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Results on Common Fixed Points in Strong-Composed-Cone Metric Spaces

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Abstract. The current study aims to propose several generalizations of a strong b-metric space which is called Strong-composed cone metric spaces. Therefore, to illustrate the concept of these generalizations, the study provides examples of Strong-composed cone metric space, which are neither a Strong-controlled metric type space nor Strong b-metric space, also redefined with cone metric spaces. Finally, the study demonstrates the uniqueness of some fixed-point results involving some general structures of nonlinear rational contractions with applications.

1. Introduction

In recent years, there has been a surge in interest in fixed point theorem (FPT). Its modification depends on tools of triangular inequality of metric space via important contractions in extending the concept of the fixed-point theorem with applications. In 1989, Bakhtin [1] investigated a metric called the b-metric space (bMS), which is generalized to metric space. Many previous works in this area dealt with the important properties of (bMS), (see [2, 3]), whereas others focused their attention on (SbMS) as in Kirk [4], extending (SbMS) via some fixed-point theorems as in [5]. In 2023, Santina et al. introduced a new generalization of (SbMS) called controlled-strong b-metric type space (CSbMS), through some fixed-point theorems with famous applications [6]. In 2024, Anas et al. presented an expansion to CSbMS known as strong composed metric space (SCMS) [7] (for more details see [8–13]). Despite all of these studies, there is much work concerning the application of special contractions to (SbMS) (see [14, 15]).

There are various previous works on metric space. In 2007, Huang et al. [16] introduced the notion of cone metric space as an expanded metric space. Hussain et al. [17] presented the cone

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b-metric spaces. Shateri [18] provided some fixed-point theorems on double controlled cone metric spaces. Subsequently, Anas et al. in [19] introduced type I and II composed cone metric space, and in [20] the extension of double-composed metric space to double composed metric like space (for more details refer to [21–26]). Moreover, in 2020, Lateef [10] proved Fisher type fixed point results in controlled metric spaces. Later, several authors including Dass and Gupta [27] and Jaggi [28] discussed their results utilizing a contraction condition of the rational type. The authors in [29] gave a generalization of rational contractions in double controlled metric space for common fixed-point theorem (for more details see [7,25,29,31]).

The objectives of this study establish an extended concept of CSbMS called strong-composed cone metric space (SCCMS), which satisfies the inequality: $\zeta_f(a,b) \leq \zeta_f(a,c) + f(\zeta_f(c,b))$, where $f : P \to P$ is an auxiliary nonconstant function, P is cone and $a,b,c \in \Gamma$, represent the reverse in not necessarily true example, while CSbMS does not imply to SbMS in cone metric space. Further, the first step in this study is employing the concept of four mappings in common fixed points results via numerical contraction, using the study of Matkowski [32]. In addition, utilizing the study of Karami [11], a new generalization of ϕ-contraction for four maps is created and rational. Finally, the study introduces an application of polynomial and nonlinear integral equations which support the fixed-point theorems within these new spaces.

2. Preliminaries

This section presents some notations and basic concepts of definitions and lemmas from earlier research. These concepts are then employed throughout the main findings of this study.

Definition 2.1. *[16] Let E be a real Banach space and $P \subset E$. P is called a cone if it satisfies the following conditions:*

(P1) $\{0_E\} \neq P$ *is nonempty closed,*

(P2) $\alpha_1 a + \alpha_2 b \in P$ *for all* $a,b \in P$, *where* $\alpha_1, \alpha_2 \geq 0$,

(P3) $P \cap (-P) = \{0_E\}$.

Considering a cone P, a partial ordering $\preceq$ on E can be defined with respect to P by $a \preceq b$ if and only if $b - a \in P$. Here, $a \prec b$ indicates that $a \preceq b$ and $a \neq b$, but $a \ll b$ stands for $b - a \in intP$, such that $intP$ denotes the interior of P.

Let E be a Banach space, P be a cone in E such as $intP \neq \phi$ and $\preceq$ be a partial ordering of P. The cone P is called normal if there exists a constant number $M > 0$ such that for all $a,b \in E$ and $0 \preceq a \preceq b$ implies that $||a|| \leq M||b||$ or equivalently, if
$inf\{||a+b|| | a,b \in P, ||a|| = ||b|| = 1\} > 0$ for non-normal cone, (e.g., see [18]). Moreover, P is called solid if $intP \neq \phi$.

Now, some basic notations of cone metric spaces are presented with their properties.
Abdeljawad et al. [9] present the double controlled type-metric spaces. Moreover, the expanded-on cone metric space is introduced as follows:

Definition 2.2. *[18] Let Γ be a nonempty set and $\omega_1, \omega_2 : \Gamma \times \Gamma \to [1, \infty)$. A function $\sigma : \Gamma \times \Gamma \to E$, if for all $a, b, c \in \Gamma$, satisfying the following conditions:*

(σ1) $\sigma(a, b) = 0_E$ *if and only if* $a = b$,

(σ2) $\sigma(a, b) = \sigma(b, a)$,

(σ3) $\sigma(a, b) \leq \omega_1(a, c)\sigma(a, c) + \omega_2(c, b)\sigma(c, b)$.

Then, the pair (Γ, σ) is called a double controlled-cone-metric type space (DCCMTS), and it is called a controlled strong-cone-metric type space if $\omega_1 = 1$ or $\omega_2 = 1$, not both (for short, CSCMTS), and strong cone b-metric space (SCbMS) if a function says $\omega_2 = s, s \geq 1$.

Example 2.1. *Let $E = C(\mathbb{R}), P = \{\varphi(t) \in E : \varphi(t) \geq 0, t \in [0, 1]\}$ and $\Gamma = [1, \infty)$ and define $\sigma(a, b) = \{|a - b|, 2|a - b| - 1\}\varphi(t)$ for all $a, b \in \Gamma$ and $t \in [0, 1]$ such that $\omega_2(a, b) = \{a, b\} + 1$.*
Then, (Γ, ω) is a CSCMTS, but not a controlled strong b-metric type space.

Anas et al. [7] introduced the strong-composed metric type space, which is the triangular inequality, exhibited by all $a, b, c \in \Gamma, \mathcal{S}_\psi : \Gamma \times \Gamma \to \mathbb{R}^+$, $\mathcal{S}_\psi(a, b) \leq \mathcal{S}_\psi(a, c) + \psi(\mathcal{S}_\psi(c, b))$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ nonconstant function. Now, the strong-composed cone-metric space is presented as the follows:

Definition 2.3. *Let Γ be a nonempty set. A mapping $\zeta_f : \Gamma \times \Gamma \to E$ is a strong composed cone-metric if for all $a, b, c \in \Gamma$, there exists a nonconstant function $f : P \to P$, satisfying the following necessaries:*

(ζ1) $\zeta_f(a, b) = 0_E$ *if and only if* $a = b$,

(ζ2) $\zeta_f(a, b) = \zeta_f(b, a)$,

(ζ3) $\zeta_f(a, b) \leq \zeta_f(a, c) + f\Big(\zeta_f(c, b)\Big)$.

Then, the triple (Γ, ζ_f, f) is called a strong composed cone-metric space (SCCMS) with regarly f. Obviously, every CSCMTS is a SCCMS, but the reverse is not necessarily true, the following example clarifies this case.

