Int. J. Anal. Appl. (2025), 23:39

International Journal of Analysis and Applications

Optimized Three-Stage EPQ Model Incorporating Time-Dependent Deterioration and Trapezoidal Demand Dynamics

R. Suvetha[1], K. Rangarajan[1], P. Rajadurai[2,*], M. Kaviyarasu[3], Mohammad Alqahtani[4,*]

[1]*Department of Mathematics, Saveetha School of Engineering, SIMATS Deemed University, Chennai, Tamil Nadu, India*

[2]*Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed University, Kumbakonam, Tamilnadu, India*

[3]*Department of Mathematics, Vel Tech Rangarajan Dr. Saguthala R&D Institute of Science and Technology, Chennai, Tamilnadu, India*

[4]*Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh, Saudi Arabia.*

[*]*Corresponding authors:* psdurai17@gmail.com, m.alqahtani@seu.edu.sa

Abstract. In the world of economic evolution, world economics depends on the manufacturing industries for their source of income. Besides this, all manufacturing industries aim to achieve maximum profit with minimum cost. An optimal three-stage economic production quantity model with time-dependent deterioration and trapezoidal and triangular demands is described in this paper. Shortages are not allowed; all cost values depend on time, and demand depends on trapezoidal and triangular demand. However, trapezoidal demand is described mainly in this paper. Practically, many products like automobiles, electronic devices, vegetables, biomedicines, fruits, fancy products, dairy products, etc. exhibit trapezoidal demand. A direct inverse relationship between the trapezoidal demand and production rate describes the demand rate. This study intends to delight consumers and reduce total costs. Our conclusions are illustrated using numerical examples, extensive prediction with the help of MATLAB software R2021b, and sensitivity analysis for all parameters.

1. Introduction

In our daily lives, deteriorating items are prevalent issues. Recently, many researchers have investigated inventory models for decaying objects. Wee [35] defined degrading items as those that over time become damaged, evaporative, invalid, expired, decayed, devalued, and so on. Based on

Received: Nov. 17, 2024.

2020 *Mathematics Subject Classification.* 90B05; 90B50;

Key words and phrases. Trapezoidal demand; Total cost; Optimality.

https://doi.org/10.28924/2291-8639-23-2025-39

ISSN: 2291-8639

the explanation deteriorating items can be categorized into two different groups. Vegetables, meat, fruit, flowers, medicines, and other such items fall into the first category of goods that deteriorate, expire, become damaged, or evaporate over time. The second category deals with the introduction of new products, such as mobile phones, computer chips, fashion products, and seasonal goods, among other such items. Whitin [36] was the first to investigate the issue of degrading inventory products. He focused on the deterioration of fancy products at the end of the storage period. Finally, Ghare and Schrader [10] found that the use of degrading goods was strongly related to a negative exponential function of time in their studies. Inventory models for degrading objects were explored in various areas by many researchers, including Cheng et al. [6], Hung [14], Cheng and Wang [5], and Deng et al. [9].

In the worldwide commercial market, demand has been demonstrated to be one of the most important considerations when making decisions about inventory and production. Recently, it has been classified into deterministic demand and probabilistic demand. There have been studies on many different kinds of consumption tendency forms, such as time-dependent demand (Resh et al. [29]; Hennery, [12]; Mishra et al. [19], Akhilesh Kumar et al. [3]), constant demand (Padmanabhan [22]; Chuang & Lin, [8], price-stock dependent demand (Wee & Law [34]; Abad [1] and [2], Pervin et al. [23], Pervin et al. [26]), ramp type demand (Hill [13], Mandal [17], Chakraborty et al. [4]), stochastic demand (Mohammedi et al. [20], Roy et al. [31]), quadratic demand (Pervin et al. [24]), composite demand (Pervin et al. [25]), variable demand (Roy et al. [30]), exponential demand (Preethi Jawla and Singh [27], Rajendra Kumar [28]).

