu_S(t)] \cup ([\chi_S(t)] \cup [\gamma_S(t)])$

(iv) $([\mu_S(t)] \cap [\chi_S(t)]) \cap [\gamma_S(t)] = [\mu_S(t)] \cap ([\chi_S(t)] \cap [\gamma_S(t)])$

(v) $[\mu_S(t)] \cup ([\chi_S(t)] \cap [\gamma_S(t)]) = ([\mu_S(t)] \cup [\chi_S(t)]) \cap ([\mu_S(t)] \cup [\gamma_S(t)])$

(vi) $[\mu_S(t)] \cap ([\chi_S(t)] \cup [\gamma_S(t)]) = ([\mu_S(t)] \cap [\chi_S(t)]) \cup ([\mu_S(t)] \cap [\gamma_S(t)])$.

Proposition 2.1. *Let* $[|\mu_S(t)|]$ *be* $n \times n$ *complex adjacency matrices. Then*

(i) $[\mu_S(t)] \cup [\mu_S(t)] = [\mu_S(t)]$

(ii) $[\mu_S(t)] \cap [\mu_S(t)] = [\mu_S(t)]$.

Proposition 2.2. *Let* $[\mu_S(t)]$ *and* $[\chi_S(t)]$ *be* $n \times n$ *complex* ***adjacency*** *matrices. Then De Morgan's laws hold for these matrices*

$$(1)\ \ ([\mu_S(t)] \cup [\chi_S(t)])^c = [\mu_S(t)]^c \cap [\chi_S(t)]^c$$
$$(2)\ \ ([\mu_S(t)] \cap [\chi_S(t)])^c = [\mu_S(t)]^c \cup [\chi_S(t)]^c$$

Proof. (i)

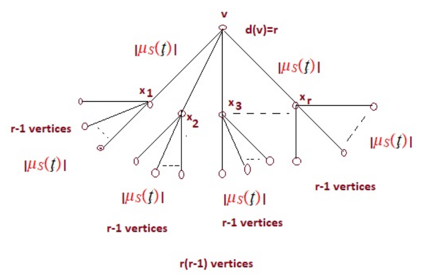
$$\begin{aligned}
([\mu_S(t)] \cup [\chi_S(t)])^c &= [\max \mathrm{Re}\{\mu_S(t), \chi_S(t)\}\} + \max\{imaginary\,\{\mu_S(t), \chi_S(t)\}\}]^c \\
&= [\mu_S(t)]^{UCAM}_{m\times n} - \begin{bmatrix} \max \mathrm{Re}\{\mu_S(t), \chi_S(t)\}\} \\ + \max\{imaginary\,\{\mu_S(t), \chi_S(t)\}\} \end{bmatrix} \\
&= \min\{[\mu_S(t)]^{UCAM}_{m\times n} - \begin{bmatrix} \max \mathrm{Re}\{\mu_S(t), \chi_S(t)\}\} \\ + \max\{imaginary\,\{\mu_S(t), \chi_S(t)\}\} \end{bmatrix}\} \\
&= [|\mu_S(t)|]^c \cap [|\chi_S(t)|]^c
\end{aligned}$$

□

(ii) It can be proved similarly.

Theorem 2.1. *If* x *and* y *are any tow vertices of a graph* $G(V, E)$ *such that* $| N(x) \cap N(y) |=| \mu_S(t) |$ *and* $d(v) = r$ *for each* v *of* G, *and* G *has smallest cycle of length five, where* $\mu_S(t)$ *is a complex fuzzy set. Find order and size the graph* G.

Proof. Let us draw the general complex fuzzy graph as □



The total number of vertices in above graph is obviously

$$= 1 + r + r(r-1) = 1 + r + r^2 - r = r^2 + 1$$

Let there exists a graph G which satisfies the given conditions. Let A be a matrix for G.