Example 2.2. *Let $E = C(\mathbb{R}^2), P = \{\varphi(t) \in E : \varphi(t) \geq 0, t \in [0, 1]\}$ and $(\Gamma, \mathcal{S}_b)$ be a strong b-metric space via $s > 1$ and let $\zeta_f(a, b) = \Big((\mathcal{S}_b(a, b))e^t, 0\Big)$ for all $a, b \in \Gamma$ and $t \in [0, 1]$.*
It is enough to show that (ζ3). Since $Sinh^{-1}(w)$ is an increasing function, hence for all $a_1, a_2 \geq 0$, this undergoes (by a part 4 in [11]):

$$Sinh^{-1}(a_1 + a_2) \leq Sinh^{-1}(a_1) + Sinh^{-1}(a_2). \tag{2.1}$$

Therefore, for all $a, b, c \in \Gamma$, the following result is obtained:

$$\begin{aligned}
(\mathcal{S}_b(a, b))e^t &\leq \Big(\mathcal{S}_b(a, c) + s\mathcal{S}_b(a, b)\Big)e^t \\
&\leq \mathcal{S}_b(a, c)e^t + s\mathcal{S}_b(c, b)e^t \\
&\leq Sinh^{-1}\Big(\mathcal{S}_b(a, c)\Big)e^t + Sinh^{-1}\Big(sSinh\Big(\mathcal{S}_b(c, b)\Big)\Big)e^t \\
&\leq Sinh^{-1}\Big(\mathcal{S}_b(a, c)\Big)e^t + Sinh^{-1}\Big(sSinh\Big(\mathcal{S}_b(c, b)e^t\Big)\Big)e^t
\end{aligned}$$

Thus,

$$\zeta_f(a, b) \leq \zeta_f(a, c) + f(\zeta_f(c, b)),$$

where $f(w) = \Big(Sinh^{-1}(sSinh(w_1))e^t, 0\Big), w \in P$. *It is clear that it is not a CSCMTS. But* (Γ, ζ_f) *is an SCCMS.*

Remark 2.1. *The assumption in example 2.2 can be interchanged if* (Γ, σ_w) *is considered a controlled strong b-metric space via* $w : \Gamma \times \Gamma \to [1, \infty)$. *Also, the same result is obtained such that* $f(w) = \Big(Sinh^{-1}(w(a,b)Sinh(w_1))e^t, 0\Big)$.

Example 2.3. *Let* $(\Gamma, \mathcal{S})$ *be a* $\mathcal{S}$*-metric space, then*

$$\mathcal{S}(a,a,b) \le 2\mathcal{S}(a,a,c) + \mathcal{S}(b,b,c), \forall a,b,c \in \Gamma. \tag{2.2}$$

Assume that $E = C(\mathbb{R}), P = \{\varphi(t) \in E : \varphi(t) \ge 0, t \in [0,1]\}$. *SCCMS is defined by* $\zeta_f(a,b) = Sinh^{-1}\Big(\mathcal{S}(a,a,b)\Big)e^t$ *for all* $a, b \in \Gamma, t \in [0,1]$, *where* $\varphi(t) = e^t$.
Clearly, $(\zeta 1)$ *and* $(\zeta 2)$ *are held. It is clear that* $(\zeta 3)$ *by* (2.2), *the result is:*

$$\begin{aligned}\zeta_f(a,b) = Sinh^{-1}\Big(\mathcal{S}(a,a,b)\Big)e^t &\le Sinh^{-1}\Big(2\mathcal{S}(a,a,c) + \mathcal{S}(b,b,c)\Big)e^t \\ &\le Sinh^{-1}\Big(2\mathcal{S}(a,a,c)\Big)e^t + Sinh^{-1}\Big(\mathcal{S}(b,b,c)\Big)e^t \\ &\le \zeta_f(c,b) + f\Big(\zeta_f(a,c)\Big),\end{aligned}$$

where $f(w) = Sinh^{-1}\Big(2Sinh(w)\Big)e^t$, *and* $w \in P, t \in [0,1]$.

First, open and closed balls are defined in SCCMS.

Definition 2.4. *Let us choose* $a \in \Gamma$ *and for some* $0_E \le c$ *defined*
$\mathcal{B}(a,c) = \{b \in \Gamma : \zeta_f(a,b) \ll c\}$ *and* $\overline{\mathcal{B}(a,c)} = \{b \in \Gamma : \zeta_f(a,b) \le c\}$ *are called open and closed balls, respectively.*

Next, the notion of convergence is defined in SCCMS.

Definition 2.5. *Let* (Γ, ζ_f) *be an SCCMS and E be a real Banach space via a cone P. Then:*

1. $\{a_n\}$ *in* Γ *converges to a if for every* $c \in E$ *with* $0_E \le c$, *there is* $N \in \mathbb{N}$ *such that for all* $n \ge N, \zeta_f(a_n, a) \ll c$. *It is denoted as* $\lim_{n\to\infty} a_n = a$.
2. $\{a_n\}$ *in* Γ *is said to be Cauchy if for every* $c \in E$ *with* $0_E \le c$, *there is* $N \in \mathbb{N}$ *such that for all* $n, m \ge N, \zeta_f(a_n, a_m) \ll c$.
3. (Γ, ζ_f) *is said to be a complete SCCMS if every Cauchy sequence in* Γ *converges to some point in* Γ.

Lemma 2.1. *Let* (Γ, ζ_f) *be an SCCMS with respect to* f, P *be a normal cone with normal constant M. Let* $\{a_n\}$ *be a sequence in* Γ. *Then,* $\{a_n\}$ *converges to a if and only if* $\zeta_f(a_n, a) = 0_E$.

Proof. By ([19], Lemma 1.5) just taking $\psi_1 = I$ and $\psi_2 = f$, an SCCMS exists. □

Lemma 2.2. *Let* (Γ, ζ_f) *be an SCCMS with respect to* f, P *be a normal cone with normal constant M. Let* $\{a_n\}$ *be a sequence in* Γ *such that* $\{a_n\}$ *converges to a and b. If f is bounded, then* $a = b$. *That is, the limit of* $\{a_n\}$ *is unique.*

Proof. By ([19], Lemma 1.6) just taking $\psi_1 = I$ and $\psi_2 = f$, an SCCMS exists. □

Proposition 2.1. *Let* (Γ, ζ_f) *be an SCCMS, then for all* $a, b, c \in \Gamma$

$$|\zeta_f(a,b) - \zeta_f(c,d)| \leq f\Big(\zeta_f(a,d)\Big) + f\Big(\zeta_f(b,c)\Big).$$

Proof. Utilizing the axiom $(\zeta 3)$, then

$$\begin{aligned}\zeta_f(a,b) &\leq \zeta_f(a,c) + f\Big(\zeta_f(c,b)\Big)\\ &\leq \zeta_f(c,d) + f\Big(\zeta_f(d,a)\Big) + f\Big(\zeta_f(b,c)\Big)\end{aligned}$$

It implies that

$$\zeta_f(a,b) - \zeta_f(c,d) \leq f\Big(\zeta_f(a,d)\Big) + f\Big(\zeta_f(b,c)\Big). \tag{2.3}$$

A similar argument shows that

$$\zeta_f(c,d) - \zeta_f(a,b) \leq f\Big(\zeta_f(a,d)\Big) + f\Big(\zeta_f(b,c)\Big). \tag{2.4}$$

The desired result is obtained. □

Remark 2.2. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be a sequences in* Γ *such that* $\lim_{n\to\infty} \zeta_f(a_n, a) = 0_E$ *and* $\lim_{m\to\infty} \zeta_f(b_m, b) = 0_E$*, then by Proposition 2.1, the following result is obtained:*
$\lim_{n,m\to\infty} \zeta_f(a_n, b_m) = \zeta_f(a,b)$*, where* f *is bounded; this means that,* ζ_f *is continuous.*

Lemma 2.3. *[4] The strong b-metric space is normal.*

Lemma 2.4. *An SCCMS is normal, where* f *is bounded.*

Proof. Let (Γ, ζ_f) be an SCCMS. If $a, b \in \Gamma$ such that $a \neq b$, then $\mathcal{V} := \mathcal{B}(a, \frac{c}{2(M+1)})$ and $\mathcal{W} := \mathcal{B}(b, \frac{c}{2(M+1)})$ are disjoint neighborhoods of a and b, respectively. Then, assume that $\mathcal{V} \cap \mathcal{W} \neq \phi$, hence, there exists $d \in \mathcal{V} \cap \mathcal{W}$. Thus, by utilizing $\zeta_f(a,d) \prec \frac{c}{2(M+1)}$ and $\zeta_f(b,d) \prec \frac{c}{2(M+1)}$, where $c = \zeta_f(a,b)$, the following result is obtained:

$$\begin{aligned}c = \zeta_f(a,b) &\leq \zeta_f(a,d) + f\Big(\zeta_f(d,b)\Big)\\ &\prec \frac{c}{2(M+1)} + f\Big(\zeta_f(d,b)\Big).\end{aligned} \tag{2.5}$$