Among them, trapezoidal demand is a specific type of time-dependent demand. This study proposed a three-stage production inventory model to minimize total cost and deterioration for deteriorating items with trapezoidal demand. The demand function of the trapezoidal type has three stages: initial stage, the product of demand is increased, second stage the product of demand is constant and third stage the product of demand is declined. For example, fruits, pharmaceutical items, vegetables, electronic goods, seafood products, etc., frequently exhibit this demand pattern. Particularly, cool beverages, and ice creams that generally used in all weather conditions, but sell are more during the summer season. But its sales are sluggish during the cold and rainy season. Cheng and Wang [5] initially proposed trapezoidal demand in the modeling of an inventory control problem. Using partially backlogged shortages as well as the impact of degradation, Cheng et al. [6] enhanced the model of Cheng and Wang [5]. Later on, Lin [15] constructed an inventory model with the trapezoidal demand pattern. Singh and Pattanayak [32], and Chuang et al. [7] subsequently examined inventory models considered the trapezoidal demand for decaying products.

Mishra [18] established an inventory model for degrading items considering preservation technology for deterioration and trapezoidal demand. Wu et al. [37] established two-level inventory models that included a trapezoidal demand, fully backlogged, and time-dependent degradation. Vandana and Srivastava [33] recently established an inventory model for degrading items with an extensive backlog and trapezoidal demand under inflationary conditions. Xu et al. [39] examined an inventory model for nonperishable items with a backlog partially and trapezoidal demand. Kumar [15] investigated a fuzzy inventory model of trapezoidal demand and time-dependent holding costs with shortages. Mandal et al. [17] established an inventory model to prevent technology from deterioration and trapezoidal demand under shortages. The ordering model and joint pricing with decaying items and partially backlogged shortfalls under trapezoidal demand were explored by Xu et al. [38].

This article is organized as follows: In section 2, explains the motivation and contribution of the proposed work. Section 3, describes the research gap in the proposed model. Section 4, explains the problem description of the proposed model. In section 5, notations and assumptions, which are used throughout this article are described. In section 6, mathematical modeling and its solution to minimize the total inventory cost is established. In sections 7 and 8, computational algorithm and procedure for the proposed model. In section 9, numerical examples, section 10 derives sensitivity analysis, graphical representation, and their observations for models are provided. Section 11, explains the overall findings of the work. In section 12, the conclusion, real-life applications, and future research are discussed. In the Appendix, derived hessian matrix of the proposed model.

2. Motivation and Contribution

The motivation of this paper is to maximize profits for manufacturers by implementing an optimal three-stage EPQ model. This model considers time-dependent deterioration and both triangular and trapezoidal demand patterns. While triangular demand is considered, the main focus is on trapezoidal demand to achieve profitable growth and stable economic progress during periods of fluctuation.

This paper presents a three-stage EPQ inventory model that incorporates time-dependent deterioration and both triangular and trapezoidal demand. The model aims to reduce the deteriorating rate and associated costs, as well as the holding cost, optimum cost, production period, and production cost. By minimizing these factors, the overall total cost of production can be reduced. Additionally, this model not only reduces costs but also decreases production time, resulting in a more efficient and cost-effective process. In comparison, other inventory models that are based on constant demand, price, and time-dependent factors often have high initial costs and minimal production outcomes and gains.