Let $a_{ij} = \mid \mu_S(ţ) \mid$ if $i \sim j$. $a_{ii} = 0$ if $i \nsim i$ that is $i = i$. A is $n \times n$ matrix and is symmetric that is $A^T = A$. Sum of the ith row

$$\sum_{j=1}^{n} a_{ij} = a_{i1} + a_{i2} + ... + a_{in}$$

This show the degree of vertex i in G. Let

$$P = A^2 = A \times A, \text{ where } p_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$$

Thus

$$a_{ik}.a_{kj} = \mid \mu_S(ţ) \mid^2 \Longleftrightarrow a_{ik} = \mid \mu_S(ţ) \mid \text{ and } a_{kj} = \mid \mu_S(ţ) \mid$$

$$p_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} = \mid N(i) \cap N(j) \mid$$

Now if $a_{ij} = 0$, then $p_{ij} = 1$, for $ij \notin E$. Moreover if $a_{ij} = 1$ then $p_{ij} = 0$, for $ij \in E$. Thus

$$a_{ij} = 0, P_{ij} = \mid N(x) \cap N(y) \mid = \mid N(i) \mid = r \text{ and } P_{ir} = r$$

Now as A is symmetric so

$$a_{ik} = a_{ik} p_{ii} = \sum_{k=1}^{n} a_{ik}.a_{ki} = \sum_{k=1}^{n} a_{ik} a_{ik} = \sum_{k=1}^{n} a_{ik}^2 = \sum_{k=1}^{n} a_{ik}$$

Since $p_{ii} = d(i)$, $a_{ii} = 0$ and $p_{ii} = r$. Thus

$$A + A^2 = J + (r-1)I$$

If A is symmetric then all its eigenvalues of A are real. Note that A is called eigenvalue of A if $Av = \lambda v$, as $A = A^T$. Thus $A^T v = \lambda v \Longrightarrow Av = \bar{A} = \lambda v$

$$A\vec{v} = A\vec{v}$$

Let

$$\overrightarrow{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}_n ; d_i = 1 \text{ for all } i$$

$$A\overrightarrow{d} = \sum_{j=1}^{n} a_{ij} d_j = r\overrightarrow{d}, \; as \sum_{j=1}^{n} a_{ij} = r$$

where r is the eigenvalue and $\overrightarrow{d}$ is the eigenvector.
Now

$$\begin{aligned} A + A^2 &= J + (r-1)I \\ A\overrightarrow{d} + A^2\overrightarrow{d} &= J\overrightarrow{d} + (r-1)I\overrightarrow{d} \\ r\overrightarrow{d} + r^2\overrightarrow{d} &= n\overrightarrow{d} + (r-1)\overrightarrow{d} \\ (r + r^2)\overrightarrow{d} &= n\overrightarrow{d} + r\overrightarrow{d} - \overrightarrow{d} \\ r + r^2 &= n + r - 1 \\ n &= r^2 + 1 \end{aligned}$$

Let λ be an arbitrary any other eigenvalue and v be the eigenvector $A\overrightarrow{v} = \lambda\overrightarrow{v}, A^2\overrightarrow{v} = \lambda\overrightarrow{v}$

$$\begin{aligned} \lambda\overrightarrow{v} + \lambda^2\overrightarrow{v} &= J\overrightarrow{v} + (r-1)\overrightarrow{v} \\ (\lambda + \lambda^2)\overrightarrow{v} &= 0 + (r-1)\overrightarrow{v} \\ (\lambda + \lambda^2) &= r - 1 \\ \lambda + \lambda^2 + 1 - r &= 0 \\ \lambda &= \frac{-1 \mp \sqrt{4r-3}}{2} \\ \lambda_1 &= \frac{-1 + \sqrt{4r-3}}{2}, \lambda_2 = \frac{-1 - \sqrt{4r-3}}{2} \\ let\ S &= \sqrt{4r-3} \\ \lambda_1 &= \frac{-1+S}{2}, \lambda_2 = \frac{-1-S}{2} \end{aligned}$$