Since f is bounded, then there is $M > 0$ such that

$$\|f\Big(\zeta_f(d,b)\Big)\| \leq M\|\zeta_f(d,b)\|. \tag{2.6}$$

Utilizing the norm in (2.5) and (2.6) results in:

$$c = \|\zeta_f(a,b)\| < \frac{c}{2(M+1)} + M\frac{c}{2(M+1)} = \frac{c}{2} < c.$$

Hence, this represents a contradiction, so our claim holds. Therefore, it is concluded that Γ is Hausdorff.
Now, let $\mathcal{V}$ and $\sqsupseteq$ be disjoint closed sets and let $\zeta_f(a, \mathcal{V}) := inf_{d\in\mathcal{V}}\zeta_f(a,d)$ and $\zeta_f(b, \mathcal{W}) := inf_{g\in\mathcal{W}}\zeta_f(a,g)$. Define the sets
$V' := \{a \in \Gamma : \zeta_f(a, \mathcal{V}) \prec \zeta_f(a, \mathcal{W})\}$ and $W' := \{b \in \Gamma : \zeta_f(b, \mathcal{W}) \prec \zeta_f(b, \mathcal{V})\}$. □

Definition 2.6. *[33] Let* Φ *be the set of all continuous self-maps* ϕ *of* P*, satisfying*

(1) *ϕ is monotonically increasing,*
(2) *$\phi(w) = 0_E$ if and only if $w = 0_E$.*

Then, it is called an altering distance function on the cone P.

Let Ψ be the family of all mappings $\psi : [0,\infty) \to [0,\infty)$, satisfying the condition $t \leq \psi(t)$ for each $t \in [0,\infty)$, and ψ' is increasing (the derivative of ψ) [11].

Lemma 2.5. *[7] If $\psi \in \Psi$, then for all $a, b \in [0,\infty)$, the result is:*

$$|\psi^{-1}(a) - \psi^{-1}(b)| \leq \psi^{-1}(|a-b|) \leq |a-b| \leq \psi(|a-b|) \leq |\psi(a) - \psi(b)|.$$

In particular, if $b = 0$, that is, $|\psi^{-1}(a)| \leq \psi^{-1}(|a|) \leq |a| \leq \psi(|a|) \leq |\psi(a)|$.

Lemma 2.6. *[20] Let $\psi \in \Psi$, then for all $a \in [0,1]$ and $0 < \mathfrak{q} \leq 1 \leq \mathfrak{p}$, the following result is obtained:*

(1) $\Big(\psi(a^{\mathfrak{p}})\Big)^{\frac{1}{\mathfrak{p}}} \leq \psi(a) \leq \Big(\psi(a^{\mathfrak{q}})\Big)^{\frac{1}{\mathfrak{q}}}$.
(2) $\Big(\psi^{-1}(a^{\mathfrak{q}})\Big)^{\frac{1}{\mathfrak{q}}} \leq \psi^{-1}(a) \leq \Big(\psi^{-1}(a^{\mathfrak{p}})\Big)^{\frac{1}{\mathfrak{p}}}$.

3. The Main Results

This section presents four common fixed-point results with two ϕ-contractions in SCCMS.

Theorem 3.1. *Let (Γ, ζ_f) be a complete SCCMS with functions $f : P \to P$ and P be a normal cone via normal constant M. Consider $T_1, T_2, T_3, T_4 : \Gamma \to \Gamma$ be a self-mapping such that*

(1) *$T_1(\Gamma) \subseteq T_4(\Gamma)$ and $T_2(\Gamma) \subseteq T_3(\Gamma)$,*
(2) *The pairs (T_1, T_4) and (T_2, T_3) are compatible,*
(3) *T_i is continuous for all $i = 1, \cdots, 4$,*
(4) *For all $a, b \in \Gamma$,*

$$\phi\Big(\zeta_f(T_1a, T_2b)\Big) \leq \lambda_1 \phi\Big(\aleph_1(a,b)\Big) + \lambda_2 \phi\Big(\aleph_2(a,b)\Big), \tag{3.1}$$

where, $\phi \in \Phi$, $0 < \lambda_1 + \lambda_2 < 1$, and

$$\aleph_1(a,b) = Max\Bigg\{\zeta_f(T_3a, T_4b), \zeta_f(T_3a, T_1a), \zeta_f(T_4b, T_2b), \frac{\zeta_f(T_3a, T_1a)\zeta_f(T_4b, T_2b)}{1 + \zeta_f(T_3a, T_4b)},$$
$$\frac{\zeta_f(T_3a, T_1a)[1 + \zeta_f(T_4b, T_2b)]}{1 + \zeta_f(T_3a, T_4b)}, \frac{[\zeta_f(T_3a, T_1a) + \zeta_f(T_4b, T_2b)]\zeta_f(T_1a, T_2b)}{1 + \zeta_f(T_1a, T_2b) + \zeta_f(T_3a, T_4b)}\Bigg\},$$

$$\aleph_2(a,b) = Min\Big\{\zeta_f(T_1a, T_4b), \zeta_f(T_2b, T_3a)\Big\}.$$

(5) $\sum_{i=m}^{n-2} \|f\Big(\lambda_1^i \phi(\zeta_f(b_1, b_0))\Big)\| + \|\lambda_1^{n-1} \phi\Big(\zeta_f(b_1, b_0)\Big)\| \to 0.$

Then, $T_i, i = 1, \cdots, 4$ have a unique common fixed point in Γ.

Proof. Let $a_0 \in \Gamma$ be arbitrary. By the assumption $T_2(\Gamma) \subseteq T_3(\Gamma)$, so there exists a_1 in Γ such that $T_2a_0 = T_3a_1$ and also as $T_1(\Gamma) \subseteq T_4(\Gamma)$, that is $T_1a_1 \in T_4(\Gamma)$, hence taking $a_2 \in \Gamma$, where $T_1a_1 = T_4a_2$. In general, the following result is obtained:

$$T_1a_{2n+1} = T_4a_{2n+2} \quad \text{and} \quad T_2a_{2n} = T_3a_{2n+1}, \forall n \in \mathbb{N}. \tag{3.2}$$

A sequence b_n is obtained in Γ such that

$$b_{2n+1} = T_1 a_{2n+1} = T_4 a_{2n+2} \quad \text{and} \quad b_{2n} = T_2 a_{2n} = T_3 a_{2n+1}, \forall n \in \mathbb{N}. \tag{3.3}$$

Next, the results prove that $\{b_n\}$ is a Cauchy sequence in Γ. It is deduced that

$$\phi\Big(\zeta_f(b_{2n+1}, b_{2n})\Big) = \phi\Big(\zeta_f(T_1 a_{2n+1}, T_2 a_{2n})\Big) \preceq \lambda_1 \phi\Big(\aleph_1(a_{2n+1}, a_{2n})\Big) + \lambda_2 \phi\Big(\aleph_2(a_{2n+1}, a_{2n})\Big),$$

where

$$\begin{aligned}
\aleph_1(a_{2n+1}, a_{2n}) &= Max\Big\{\zeta_f(T_3 a_{2n+1}, T_4 a_{2n}), \zeta_f(T_3 a_{2n+1}, T_1 a_{2n+1}), \zeta_f(T_4 a_{2n}, T_2 a_{2n}), \\
&\frac{\zeta_f(T_3 a_{2n+1}, T_1 a_{2n+1})\zeta_f(T_4 a_{2n}, T_2 a_{2n})}{1 + \zeta_f(T_3 a_{2n+1}, T_4 a_{2n})}, \frac{\zeta_f(T_3 a_{2n+1}, T_1 a_{2n+1})[1 + \zeta_f(T_4 a_{2n}, T_2 a_{2n})]}{1 + \zeta_f(T_3 a_{2n+1}, T_4 a_{2n})}, \\
&\frac{[\zeta_f(T_3 a_{2n+1}, T_1 a_{2n+1}) + \zeta_f(T_4 a_{2n}, T_2 a_{2n})]\zeta_f(T_1 a_{2n+1}, T_2 a_{2n})}{1 + \zeta_f(T_1 a_{2n+1}, T_2 a_{2n}) + \zeta_f(T_3 a_{2n+1}, T_4 a_{2n})}\Big\}, \\
&= Max\Big\{\zeta_f(b_{2n}, b_{2n-1}), \zeta_f(b_{2n}, b_{2n+1}), \zeta_f(b_{2n-1}, b_{2n}), \frac{\zeta_f(b_{2n}, b_{2n+1})\zeta_f(b_{2n-1}, b_{2n})}{1 + \zeta_f(b_{2n}, b_{2n-1})}, \\
&\frac{\zeta_f(b_{2n}, b_{2n+1})[1 + \zeta_f(b_{2n-1}, b_{2n})]}{1 + \zeta_f(b_{2n}, b_{2n-1})}, \frac{[\zeta_f(b_{2n}, b_{2n+1}) + \zeta_f(b_{2n-1}, b_{2n})]\zeta_f(b_{2n+1}, b_{2n})}{1 + \zeta_f(b_{2n+1}, b_{2n}) + \zeta_f(b_{2n}, b_{2n-1})}\Big\}, \\
&\preceq Max\{\zeta_f(b_{2n}, b_{2n-1}), \zeta_f(b_{2n}, b_{2n+1})\}.
\end{aligned} \tag{3.4}$$