3. Literature Review

Table 1. Related work of the proposed model

Author Name	Production Level	Demand Type	Backlog	Deterioration
Akhilesh Kumar et al. [3]	-	Time-dependent	No backlog	-
Mandal et al. [17]	Single level	Ramp-type	Complete backlog	Constant
Xu et al. [39]	Single level	Trapezoidal	Partial backlog	Constant
Xu et al. [38]	Single level	Trapezoidal	Partial backlog	Constant
Chakraborty et al. [4]	Two level	Ramp-type	Partial backlog	Weibull distribution
Padmanabhan [22]	Two level	Constant	Partial backlog	-
Wee [35]	Single level	Constant	Partial backlog	Constant
Preethi et al. [27]	Single level	Exponential	Partial backlog	-
Wu et al. [37]	Single level	Trapezoidal	Complete backlog	Time-dependent
Hung [14]	Two level	Ramp-type	Partial backlog	Weibull distribution
Chuang et al. [7]	Single level	Constant	With & Without backlog	Constant
Lin [16]	Two level	Trapezoidal	Complete backlog	Constant
Chuang et al. [8]	Two level	Ramp-type	Complete backlog	Constant
Cheng et al. [6]	Single level	Trapezoidal	Complete backlog	Constant
Cheng et al. [5]	Two level	Trapezoidal	Partial backlog	Time-dependent
Pervin et al. [23]	Two level	Stock dependent	Partial backlog	Constant
Pervin et al. [24]	-	Quadratic	Complete backlog	Constant
Pervin et al. [26]	Single level	Price and stock dependent	Partial backlog	Constant
Pervin et al. [25]	Single level	Composite	No backlog	Weibull-distribution
Rajendra Kumar et al. [28]	-	Exponential	Partial backlog	-
Kumar [15]	-	Time-dependent	Partial backlog	Constant
Mandal et al. [17]	Single level	Trapezoidal	Complete backlog	Constant
Mishra [18]	-	Trapezoidal	No backlog	Constant
Mishra et al. [19]	Single level	Time-dependent	Partial backlog	Constant
Vandana et al. [33]	Two level	Trapezoidal	Complete backlog	Constant
Proposed Model	Three level	Trapezoidal	No backlog	Time-dependent

Based on Table 1, most of the researchers constructed models of production inventory that included time-dependent, price-dependent, and constant demand. In this proposed model, a three-stage production inventory model with trapezoidal demand, production cost, deteriorating cost, and holding cost values depending on time has developed.

This paper intends to establish a deterministic production inventory model with the most crucial decision parameter being the demand for trapezoidal type. This model intends to minimize total cost and reduce deterioration rate and production period time for a three-stage production inventory system that follows trapezoidal demand. Also, a finite planning horizon and the production rate are inversely proportional to the demand rate.

4. Problem Description

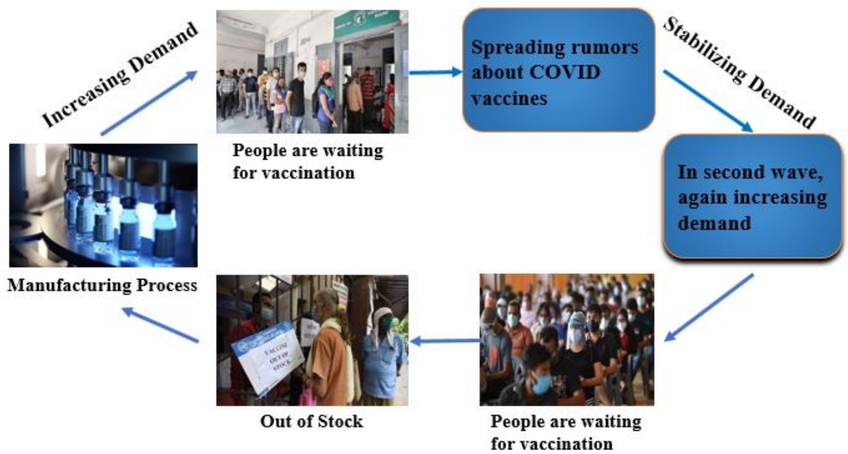

Figure 1. Description of the proposed model

Figure 1, explains the impact of COVID-19, people have volunteered to get vaccinated to prevent the spread of the disease and protect themselves. However, due to the severity of COVID-19, there was demand for vaccines was very high. After that, the people were not interested in coming forward for preventive injections because of the rumors spread about those drugs. So the demand for those drugs also remained steady but not decreased. Later, as the virulence of corona decreased, the need for vaccinations also decreased. When the second wave of COVID is even more intense than before, the need for vaccinations starts to rise again for preventive measures. Then, as the severity of the coronavirus decreased all the people got vaccinated. So the demand for preventive medicine also started to decrease. In the pandemic situation, the demand for preventive medicines like Covaxin and Covishield was very high. During that time even though the people eagerly came forward to get vaccinated in large numbers, the government found it difficult to provide enough medicines to the people and suffered a lot. For this insufficiency, some strikes also happened at that time. If we used trapezoidal demand in such a difficult crisis, that critical situation can be avoided. A good solution can be reached by applying this trapezoidal demand method to not cause this type of crisis in the future.