(a). The eigenvalues λ_1, λ_2, r

(b). Let m_1 is the multiple of λ_1 and m_2 is the multiple of λ_2. Number of vectors$= m_1 + m_2 + 1 = n = r^2 + 1$

(c). Sum of eigenvalues is the traces of matrices and trace of $(A) = 0 + ... + 0 = 0$. Thus

$$\begin{aligned}
\lambda_1 m_1 + \lambda_2 m_2 + r &= 0 \\
m_1 + m_2 - r^2 &= 0 \\
\lambda_1 m_1 + \lambda_2 m_2 + r &= m_1 + m_2 - r^2 \\
m_1(\lambda_1 - 1) + m_2(\lambda_2 - 1) + r^2 + r &= 0 \\
m_1(S-3) + m_2(-S-3) + 2r^2 + 2r &= 0 \\
S(m_1 - m_2) - 3(m_1 + m_2) + 2r^2 + 2r &= 0 \\
S(m_1 - m_2) - 3r^2 + 2r^2 + 2r &= 0 \\
S(m_1 - m_2) - r^2 + 2r &= 0 \\
r^2 - 2r - S(m_1 - m_2) &= 0
\end{aligned}$$

We know that r is an integer (degree) and $S = \sqrt{4r-3}$, $4r-3$ is also an integer hence S is the square root of an integer and for square root of an integer there are two cases.

(i). Either an integer.

(ii). Irrational.

If S is irrational, then $r^2 - 2r - S(m_1 - m_2) = 0$ which is a contradiction.

So if S is rational then $m_1 - m_2 = 0 \Longrightarrow m_1 = m_2 \therefore r^2 - 2r = 0 \Longrightarrow r = 0, r = 2$

(ii). If S is an integer then $r = \frac{S^2+3}{4}$ put in $r^2 - 2r - S(m_1 - m_2) = 0$,

$$\begin{aligned}
\frac{S^2 + 9 + 6S^2}{4} - \frac{2S^2 - 6}{4} - S(m_1 - m_2) &= 0 \\
S^4 + 6S^2 + 9 - 8S^2 - 6.4 - 16S(m_1 - m_2) &= 0 \\
S^4 - 2S^2 - S(16m_1 - 16m_2) - 15 &= 0 \\
S^4 - 2S^2 - 16S(m_1 - m_2) &= 15
\end{aligned}$$

This equation can be solved if S is divided 15 so possible values of S are $S = 1, 3, 5, 15$

$$\begin{aligned}
S &= 1 \Longrightarrow r = 1 \\
S &= 3 \Longrightarrow r = 3 \\
S &= 5 \Longrightarrow r = 7 \\
S &= 15 \Longrightarrow r = 57 \text{ and so on}
\end{aligned}$$

3. Applications of Complex Fuzzy Dynamic Graphs in Signals

Here in this section we are going to discuss a real life application of newly defined complex fuzzy dynamic graphs using the concepts of complex fuzzy sets by introducing an algorithm. This will show that how our these graphs and the eigenvalues of their complex fuzzy dynamic graphs offer real life applications. Specifically the average absolute eigenvalues explain how to distinguish a reference clean signal mixed up with noise. A radar (Radio Detection and Ranging) is an device which can detect surrounding objects using radio waves.