Furthermore,

$$\begin{aligned}
\aleph_2(a_{2n+1}, a_{2n}) &= Min\{\zeta_f(T_1 a_{2n+1}, T_4 a_{2n}), \zeta_f(T_2 a_{2n}, T_3 a_{2n+1})\} \\
&= Min\{\zeta_f(b_{2n+1}, b_{2n-1}), \zeta_f(b_{2n}, b_{2n})\} = 0_E
\end{aligned} \tag{3.5}$$

Thus,

$$\phi\Big(\zeta_f(b_{2n+1}, b_{2n})\Big) \preceq \lambda_1 \phi\Big(\aleph_1(a_{2n+1}, a_{2n})\Big) + \lambda_2 \phi\Big(\aleph_2(a_{2n+1}, a_{2n})\Big), \tag{3.6}$$

where $\aleph_1(a_{2n+1}, a_{2n}) = Max\{\zeta_f(b_{2n}, b_{2n-1}), \zeta_f(b_{2n}, b_{2n+1})\}$, and $\aleph_2(a_{2n+1}, a_{2n}) = 0_E$.

Case 1. If $\aleph_1(a_{2n+1}, a_{2n}) = \zeta_f(b_{2n}, b_{2n+1})$, then by (3.6), it is deduced that

$$\phi(\zeta_f(b_{2n+1}, b_{2n})) \preceq \lambda_1 \phi\Big(\zeta_f(b_{2n}, b_{2n+1})\Big) \prec \phi\Big(\zeta_f(b_{2n}, b_{2n+1})\Big), \tag{3.7}$$

clarifying a contradiction.

Case 2. If $\aleph_1(a_{2n+1}, a_{2n}) = \zeta_f(b_{2n}, b_{2n-1})$, based on (3.6), it is concluded that

$$\phi\Big(\zeta_f(b_{2n+1}, b_{2n})\Big) \preceq \lambda_1 \phi\Big(\zeta_f(b_{2n}, b_{2n-1})\Big) \preceq \cdots \preceq \lambda_1^n \phi\Big(\zeta_f(b_1, b_0)\Big). \tag{3.8}$$

Applying it recursively, the result is:

$$\phi\Big(\zeta_f(b_n, b_{n+1})\Big) \preceq \lambda_1^n \phi\Big(\zeta_f(b_1, b_0)\Big), \tag{3.9}$$

For $m < n$, and $n, m \in \mathbb{N}$, the following result is obtained:

$$\begin{aligned}\zeta_f(b_m, b_n) &\leq f\Big(\zeta_f(a_m, a_{m+1})\Big) + \zeta_f(a_{m+1}, a_n)\\ &\leq f\Big(\zeta_f(a_m, a_{m+1})\Big) + f\Big(\zeta_f(a_{m+1}, a_{m+2})\Big) + \zeta_f(a_{m+2}, a_n)\\ &\vdots\\ &\leq \sum_{i=m}^{n-2} f\Big(\zeta_f(a_i, a_{i+1})\Big) + \zeta_f(a_{n-1}, a_n). \end{aligned} \tag{3.10}$$

Employing (3.9) in (3.10) leads to:

$$\|\phi\Big(\zeta_f(b_m, b_n)\Big)\| \leq M\left[\sum_{i=m}^{n-2} \|f\Big(\lambda_1^i \phi\Big(\zeta_f(b_1, b_0)\Big)\Big)\| + \|\lambda_1^{n-1}\phi\Big(\zeta_f(b_1, b_0)\Big)\|\right].$$

Thus, as $n, m \to \infty$, and by using item 5, the result is:

$$\|\zeta_f(b_m, b_n)\| = 0.$$

Therefore, the sequence $\{b_n\}$ is called Cauchy. Since (Γ, ζ_f) is a complete SCCMS, there must exist an element $b \in \Gamma$ such that $\{b_n\} \to b$. Consider

$$\zeta_f(b_n, b) = 0_E. \tag{3.11}$$

It implies that

$$T_1 a_{2n+1} = T_2 a_{2n} = T_3 a_{2n+1} = T_4 a_{2n+2} = b.$$

It is argued that $T_i b = b$, and by the assumption items $2, 3$ are continuous and compatible for all $i = 1, \cdots, 4$. Hence,

$$T_4 b = T_4 T_2 a_{2n} = T_4 T_2 a_{2n} = T_2 T_4 a_{2n} = T_2 T_4 a_{2n} = T_2 b. \tag{3.12}$$

Moreover, it is concluded that

$$T_1 b = T_1 T_3 a_{2n+1} = T_1 T_3 a_{2n+1} = T_3 T_1 a_{2n+1} = T_3 T_1 a_{2n+1} = T_3 b. \tag{3.13}$$

Employing the condition 4 of inequality (3.1), let $p = T_4 b = T_2 b$ and $q = T_1 b = T_3 b$, the following result is obtained:

$$\phi\Big(\zeta_f(q, p)\Big) = \phi\Big(\zeta_f(T_1 b, T_2 b)\Big) \leq \lambda_1 \phi\Big(\aleph_1(b, b)\Big) + \lambda_2 \phi\Big(\aleph_2(b, b)\Big),$$

where

$$\begin{aligned}\aleph_1(b, b) = Max\Bigg\{&\zeta_f(T_3 b, T_4 b), \zeta_f(T_3 b, T_1 b), \zeta_f(T_4 b, T_2 b), \frac{\zeta_f(T_3 b, T_1 b)\zeta_f(T_4 b, T_2 b)}{1 + \zeta_f(T_3 b, T_4 b)},\\ &\frac{\zeta_f(T_3 b, T_1 b)[1 + \zeta_f(T_4 b, T_2 b)]}{1 + \zeta_f(T_3 b, T_4 b)}, \frac{[\zeta_f(T_3 b, T_1 b) + \zeta_f(T_4 b, T_2 b)]\zeta_f(T_1 b, T_2 b)}{1 + \zeta_f(T_1 b, T_2 b) + \zeta_f(T_3 b, T_4 b)}\Bigg\},\\ = \zeta_f(q, p).&\end{aligned}$$

It is noted that $\aleph_2(b,b) = Min\Big\{\zeta_f(T_1b,T_4b),\zeta_f(T_2b,T_3b)\Big\} = \zeta_f(q,p)$. Subsequently,

$$\phi\Big(\zeta_f(q,p)\Big) \preceq (\lambda_1+\lambda_2)\phi\Big(\zeta_f(q,p)\Big) \prec \phi\Big(\zeta_f(q,p)\Big),$$

where $\lambda_1+\lambda_2 < 1$. That is a contradiction. Thus, $\phi\Big(\zeta_f(q,p)\Big) = 0_E$, that is; $p = q = T_1b = T_2b = T_3b = T_4b$.

Additionally, it is deduced that $T_1q = T_1T_3b = T_3T_1b = T_3q$ and $T_2p = T_2T_4b = T_4T_2b = T_4p$.