5. Notations and Assumptions

5.1. Notations.

- Consider maximum inventory levels are Q_1, Q_2 and Q_3 at time t_1, t_2 and t_3.
- C_p is the production cost of an inventory management system.
- C_h is the cost of unit-level holding.
- C_d is the depreciation cost per unit.
- μ_1 and μ_2 is the trapezoidal time period per unit.
- δ_1 and δ_2 is the trapezoidal demand parameter.
- θ is the deterioration rate per unit.
- T is the total production period.
- C_o is the ordering cost per unit time.

5.2. Assumptions.

- Inventory models deal with a single item. The replenishment rate is countable and the lead time is negligible. Demand directly affects the rate of production. Planning horizons can be as long as you like.
- The function $I(t)$ represents a stock level at any point in time between $[0, T]$.
- In an inventory system considered to be no backlog.
- The demand is appraised to be trapezoidal type, say $D(t)$, where

$$D(t) = \begin{cases} \delta_1 + \delta_2 t; & 0 \leq t \leq \mu_1 \\ \delta_1 + \delta_2 \mu_1; & \mu_1 \leq t \leq \mu_2 \\ \delta_1 - \delta_2 t; & \mu_2 \leq t \leq T. \end{cases}$$

  where δ_1 and δ_2 are scaling variables for demand rate. During $[0, \mu_1]$ demand increases concerning time, then it stabilizes during $[\mu_1, \mu_2]$, and thereafter it decreases as t increases during $[\mu_2, T]$.
- There must be no variation in the rate of degradation.
- Inventory system is pretended to be a manufacturing inventory system, with an initial stock level is zero at time $t = 0$. The manufacturing process begins at time $t = 0$ and continues until time $t = t_1$. The stock level reaches its first maximum say Q_1 at time $t = t_1$. Similarly, at time $t = t_2$ and $t = t_3$ the stock level attains its second and third maximum levels say, Q_2 and Q_3.

6. Mathematical Model with Solution

Model: $0 \leq t_1 \leq \mu_1 \leq \mu_2 \leq t_2 \leq t_3 \leq T$

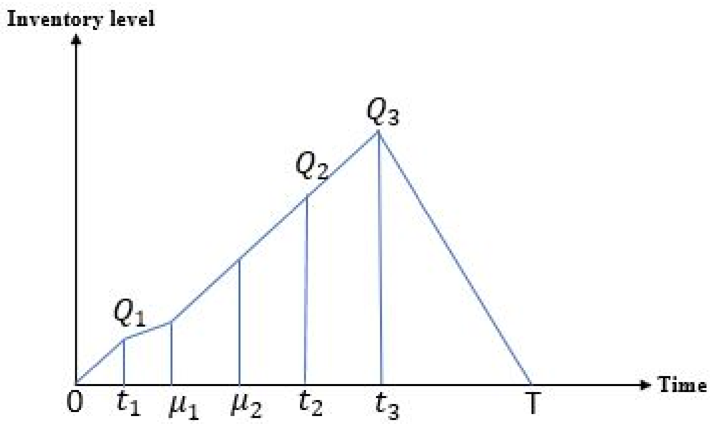

Figure 2. Inventory system of the proposed model.

Figure 2, shows how the inventory level has changed over time. Then t_1, t_2, t_3 and T are the successive periods from the start to the end of production respectively. Among these, μ_1 and μ_2 are trapezoidal period parameters and the interval time between μ_1 and μ_2 is referred to as the period when trapezoidal demand is observed. At time $t = 0$, the first manufacturing setup has no inventory. The stock level rises during time t_1 as a result of production inferior demand and diminution until the extremum stock level is attained at time $t = t_1$. During the interval $(0, t_1)$ production and demand increase at the rate of $P - D$. During the interval (t_1, t_2) production and demand rises with respect to "x" time of $P - D$. At time t_2 and t_3, demand and production increase at the rate of "y" time of $P - D$ respectively, where x and y are positive constants. Therefore, the maximum inventory level in t_2 and t_3 is equal to $x(P - D)t_2$ and $y(P - D)t_3$ respectively. During the period t_1 to t_2, we apply the trapezoidal demand as a result of production less cost.