The average eigenvalue of a complex fuzzy Laplacian matrix is the extension of the fuzzy Laplacian matrix of a graph to incorporate fuzzy information by introducing the concepts of amplitude and phase terms. In this matrix, each entry shows the strength of the complex fuzzy relationship between vertices in the complex fuzzy graph. The average eigenvalues have large number of application in various fields in social/biological networks data clustering, traffic flow, Bioinformatics, Sociology, Anthropology, wireless sensor networks, quantum information theory, image processing, social sciences, and computer science, network stability and signals and systems. If vertices in social/biological networks have complex fuzzy memberships to different communities, the average absolute eigenvalue can capture the strength of these memberships and help identify communities in a more nuanced way. The absolute average eigenvalue of the complex fuzzy Laplacian matrix can be applied to assess the stability of complex fuzzy networks. Certain range of absolute eigenvalues are needed for stability in network and deviations from such values may show high instability in the network. Absolute eigenvalues play vital role in community detection algorithms. Mostly communities are associated with certain eigenvalues and their absolute average eigenvalue could help in identifying the overall community structure. Complex fuzzy graphs represent signals flow in communication systems and the eigen values of their Laplacian matrices are used for detection of errors while the average absolute eigenvalues are used for efficiently design of error detections. Complex fuzzy graphs are useful for modeling the relations between different parts of images and the eigenvalues of their Laplacian matrices can aid in image segmentation while the absolute average eigenvalues give information about the homogeneity of image regions. Eigenvalues are widely used in clustering algorithms. Complex fuzzy graphs can represent relationships in multidimensional data. Analyzing the eigenvalues, particularly the absolute average eigenvalue, can assist in determining the optimal number of clusters and the quality of the clustering in complex fuzzy data sets. The absolute average eigenvalues are used in studying some specific type of quantum systems. The applications mentioned above elaborate the diversified usefulness of the absolute average eigenvalues of the complex fuzzy Laplacian matrices in several branches of sciences, making these concepts as useful tools to analyze systems representing by the complex fuzzy graphs. In robotics and sensor networks, spectral methods are used for localization and mapping. Eigenvalues of the Laplacian matrix, including the average eigenvalue, help in determining the relative positions of robots or sensors in an environment.

The fuzzy absolute average eigenvalues of the Laplacian matrix within [0, 1], specify different informations for the fuzzy relationships between clean and noisy signals in any network. If the fuzzy absolute average eigenvalue of the Laplacian matrix of the complex fuzzy graph lie with in the range [0.6,1], then this will generally indicates the moderate to strong complex fuzzy relations between vertices in the complex fuzzy graphs that is the connections between vertices are strong or can be considered similar to their vertices are neighborhood vertices. As far as the applications are concerning in signals, the absolute average eigenvalue approaching to 1 provide the indications of strong relation between clean and noisy signals and this situation has the potential to separate clean signals from noisy signals and could be identify as reliable reference signals. This eigenvalue analysis is useful to reveal dominant patterns and structures in the graph, which are helping for identification of vertices (signals) which are relatively stable and can be identifies as references signals in the presence of noise in the environment. Graphs of signals with such complex fuzzy relationships are strength full in noisy environments. The strong connections between vertices make the graph more resilient to noise, allowing for the identification of reliable reference signals even when the signals are corrupted to some extent. More applications of the absolute average eigenvalues could be find in telecommunications, image processing, sensor networks, and bioinformatics, where identifying stable references in the presence of uncertainties is crucial. While a fuzzy absolute average eigenvalue near to 0 guarantee the existence of very weak relation between clean and distort signals in the specific network of consideration. The very low value eigenvalue indicates that the fuzzy relation between clean and noisy signals are almost negligible that is the connection between clean signals and noisy signals is nearly zero in the network and with such undernourished fuzzy relation it is quite difficult to separate clean signals from distort signals. Thus for the identification of clean signals we need advanced methods and procedures of signals processing and such techniques include advanced statistical algorithms or machine learning algorithms for extracting reasonable and reliable information from the provided data and to identify reliable reference signals. The fuzzy absolute average eigen value near to 0.5 provide sensible and midst ambiguous relationships between clean and noisy signals. The middle value suggests that there exist moderate fuzzy relationships between clean and noisy signals. While not exceptionally strong, these relationships have more potential for identifying patterns and connections compared to the case with a value of near to 0. In this case there is way to separate clean signals from noisy ones. Thus there are certain signals making then reliable to consider as reference signals. This situation has the potential to analyze and reveal some dominant patterns or structures in the network. While the relationships are moderate, specific signals might stand out as relatively reliable amidst the noise. Further analysis and signal processing techniques can enhance the identification of these signals. Usually signals in audio and image processing are interfered by noise. The eigenvalue analysis is an easy and reliable approach to identify reference signals amidst the noise.