Moreover, by inequality (3.1), the following result is found:

$$\phi\Big(\zeta_f(T_1q,T_1b)\Big) = \phi\Big(\zeta_f(T_1q,T_2b)\Big) \preceq \lambda_1\phi\Big(\aleph_1(q,b)\Big) + \lambda_1\phi\Big(\aleph_2(q,b)\Big), \tag{3.14}$$

where,

$$\begin{aligned}\aleph_1(q,b) &= Max\Bigg\{\zeta_f(T_3q,T_4b),\zeta_f(T_3q,T_1q),\zeta_f(T_4b,T_2b),\frac{\zeta_f(T_3q,T_1q)\zeta_f(T_4b,T_2b)}{1+\zeta_f(T_3q,T_4b)},\\ &\frac{\zeta_f(T_3q,T_1q)[1+\zeta_f(T_4b,T_2b)]}{1+\zeta_f(T_3q,T_4b)},\frac{[\zeta_f(T_3q,T_1q)+\zeta_f(T_4b,T_2b)]\zeta_f(T_1q,T_2b)}{1+\zeta_f(T_1q,T_2b)+\zeta_f(T_3q,T_4b)}\Bigg\},\\ &= Max\Bigg\{\zeta_f(T_1q,T_1b),\zeta_f(T_1q,T_1q),\zeta_f(T_2b,T_2b),\frac{\zeta_f(T_1q,T_1q)\zeta_f(T_2b,T_2b)}{1+\zeta_f(T_1q,T_1b)},\\ &\frac{\zeta_f(T_1q,T_1q)\Big[1+\zeta_f(T_2b,T_2b)\Big]}{1+\zeta_f(T_1q,T_1b)},\frac{\Big[\zeta_f(T_1q,T_1q)+\zeta_f(T_2b,T_2b)\Big]\zeta_f(T_1q,T_1b)}{1+\zeta_f(T_1q,T_1b)+\zeta_f(T_1q,T_1b)}\Bigg\}\\ &= \zeta_f(T_1q,T_1b),\end{aligned}$$

and

$$\aleph_2(q,b) = Min\Big\{\zeta_f(T_1q,T_4b),\zeta_f(T_2b,T_3q)\Big\} = \zeta_f(T_1q,T_1b).$$

It leads to $\phi\Big(\zeta_f(T_1q,T_1b)\Big) \preceq (\lambda_1+\lambda_2)\phi\Big(\zeta_f(T_1q,T_1b)\Big)$, whenever $\lambda_1+\lambda_2 < 1$, which implies a contradiction, so that $T_1q = T_3q = T_1b = q$.

By following the same method of proof in the mentioned equality (3.12), thus $T_2p = T_4p = T_2b = p$. That is, $p = q = T_1q = T_2q = T_3q = T_4q$. Let $T_i, i = 1,\cdots,4$ have a different fixed-point say r, then

$$\phi\Big(\zeta_f(q,r)\Big) = \phi\Big(\zeta_f(T_1q,T_2r)\Big) \preceq \lambda_1\phi\Big(\aleph_1(q,r)\Big) + \lambda_2\phi\Big(\aleph_2(q,r)\Big),$$

where,

$$\begin{aligned}\aleph_1(q,r) &= Max\Bigg\{\zeta_f(T_3q,T_4r),\zeta_f(T_3q,T_1q),\zeta_f(T_4r,T_2r),\frac{\zeta_f(T_3q,T_1q)\zeta_f(T_4r,T_2r)}{1+\zeta_f(T_3q,T_4r)},\\ &\frac{\zeta_f(T_3q,T_1q)[1+\zeta_f(T_4r,T_2r)]}{1+\zeta_f(T_3q,T_4r)},\frac{[\zeta_f(T_3q,T_1q)+\zeta_f(T_4r,T_2r)]\zeta_f(T_1q,T_2r)}{1+\zeta_f(T_1q,T_2r)+\zeta_f(T_3q,T_4r)}\Bigg\}\\ &= \zeta_f(q,r),\end{aligned}$$

and

$$\aleph_2(q,r) = Min\Big\{\zeta_f(T_1q,T_4r),\zeta_f(T_2r,T_3q)\Big\} = \zeta_f(q,r).$$

Subsequently, $\phi\Big(\zeta_f(q,r)\Big) \leq (\lambda_1+\lambda_2)\phi\Big(\zeta_f(q,r)\Big) \prec \phi\Big(\zeta_f(q,r)\Big)$, where $\lambda_1+\lambda_2<1$. Therefore, $q=r$ and it is concluded that the mappings $T_i, i=1,\cdots,4$ have a unique common fixed-point. □

Example 3.1. *Consider the set* $E=C(\mathbb{R}), P=\{\varphi(t)\in E:\varphi(t)\geq 0, t\in[0,1]\}$ *and SCCMS by* $\zeta_f(a,b) = Sinh^{-1}\Big(\mathcal{S}_b(a,b)\Big)e^t$ *for all* $a,b\in\Gamma$ *and* $t\in[0,1]$. *Let* $\Gamma=[1,\infty)$ *and* $\mathcal{S}_b$ *be CSbMS defined by* $\mathcal{S}_b(a,b) = Max\{|a-b|, 2|a-b|-1\}$ *with* $f(w)=Sinh^{-1}\Big((a+b+1)Sinh(w_1)\Big)e^t$.
Then, $\phi(w)=Sinh(w)$ *and* $T_1(a)=\frac{a+4}{5}, T_2(a)=\frac{a+3}{4}, T_3(a)=\frac{a+1}{2}, T_4(a)=\frac{a+2}{3}$ *are defined as continuous.*
Here, $T_1(\Gamma)\subseteq T_4(\Gamma)$ *and* $T_2(\Gamma)\subseteq T_3(\Gamma)$. *Also, the pairs* (T_1,T_4) *and* (T_2,T_3) *are compatible. by taking* $\lambda_1=\frac{1}{5}$ *and* $\lambda_2=\frac{1}{4}$, *it is deduced that*

$$\phi\Big(\zeta_f(T_1a,T_2b)\Big) = Sinh\left(Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+4}{5},\frac{b+3}{4}\right)\right)e^t\right) \leq \frac{1}{5}Sinh\Big(\aleph_1(a,b)\Big)+\frac{1}{4}Sinh\Big(\aleph_2(a,b)\Big),$$

where,

$$\begin{aligned}
&\aleph_1(a,b)\\
&=Max\left\{Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+1}{2},\frac{b+2}{3}\right)\right)e^t, Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+1}{2},\frac{a+4}{5}\right)\right)e^t, Sinh^{-1}\left(\mathcal{S}_b\left(\frac{b+2}{3},\frac{b+3}{4}\right)\right)e^t,\right.\\
&\frac{Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+1}{2},\frac{a+4}{5}\right)\right)Sinh^{-1}\left(\mathcal{S}_b\left(\frac{b+2}{3},\frac{b+3}{4}\right)\right)e^{2t}}{1+Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+1}{2},\frac{b+2}{3}\right)\right)e^t}, \frac{Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+1}{2},\frac{a+4}{5}\right)\right)e^t\left[1+Sinh^{-1}\left(\mathcal{S}_b\left(\frac{b+2}{3},\frac{b+3}{4}\right)\right)e^t\right]}{1+Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+1}{2},\frac{b+2}{3}\right)\right)e^t},\\
&\left.\frac{\left[Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+1}{2},\frac{a+4}{5}\right)\right)+Sinh^{-1}\left(\mathcal{S}_b\left(\frac{b+2}{3},\frac{b+3}{4}\right)\right)\right]Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+4}{5},\frac{b+3}{4}\right)\right)e^{2t}}{1+Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+4}{5},\frac{b+3}{4}\right)\right)e^t+Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+1}{2},\frac{b+2}{3}\right)\right)e^t}\right\},\\
&\aleph_2(a,b)=Min\left\{Sinh^{-1}\left(\mathcal{S}_b\left(\frac{a+4}{5},\frac{b+2}{3}\right)\right)e^t, Sinh^{-1}\left(\mathcal{S}_b\left(\frac{b+3}{4},\frac{a+1}{2}\right)\right)e^t\right\}.
\end{aligned}$$