Algorithm 1

Step 1.

If a receiver gets some signals $S_1(t), S_2(t), S_3(t), ..., S_m(t)$ from any source. Each signal is sampled N times by the receiver. Then $S_i(t)$ (i varies from 1 to m) signals can be recognized with respect to R, where R is a reference signal. Assume that both $S_i(t)$ and R are considered as n times.

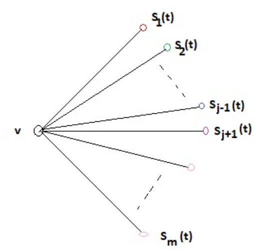
Assume that $S_j(t)$ is i-th signal, where $1 \leq t \leq m$. Since all the signals cannot be received exactly on digital receiver so let $|\mu_{S_j}(t)|$ be representing their receiving approximations on the receiver.

Then each $S_j(t)$ in terms of discrete complex Fourier transform as given as follows:

$$S_j(t) = \frac{1}{n}\sum_{s=1}^{n} g_{j,s} \cdot e^{\frac{2\pi i(s-1)(t-1)}{n}},$$

where $g_{j,n}$ is the complex Fourier coefficients of signals S_j, and $S_j(t)$ varies from 1 to m.

Now draw the graphs by taking all combinations of signals considering one signal as missing signal in each combination, without lost of generality, let the ignoring signal be $S_j(t)$.



$G(S_j(t))$

The complex adjacency matrices for graphs $G(S_j(t))$ are given below

$$A_i[G(S_j(t))] = \begin{bmatrix} 0 & \mu_{S_1}(t) & ... & \mu_{Sm}(t) \\ \mu_{S_1}(t) & 0 & ... & 0 \\ . & . & 0 & . \\ \mu_{S_m}(t) & 0 & ... & 0 \end{bmatrix}, \text{ for } 1 \le i \le m$$

Step 2. Take intersection of complex adjacency matrices to get

$$A[G(S_j(t))] = A_1[G(S_j(t)) \cap A_2[G(S_j(t)) \cap ... \cap A_m[G(S_j(t))$$

Step 3. Find the complex fuzzy diagonal matrix D

$$D = \begin{bmatrix} \mu_{S_1}(1) & 0 & ... & 0 \\ 0 & \mu_{S_2}(2) & ... & |0 \\ . & . & . & . \\ 0 & 0 & ... & \mu_{S_m}(m) \end{bmatrix}$$

Step 4. Find complex fuzzy Laplacian matrix using the know criteria for Laplacian.

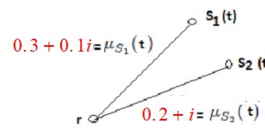
Step 5. Find eigenvalues of complex fuzzy Laplacian matrix.

Step 6. Find absolute average eigenvalue. The fuzzy absolute average eigenvalue of the Laplacian matrix is used to identify average strength between distort and clean signals. The obtained eigenvalue is inversely proportional to the said strength which means that higher the strength between clean and noisy signals less the eigenvalue.

Assessment: Identifications of Reference Signals within the Less Noisy Environments

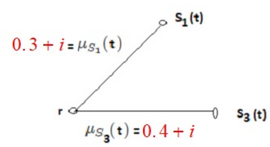
To analyzed the algorithm and its effectiveness of reliability, we are considering a case study of signals in noisy environments.