Hence, Theorem 3.1 is fulfilled and $a_0=1\in\Gamma$ *is a common fixed point.*

Corollary 3.1. *Let* (Γ,ζ_f) *be a complete SCCMS with functions* $f:P\to P$ *and* P *be a normal cone via normal constant* M. *Consider* $T_1,T_2:\Gamma\to\Gamma$ *to be a continuous self-mapping such that, for all* $a,b\to\Gamma$,

$$\phi\Big(\zeta_f(T_1a,T_2b)\Big) \leq \lambda_1\phi\Big(\aleph_1(a,b)\Big)+\lambda_2\phi\Big(\aleph_2(a,b)\Big),$$

where, $\phi\in\Phi$, $0<\lambda_1+\lambda_2<1$, *and*

$$\begin{aligned}
\aleph_1(a,b) = Max&\left\{\zeta_f(T_2a,T_1b),\zeta_f(T_2a,T_1a),\zeta_f(T_1b,T_2b),\frac{\zeta_f(T_2a,T_1a)\zeta_f(T_1b,T_2b)}{1+\zeta_f(T_2a,T_1b)},\right.\\
&\left.\frac{\zeta_f(T_2a,T_1a)\left[1+\zeta_f(T_1b,T_2b)\right]}{1+\zeta_f(T_2a,T_1b)},\frac{\left[\zeta_f(T_2a,T_1a)+\zeta_f(T_1b,T_2b)\right]\zeta_f(T_1a,T_2b)}{1+\zeta_f(T_1a,T_2b)+\zeta_f(T_2a,T_1b)}\right\},\\
\aleph_2(a,b) = Min&\Big\{\zeta_f(T_1a,T_1b),\zeta_f(T_2b,T_2a)\Big\}.
\end{aligned}$$

$$\sum_{i=m}^{n-2}\|f\Big(\lambda_1^i\phi\Big(\zeta_f(b_1,b_0)\Big)\Big)\|+\|\lambda_1^{n-1}\phi\Big(\zeta_f(b_1,b_0)\Big)\|\to 0.$$

Then, T_1 *and* T_2 *have a unique common fixed point in* γ.

Proof. Immediately from Theorem 3.1 by taking $T_1 = T_4$ and $T_2 = T_3$. □

Corollary 3.2. *Let (γ, ζ_f) be a complete SCCMS with functions $f : P \to P$ and P be a normal cone via normal constant M. Consider $T_1, T_4 : \Gamma \to \Gamma$ to be a self-mapping such that,*

(1) $T_1(\Gamma) \subseteq T_4(\Gamma)$,
(2) *The pair (T_1, T_4) is compatible,*
(3) *T_1 and T_4 are continuous,*
(4) *For all $a, b \in \Gamma$,*

$$\phi\Big(\zeta_f(T_1a, T_1b)\Big) \leq \lambda_1\phi\Big(\aleph_1(a,b)\Big) + \lambda_2\phi(\aleph_2(a,b)\Big), \tag{3.15}$$

where, $\phi \in \Phi$, $0 < \lambda_1 + \lambda_2 < 1$, and

$$\aleph_1(a,b) = Max\Bigg\{\zeta_f(T_4a, T_4b), \zeta_f(T_4a, T_1a), \zeta_f(T_4b, T_1b), \frac{\zeta_f(T_4a, T_1a)\zeta_f(T_4b, T_1b)}{1+\zeta_f(T_4a, T_4b)},$$
$$\frac{\zeta_f(T_4a, T_1a)\Big[1+\zeta_f(T_4b, T_1b)\Big]}{1+\zeta_f(T_4a, T_4b)}, \frac{\Big[\zeta_f(T_4a, T_1a)+\zeta_f(T_4b, T_1b)\Big]\zeta_f(T_1a, T_1b)}{1+\zeta_f(T_1a, T_1b)+\zeta_f(T_4a, T_4b)}\Bigg\},$$

$$\aleph_2(a,b) = Min\Big\{\zeta_f(T_1a, T_4b), \zeta_f(T_1b, T_4a)\Big\}.$$

(5) $\sum_{i=m}^{n-2} \|f\Big(\lambda_1^i\phi\Big(\zeta_f(b_1, b_0)\Big)\Big)\| + \|\lambda_1^{n-1}\phi\Big(\zeta_f(b_1, b_0)\Big)\| \to 0.$

Then, T_1 and T_4 have a unique common fixed point in Γ.

Proof. By taking $T_1 = T_2$ and $T_3 = T_4$ in Theorem 3.1, the desired result is obtained. □

Corollary 3.3. *Let (Γ, ζ_f) be a complete SCCMS with functions $f : P \to P$ and P be a normal cone via normal constant M. Consider $T : \Gamma \to \Gamma$ to be a continuous self-mapping such that, for all $a, b \in \Gamma$,*

$$\phi\Big(\zeta_f(Ta, Tb)\Big) \leq \lambda_1\phi\Big(\aleph_1(a,b)\Big) + \lambda_2\phi\Big(\aleph_2(a,b)\Big), \tag{3.16}$$

where, $\phi \in \Phi$, $0 < \lambda_1 + \lambda_2 < 1$, and

$$\aleph_1(a,b) = Max\Bigg\{\zeta_f(a,b), \zeta_f(a,Ta), \zeta_f(b,Tb), \frac{\zeta_f(a,Ta)\zeta_f(b,Tb)}{1+\zeta_f(a,b)}, \frac{\zeta_f(a,Ta)\Big[1+\zeta_f(b,Tb)\Big]}{1+\zeta_f(a,b)},$$
$$\frac{\Big[\zeta_f(a,Ta)+\zeta_f(b,Tb)\Big]\zeta_f(Ta,Tb)}{1+\zeta_f(a,b)+\zeta_f(Ta,Tb)}\Bigg\},$$

$$\aleph_2(a,b) = Min\Big\{\zeta_f(Ta,b), \zeta_f(Tb,a)\Big\}.$$

$\sum_{i=m}^{n-2} \|f\Big(\lambda_1^i\phi\Big(\zeta_f(b_1, b_0)\Big)\Big)\| + \|\lambda_1^{n-1}\phi\Big(\zeta_f(b_1, b_0)\Big)\| \to 0.$
Then, T has a unique common fixed point in Γ.

Proof. Substitute $T = T_1 = T_2$ and $T_3 = T_4 = I$ in proof Theorem 3.1, where I the identity map. □

Remark 3.1. *In Corollary 3.3, any term can be taken to set the maximum and $\lambda_2 = 0$ get the specified various contractions. Therefore, referring to the expanded Banach [4], the state is satisfied:*

$$\phi\Big(\zeta_f(Ta, Tb)\Big) \leq \lambda_1\phi\Big(\zeta_f(a,b)\Big). \tag{3.17}$$

Corollary 3.4. *Let (Γ, ζ_f) be a complete SCCMS with functions $f : P \to P$ and P be a normal cone via normal constant M. Consider $T_1, T_2, T_3, T_4 : \Gamma \to \Gamma$ to be a self-mapping such that,*

(1) $T_1(\Gamma) \subseteq T_2(\Gamma) \subseteq T_3(\Gamma)$,
(2) *The pair* (T_1, T_3) *is compatible,*
(3) T_i *are continuous for all* $i = 1, \cdots, 3$,
(4) *For all* $a, b \in \Gamma$,

$$\phi\Big(\zeta_f(T_1a, T_2b)\Big) \preceq \lambda_1\phi\Big(\aleph_1(a,b)\Big) + \lambda_2\phi\Big(\aleph_2(a,b)\Big), \tag{3.18}$$

where, $\phi \in \Phi$, $0 < \lambda_1 + \lambda_2 < 1$, *and*

$$\aleph_1(a,b) = Max\left\{\zeta_f(T_3a, T_2b), \zeta_f(T_3a, T_1a), \frac{\zeta_f(T_3a, T_1a)}{1+\zeta_f(T_3a, T_2b)}, \frac{\zeta_f(T_3a, T_1a)\zeta_f(T_1a, T_2b)}{1+\zeta_f(T_1a, T_2b)+\zeta_f(T_3a, T_2b)}\right\},$$
$$\aleph_2(a,b) = Min\Big\{\zeta_f(T_1a, T_2b), \zeta_f(T_2b, T_3a)\Big\}.$$

(5) $\sum_{i=m}^{n-2} \|f\Big(\lambda_1^i\phi\Big(\zeta_f(b_1, b_0)\Big)\Big)\| + \|\lambda_1^{n-1}\phi\Big(\zeta_f(b_1, b_0)\Big)\| \to 0.$

Then, $T_i, i = 1, \cdots, 3$ *have a unique common fixed point in* Γ.