Step 1. The complex adjacency matrices for signals two and three received by the digital receiver



$$A_1 = \begin{bmatrix} 0 & 0.2+i & 0.3+0.1i \\ 0.2+i & 0 & 0 \\ 0.3+0.1i & 0 & 0 \end{bmatrix}$$

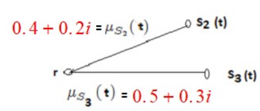
The complex fuzzy graph for signals $S_1(t)$ and $S_3(t)$ is given as



The adjacency matrix for signals two and three received by the receiver

$$\begin{bmatrix} 0 & 0.3+i & 0.4+i \\ 0.3+i & 0 & 0 \\ 0.4+i & 0 & 0 \end{bmatrix}$$

The complex fuzzy graph for signals $S_2(t)$ and $S_3(t)$ is given as
The adjacency matrix for signals two and three received by the receiver

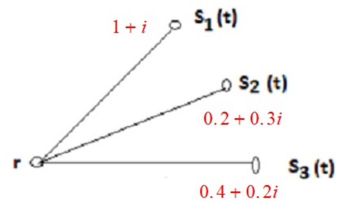


$$\begin{bmatrix} 0 & 0.4+0.2i & 0.5+0.3i \\ 0.4+0.2i & 0 & 0 \\ 0.5+0.3i & 0 & 0 \end{bmatrix}$$

Step 2. Find the minimum complex fuzzy adjacency matrix

$$A = A_1 \cap A_2 \cap A_3 = \begin{bmatrix} 0 & 0.2+0.2i & 0.3+0.1i \\ 0.2+0.2i & 0 & 0 \\ 0.3+0.1i & 0 & 0 \end{bmatrix}$$

Step 3. The the complex fuzzy diagonal for the signals $S_1(t)$, $S_2(t)$, $S_3(t)$



$$\begin{bmatrix} 1+i & 0 & 0 \\ 0 & 0.2+0.3i & 0 \\ 0 & 0 & 0.4+0.2i \end{bmatrix}$$

Step 4. The Complex fuzzy Laplacian matrix

$$L_{CFLM} = D_{CFDM} - A_{CFAM} = \begin{bmatrix} 1+i & -0.2-0.2i & -0.3-0.1i \\ -0.2-0.2i & 0.2+0.3i & 0 \\ -0.3-0.1i & 0 & 0.4+0.2i \end{bmatrix}$$

Step 5. The eigenvalues of the Laplacian matrix are: $1.130\,1 + 1.034\,3i, 0.337\,17 + 0.241\,24i, 0.132\,75 + 0.224\,44i$. Then the average eigenvalue is given as

$$\frac{(1.130\,1 + 1.034\,3i) + (0.337\,17 + 0.241\,24i) + (0.132\,75 + 0.224\,44i)}{3}$$

$$= \frac{1.60002 + 1.49998i}{3} = 0.53334 + 0.499993i$$

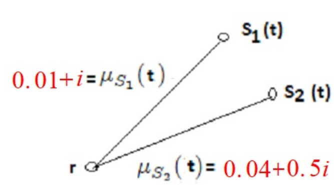
Then the absolute eigenvalue

$$|\lambda|_{CFLEV} = |0.53334 + 0.499993i| = \sqrt{.28445 + 0.249993} = 0.7311$$

Step 6. In the context of fuzzy Laplacian matrices and eigenvalue analysis, a fuzzy absolute average eigenvalue of 0.7311 indicates the average strength of the fuzzy relationships in the network. An average absolute eigenvalue of the complex fuzzy graph with value 0.7311 provide a strong comlpex fuzzy relationships between clean and noisy signals. This strength indicates that it is easy to completely separate clean signals from noisy ones.

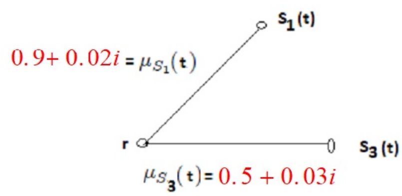
Assessment: Identifications of Reference Signals within the Larger Noisy Environments

Step 1. The complex adjacency matrix for signals two and three received by the digital receiver. The complex fuzzy graph for signals $S_1(t)$ and $S_2(t)$ is given as:



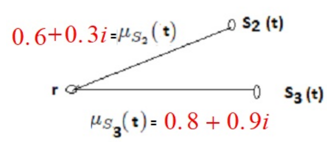
$$A_1 = \begin{bmatrix} 0 & 0.01+i & 0.04+0.5i \\ 0.01+i & 0 & 0 \\ 0.04+0.5i & 0 & 0 \end{bmatrix}$$

The complex fuzzy adjacency for signals $S_1(t)$ and $S_3(t)$ is given as



$$\begin{bmatrix} 0 & 0.9+0.02i & 0.5+0.03i \\ 0.9+0.02i & 0 & 0 \\ 0.5+0.03i & 0 & 0 \end{bmatrix}$$

The complex fuzzy adjacency matrix for signals $S_2(t)$ and $S_3(t)$ is given as

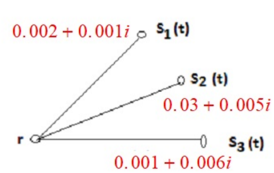


$$\begin{bmatrix} 0 & 0.6+0.3i & 0.8+0.9i \\ 0.6+0.3i & 0 & 0 \\ 0.8+0.9i & 0 & 0 \end{bmatrix}$$

Step 2. The complex fuzzy intersection of the adjacency matrices

$$A = A_1 \cap A_2 \cap A_3 = \begin{bmatrix} 0 & 0.01+0.02i & 0.04+0.03i \\ 0.01+0.02i & 0 & 0 \\ 0.04+0.03i & 0 & 0 \end{bmatrix}$$

Step 3. The complex fuzzy diagonal matrix for graph is given as



$$D_{CFDM} = \begin{bmatrix} 0.002 + 0.001i & 0 & 0 \\ 0 & 0.03 + 0.005i & 0 \\ 0 & 0 & 0.001 + 0.006i \end{bmatrix}$$

Step 4. The complex fuzzy Laplacian matrix for graph is given as

$$L_{CFLM} = D_{CFDM} - A_{CFAM} = \begin{bmatrix} 0.002 + 0.001i & -0.01 - 0.02i & -0.04 - 0.03i \\ -0.01 - 0.02i & 0.03 + 0.005i & 0 \\ -0.04 - 0.03i & 0 & 0.001 + 0.006i \end{bmatrix}$$

Step 5. Eigenvalues of the Laplacian matrix are:

$$4.497\,0 \times 10^{-2} + 3.946\,0 \times 10^{-2}i,\ 2.565\,4 \times 10^{-2} + 2.376\,3 \times 10^{-3}i, -3.762\,4 \times 10^{-2} - 2.\,983\,6 \times 10^{-2}i$$

The average eigenvalue is:

$$\begin{aligned} & \frac{0.04497 + .03946\,0i + (0.02565\,4 + 0.002376\,3i) + (-0.03762\,4 - 0.029836i)}{3} \\ = \quad & 0.03 + 0.012i \end{aligned}$$

Thus the absolute average eigenvalue is given as:

$$|\lambda|_{CFAEV} = \sqrt{0.0009 + 0.000144} = \sqrt{0.001044} = 0.03$$

Step 6. In the context of fuzzy Laplacian matrices and eigenvalue analysis, a fuzzy absolute average eigenvalue of 0.03 indicates the average strength of the fuzzy relationships in the network. The average eigenvalue 0.03 provide the indications of weak relation between clear and nosy signals which yields that it is quite difficult to separate clean signals from noise ones.

In this case we need some advanced techniques for identification of reference signals. And one of the techniques is Signal-to-Noise Ratio (SNR). This a ratio formula showing strength of a signal within the noisy environments. This formula is used to find the ratio of strength of any electronic signal or any other signal and strength of noise in the environment which effect the signal. This formula provides the comparison of the desired signal to the level of background noise showing that how much the signal is clear in the presence of noise.