Proof. Put $T_2 = T_4$ in the proof of Theorem 3.1, the result is deduced. □

Remark 3.2. *Clearly, if* $\phi(w) = w$, *then the inequality* (3.1) *in Theorem 3.1 becomes*

$$\zeta_f(T_1a, T_2b) \preceq \lambda_1\aleph_1(a,b) + \lambda_2\aleph_2(a,b),$$

where, $\lambda_1, \lambda_2 \in (0,1)$, *and* $0 < \lambda_1 + \lambda_2 < 1$, *but*

$$\aleph_1(a,b) = Max\left\{\zeta_f(T_3a, T_4b), \zeta_f(T_3a, T_1a), \zeta_f(T_4b, T_2b), \frac{\zeta_f(T_3a, T_1a)\zeta_f(T_4b, T_2b)}{1+\zeta_f(T_3a, T_4b)},\right.$$
$$\left.\frac{\zeta_f(T_3a, T_1a)\Big[1+\zeta_f(T_4b, T_2b)\Big]}{1+\zeta_f(T_3a, T_4b)}, \frac{\Big[\zeta_f(T_3a, T_1a)+\zeta_f(T_4b, T_2b)\Big]\zeta_f(T_1a, T_2b)}{1+\zeta_f(T_1a, T_2b)+\zeta_f(T_3a, T_4b)}\right\}$$
$$\aleph_2(a,b) = Min\Big\{\zeta_f(T_1a, T_4b), \zeta_f(T_2b, T_3a)\Big\}.$$

Next, the study shows some specific cases of Theorem 3.1 and Remark 3.2.

Corollary 3.5. *Suppose (Γ, ζ_f) is a complete SCCMS with functions $f : P \to P$ and P is a normal cone via normal constant M. Consider $T_1, T_2, T_3, T_4 : \Gamma \to \Gamma$ to be a self-mapping such that,*

(1) $T_1(\Gamma) \subseteq T_4(\Gamma)$ *and* $T_2(\Gamma) \subseteq T_3(\Gamma)$,
(2) *The pair* (T_1, T_4) *and* (T_2, T_3) *are compatible,*
(3) T_i *are continuous for all* $i = 1, \cdots, 4$,

(4) *For all $a, b \in \Gamma$,*

$$\zeta_f(T_1a, T_2b) \leq \lambda_1\zeta_f(T_3a, T_4b) + \lambda_2\zeta_f(T_3a, T_1a) + \lambda_3\zeta_f(T_4b, T_2b) + \lambda_4\frac{\zeta_f(T_3a, T_1a)\zeta_f(T_4b, T_2b)}{1 + \zeta_f(T_3a, T_4b)}$$
$$+ \lambda_5\frac{\zeta_f(T_3a, T_1a)\Big[1 + \zeta_f(T_4b, T_2b)\Big]}{1 + \zeta_f(T_3a, T_4b)} + \lambda_6\frac{\Big[\zeta_f(T_3a, T_1a) + \zeta_f(T_4b, T_2b)\Big]\zeta_f(T_1a, T_2b)}{1 + \zeta_f(T_1a, T_2b) + \zeta_f(T_3a, T_4b)}$$
$$+ \lambda_7 Min\Big\{\zeta_f(T_1a, T_4b), \zeta_f(T_2b, T_3a)\Big\}.$$

where, $\lambda_i \in (0, 1)$, for all $i = 1, \cdots, 7$, and $\delta = \frac{\lambda_1+\lambda_3}{1-\lambda_2-\lambda_4-\lambda_5-\lambda_6}$,

(5) $\sum_{i=m}^{n-2} \|f\Big(\delta^i\zeta_f(b_1, b_0)\Big)\| + \|\delta^{n-1}\zeta_f(b_1, b_0)\| \to 0.$

Then, $T_i, i = 1, \cdots, 4$ have a unique common fixed point in Γ.

Proof. It suffices to observe, for each $a, b \in \Gamma$ and by the same way of Theorem 3.1 with note of Remark 3.2, let take $\Delta_n = \zeta_f(b_{2n+1}, b_{2n})$ and $\Delta_{n-1} = \zeta_f(b_{2n}, b_{2n-1})$ in (3.4), (3.5) such that $\lambda_i \in (0, 1)$, for all $i = 1, \cdots, 7$, it results in that

$$\Delta_n \leq \frac{\lambda_1 + \lambda_3}{1 - \lambda_2 - \lambda_4 - \lambda_5 - \lambda_6}\Delta_{n-1} = \delta\Delta_{n-1}.$$

Therefore, by Theorem 3.1, the desired result is obtained. □

Corollary 3.6. *Suppose (Γ, ζ_f) is a complete SCCMS with functions $f : P \to P$ and P is a normal cone via normal constant M. Consider $T_1, T_2 : \Gamma \to \Gamma$ to be a self-mapping and continuous such that, for all $a, b \in \Gamma$,*

$$\zeta_f(T_1a, T_2b) \leq \lambda_1\zeta_f(T_2a, T_1b) + \lambda_2\zeta_f(T_2a, T_1a) + \lambda_3\zeta_f(T_1b, T_2b) + \lambda_4\frac{\zeta_f(T_2a, T_1a)\zeta_f(T_1b, T_2b)}{1 + \zeta_f(T_2a, T_1b)}$$
$$+ \lambda_5\frac{\zeta_f(T_2a, T_1a)\Big[1 + \zeta_f(T_1b, T_2b)\Big]}{1 + \zeta_f(T_2a, T_1b)} + \lambda_6\frac{\Big[\zeta_f(T_2a, T_1a) + \zeta_f(T_1b, T_2b)\Big]\zeta_f(T_1a, T_2b)}{1 + \zeta_f(T_1a, T_2b) + \zeta_f(T_2a, T_1b)}$$
$$+ \lambda_7 Min\{\zeta_f(T_1a, T_1b), \zeta_f(T_2b, T_2a)\}.$$

where, $\lambda_i \in (0, 1)$, for all $i = 1, \cdots, 7$, and $\delta = \frac{\lambda_1+\lambda_3}{1-\lambda_2-\lambda_4-\lambda_5-\lambda_6}$,
$\sum_{i=m}^{n-2} \|f\Big(\delta^i\zeta_f(b_1, b_0)\Big)\| + \|\delta^{n-1}\zeta_f(b_1, b_0)\| \to 0.$
Then, $T_i, i = 1, 2$ have a unique common fixed point in Γ.

Proof. By taking $T_1 = T_4$ and $T_2 = T_3$ in Corollary 3.5, the desired result is obtained. □

Remark 3.3. *By adopting the proofs provided in Corollary 3.2, 3.3, and 3.4 and with Remark 3.1, the same conclusions in the context seen in Corollary 3.5 hold the conditions.*

4. Applications

The fixed-point results play a vital role in the existence of various classes of equations, precisely, for solving differential equations, integral equations and fractional differential equations, etc. This has led to improvements in the applications of fixed-point techniques.

4.1. **Polynomial equations.**

Theorem 4.1. *Consider the equation below*

$$(a+1)^{\mathfrak{p}} + 1 = (\xi+1)a(a+1)^{\mathfrak{p}} + \xi a, \tag{4.1}$$

has a unique solution in the interval $[0,1]$ *and for* $\mathfrak{p} \in \mathbb{N}$.

Proof. Define the mapping $T : [0,1] \to [0,1]$ by $Ta = \frac{(a+1)^{\mathfrak{p}}+1}{(\xi+1)(a+1)^{\mathfrak{p}}+\xi}$ for $\mathfrak{p} \in \mathbb{N}$. Noting that a is a fixed-point if and only if there is a solution to Eq.(4.1).
By taking $\zeta_f(a,b) = |a-b|e^t$, for $t \in [0,1]$, and $f(w) = e^{(a+b+1)w+t} - e^t$, it is easy to observe that $([0,1], \zeta_f)$ is a complete SCCMS. Therefore,

$$\begin{aligned}
\zeta_f(Ta,Tb) &= \left| \frac{(a+1)^{\mathfrak{p}}+1}{(\xi+1)(a+1)^{\mathfrak{p}}+\xi} - \frac{(b+1)^{\mathfrak{p}}+1}{(\xi+1)(b+1)^{\mathfrak{p}}+\xi} \right| e^t, \\
&\leq \frac{1}{(2\xi+1)^2} \left| (a+1)^{\mathfrak{p}} - (b+1)^{\mathfrak{p}} \right| e^t, \\
&\leq \frac{n2^{n-1}}{(2\xi+1)^2} |a-b| e^t, \\
&\leq \frac{\xi}{(2\xi+1)^2} |a-b| e^t, \\
&= \lambda_1 \zeta_f(a,b).
\end{aligned}$$

such that $\xi \geq \mathfrak{p}2^{\mathfrak{p}-1}$ and $\lambda_1 = \frac{\xi}{(2\xi+1)^2} \in [0,1), \lambda_2 = 0$. Thus, all the axioms in Corollary 3.3 and Eq. (3.17) of remark 3.1 via $\phi(w) = w$ are held, so they have a UFP. □

4.2. **Non-linear integral equation.** Let us consider the nonlinear integral equation

$$\mathfrak{u}(t) = \lambda_1 \int_0^t \mathcal{G}(\mathrm{t}, w) \mathcal{F}(w, \mathfrak{u}(\mathrm{t})) dw, \mathrm{t} \in [0,1], \lambda_1 \geq 0, \tag{4.2}$$

where the functions $\mathcal{G} : [0,1]^2 \to \mathbb{R}^+$, and $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathbb{R}$ for a given.
Now, let $\Gamma = C[0,1]$ be a set of all continuous functions on $[0,1]$ endowed with the SCCMS

$$\zeta_f(\mathfrak{u}_1, \mathfrak{u}_2) = sup_{w \in [0,1]} Sinh^{-1}\Big(\left| \mathfrak{u}_1(w) - \mathfrak{u}_2(w) \right|^{\mathfrak{q}} \Big) e^{\frac{t}{3}}, \tag{4.3}$$

for each $\mathfrak{u}_1, \mathfrak{u}_2 \in C[0,1]$, $0 < \mathfrak{q} \leq 1$. Clearly, (Γ, ζ_f) is a complete SCCMS with auxiliary function $f(w) = Sinh^{-1}\Big(\Delta Sinh(w)\Big) e^{\frac{t}{3}}, \Delta = Max\{\mathfrak{u}_1, \mathfrak{u}_2\} + 1, w \in P = \{\varphi(t) \in E : \varphi(t) \geq 0, t \in [0,1]\}$.
Moreover, the mapping $T : \Gamma \to \Gamma$ is seen by

$$T\mathfrak{u}(t) = \lambda_1 \int_0^t \mathcal{G}(\mathrm{t}, w) \mathcal{F}(w, \mathfrak{u}(\mathrm{t})) dw, \forall \mathfrak{u} \in \Gamma, \mathrm{t} \in [0,1]. \tag{4.4}$$

Theorem 4.2. *Consider the integral equation in* (4.2) *for the following necessaries:*

(1) $\mathcal{F}$ *is continuous and there is such that* $\mathcal{F}(w, \mathfrak{u}_1(\mathrm{t})) - \mathcal{F}(w, \mathfrak{u}(\mathrm{t})) \leq \left| \mathfrak{u}_1(w) - \mathfrak{u}_2(w) \right|, t, w \in [0,1]$.
(2) T *is a continuous map.*

(3) *The constant* λ_1, *and function* $\mathcal{F}$ *hold the condition*

$$0 < \lambda_1 \int_0^t \mathcal{G}(\mathfrak{t}, w)dw < \left(3\lambda_1 e^{-\mathfrak{t}}\right)^{\frac{1}{\mathfrak{q}}},$$

for $\mathfrak{t} \in (0,1)$. *The integral equation in* (4.2) *has a unique solution.*

Proof. By the definition of (4.3), with Lemma 2.6, it is deduced that $(0 < \mathfrak{q} \leq 1)$

$$\begin{aligned}
\zeta_f(T\mathfrak{u}_1, T\mathfrak{u}_2) &= sup_{w\in[0,1]} Sinh^{-1}\Big(\left|T\mathfrak{u}_1(w) - T\mathfrak{u}_2(w)\right|^{\mathfrak{q}}\Big)\frac{e^{\mathfrak{t}}}{3} \\
&= sup_{w\in[0,1]} Sinh^{-1}\left(\left|\lambda_1 \int_0^{\mathfrak{t}} \mathcal{G}(\mathfrak{t}, w)\mathcal{F}(w, \mathfrak{u}_g(\mathfrak{t}))dw - \lambda_1 \int_0^{\mathfrak{t}} \mathcal{G}(\mathfrak{t}, w)\mathcal{F}(w, \mathfrak{u}(\mathfrak{t}))dw\right|^{\mathfrak{q}}\right)\frac{e^{\mathfrak{t}}}{3} \\
&= sup_{w\in[0,1]} Sinh^{-1}\left(\left|\lambda_1 \int_0^{\mathfrak{t}} \mathcal{G}(\mathfrak{t}, w)\Big(\mathcal{F}(w, \mathfrak{u}_1(\mathfrak{t})) - \mathcal{F}(w, \mathfrak{u}_2(\mathfrak{t}))\Big)dw\right|^{\mathfrak{q}}\right)\frac{e^{\mathfrak{t}}}{3} \\
&\leq sup_{w\in[0,1]} Sinh^{-1}\left(\left|\lambda_1 \int_0^{\mathfrak{t}} \mathcal{G}(\mathfrak{t}, w)|\mathfrak{u}_1(w) - \mathfrak{u}_2(w)|dw\right|^{\mathfrak{q}}\right)\frac{e^{\mathfrak{t}}}{3} \\
&= sup_{w\in[0,1]} Sinh^{-1}\left(\left|\lambda_1 \int_0^{\mathfrak{t}} \mathcal{G}(\mathfrak{t}, w)\Big(|\mathfrak{u}_1(w) - \mathfrak{u}_2(w)|^{\mathfrak{q}}\Big)^{\frac{1}{\mathfrak{q}}}dw\right|^{\mathfrak{q}}\right)\frac{e^{\mathfrak{t}}}{3} \\
&= sup_{w\in[0,1]} Sinh^{-1}\left(\left|\lambda_1 \int_0^{\mathfrak{t}} \mathcal{G}(\mathfrak{t}, w)\Big(Sinh\Big(Sinh^{-1}\Big(|\mathfrak{u}_1(w) - \mathfrak{u}_2(w)|^{\mathfrak{q}}\Big)^{\frac{1}{\mathfrak{q}}}\Big)dw\right|^{\mathfrak{q}}\right)\frac{e^{\mathfrak{t}}}{3} \\
&\leq Sinh^{-1}\left(\left|\lambda_1 \int_0^{\mathfrak{t}} \mathcal{G}(\mathfrak{t}, w)\Big(Sinh(\zeta_f(\mathfrak{u}_1, \mathfrak{u}_2))^{\frac{1}{\mathfrak{q}}}\Big)dw\right|^{\mathfrak{q}}\right)\frac{e^{\mathfrak{t}}}{3} \\
&\leq Sinh^{-1}\left(\frac{e^{\mathfrak{t}}}{3} Sinh\Big(\zeta_f(\mathfrak{u}_1, \mathfrak{u}_2)\Big)\left|\lambda_1 \int_0^{\mathfrak{t}} \mathcal{G}(\mathfrak{t}, w)dw\right|^{\mathfrak{q}}\right)
\end{aligned}$$

It is implying that

$$Sinh\Big(\zeta_f(T\mathfrak{u}_1, T\mathfrak{u}_2)\Big) \leq \lambda_1 Sinh\Big(\zeta_f(\mathfrak{u}_1, \mathfrak{u}_2)\Big).$$

where $\lambda_1 \in (0,1)$ and $\phi(w) = Sinh(w)$. Therefore, all of the conditions of Corollary 3.3 are met, and the desired results are obtained. □

5. Conclusions

This study developed a novel concept of SCCMS, which is a generalization of CSbMS and extended to SbMS in Cone metric space. It provided some results for the specifically ϕ-contraction fixed point theorems, with various rational contractions in SCCMS with some topological results. Moreover, it illustrated the application of polynomial and nonlinear integral equations. In future, the study will examine the strongly composed fuzzy metric space and high generalized contractions with establishing some new applications with non-linear (or fractional) differential equations.

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