SNR is typically expressed in decibels (dB) and can be calculated using the following formula (from Wikipedia):

$$SNR_{dB} = 10 \times \log_{10}\left(\frac{P_{\text{Signal}}}{P_{\text{Noise}}}\right) = 10\log_{10} P_{\text{Signal}} - 10\log_{10} P_{\text{noise}}$$

where P_{Signal} the power of signal while P_{noise} is the noise in the background. In this formula, P is measurable in units of power, like watts (W) or milliwatts (mW), and the signal-to-noise ratio is a simply a number.

The high value of this ratio indicates that signal is stronger as compare to existing noise in the environment while lower value shows the weak comparison to the noise, in this case it is difficult to detect the signal precisely and accurately which provide information about existence of error in the transmission of data and indicates the bad performance of the system. Different values of this ratio shows diversified indications in the systems, for instance in telecommunications, a particular SNR is needed for ensuring the clear voice communications and a specific value of SNR is required for the accuracy of decoding transmitted information. A high value of SNR is necessary for radar and medical imaging where weak signals need to be detected against a noisy background. This guaranteed signals of carrying vital information (such as a tumor in a medical image or an incoming aircraft in radar) cannot be lost in noisy background which is important for well decision making. For improvements in SNR various techniques are used as filtering, modulation schemes, and error-correcting codes to improve SNR. The purpose of such methods are, to improve the strength of the signal in presence of disturbance such as noise in the environment for the good quality of signals. In short Signal-to-Noise Ratio (SNR) is a dynamic formula that can be used to measure the quality of a signal by comparing its strength to the noise in the environment. A higher value of SNR indicates a cleaner and more reliable signal, while a small value of SNR gives a weaker signal which will be challenging to detect and interpret the signal accurately.

4. Comparison Analysis

Complex fuzzy dynamic graphs, and their associated Laplacian matrices' absolute eigenvalues, provide useful and powerful tools for modeling complex systems involving two dimensional data while fuzzy graph models provide a foundation for understanding relationships, but their one dimensional limitation reduces their capacity to represent uncertainties with two dimensional ambiguities. Fuzzy graph approaches are enhancements of traditional graphs but still they possesses flaws for accommodating all kinds of imprecise relationships, especially the data with dynamic nature. In contrast, complex fuzzy dynamic graphs possesses the capability for handling data with two dimensional ambiguities with dynamic phenomenon. The absolute average eigenvalues analysis of complex fuzzy Laplacian matrices not only reveals the underlying graph structures but also provides unique spectral understandings which play a crucial role in signals processing. As compare to the existing spectral methods and machine learning techniques, this approach provide a superior role, particularly in scenarios where signals are effected by intricate, non-linear, and two dimensional uncertain relationships within dynamic systems. This framework not only enriches theoretical foundations but also significantly enhances the accuracy and interpretability of signals processing tasks.

5. Conclusion

Dynamic natured real life phenomenon along with involvements of two dimensional data is the crucial part of study and the complex fuzzy dynamic graphs, their the exploration of their Laplacian matrices' absolute eigenvalues provide well instructed path towards deeper comprehension and more refined analysis. We explored the properties of complex graphs, their corresponding complex fuzzy Laplacian matrices with their absolute average eigenvalues. We gave several normal example and example from daily life revealing the nature and importance of the complex fuzzy dynamic graphs. By interpreting the spectral complexities encoded in the Laplacian matrices, we have gained unprecedented insights into the underlying structures of complex phenomena, allowing for more accurate and nuanced signal processing methodologies. We designed an algorithm using the properties of average eigenvalues of the Laplacain complex fuzzy dynamic graphs and discussed its applications in signals processing. This study through the intersection of complex fuzzy dynamic graphs and real-world applications has not only broadened our limits in understanding complex systems but also opened new ways to novel solutions in diverse fields, especially in signal processing.

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