From Figure 2, during time (0, T) inventory levels are governed by the following differential equations,

$$\frac{dI(t)}{dt} + \theta I(t) = (\gamma - 1)(\delta_1 + \delta_2 t); \quad 0 \le t \le t_1 \tag{6.1}$$

$$\frac{dI(t)}{dt} + \theta I(t) = x(\gamma - 1)(\delta_1 + \delta_2 t); \quad t_1 \le t \le \mu_1 \tag{6.2}$$

$$\frac{dI(t)}{dt} + \theta I(t) = x(\gamma - 1)(\delta_1 + \delta_2 \mu_1); \quad \mu_1 \le t \le \mu_2 \tag{6.3}$$

$$\frac{dI(t)}{dt} + \theta I(t) = x(\gamma - 1)(\delta_1 - \delta_2 t); \quad \mu_2 \le t \le t_2 \tag{6.4}$$

$$\frac{dI(t)}{dt} + \theta I(t) = y(\gamma - 1)(\delta_1 - \delta_2 t); \quad t_2 \le t \le t_3 \tag{6.5}$$

$$\frac{dI(t)}{dt} + \theta I(t) = (-\delta_1 + \delta_2 t); \quad t_3 \le t \le T. \tag{6.6}$$

The first-order differential equations can be solved by using initial and boundary conditions are, $I(t) = 0$ at $t = 0$, $I(t) = 0$ at $t = T$ and due to continuities at $t = t_1$, $t = \mu_1$, $t = \mu_2$, and $t = t_2$ to get the inventory levels of the above equations. The following equations are given by,

$$I(t) = \frac{(\gamma-1)(\theta\delta_1 + \theta\delta_2 t - \delta_2 + e^{-\theta t}[-\theta\delta_1 + \delta_2])}{\theta^2}, \tag{6.7}$$

$$I(t) = \frac{x(\gamma-1)(\theta\delta_1 + \theta\delta_2 t - \delta_2)}{\theta^2}, \tag{6.8}$$

$$I(t) = \frac{(\gamma-1)((1+x)(\delta_1 + \delta_2\mu_1))}{\theta}, \tag{6.9}$$

$$I(t) = \frac{x(\gamma-1)(\theta\delta_1 - \theta\delta_2 t + \delta_2)}{\theta^2}, \tag{6.10}$$

$$I(t) = \frac{y(\gamma-1)(\theta\delta_1 - \theta\delta_2 t + \delta_2)}{\theta^2}, \tag{6.11}$$

$$I(t) = \frac{-\theta\delta_1 + \theta\delta_2 t - \delta_2 + e^{\theta(T-t)}[\theta\delta_1 - \theta\delta_2 T + \delta_2]}{\theta^2}. \tag{6.12}$$

6.1. **Maximum Inventory.** The maximum inventory levels at $t = t_1$, $t = t_2$ and $t = t_3$, are deliberate as follows:

$$I(t_1) = Q_1 = \frac{(\gamma-1)(\theta\delta_1 + \theta\delta_2 t_1 - \delta_2 + e^{-\theta t_1}[-\theta\delta_1 + \delta_2])}{\theta^2}, \tag{6.13}$$

$$I(t_2) = Q_2 = \frac{x(\gamma-1)(\theta\delta_1 - \theta\delta_2 t_2 + \delta_2)}{\theta^2}, \tag{6.14}$$

$$I(t_3) = Q_3 = \frac{y(\gamma-1)(\theta\delta_1 - \theta\delta_2 t_3 + \delta_2)}{\theta^2}. \tag{6.15}$$

6.2. **Total Cost.** It is associated with the Cost of Production (PC), Cost of Holding (HC), Cost of Deterioration (DC), and Cost of Ordering (OC). Therefore,

$$TC = PC + OC + HC + DC. \tag{6.16}$$

where,

$$PC = \frac{C_p}{2}\left[\begin{array}{c} \int_0^{t_1}(\gamma-1)(\delta_1+\delta_2 t)dt + \int_{t_1}^{\mu_1} x(\gamma-1)(\delta_1+\delta_2 t)dt \\ + \int_{\mu_1}^{\mu_2} x(\gamma-1)(\delta_1+\delta_2\mu_1)dt + \int_{\mu_2}^{t_2} x(\gamma-1)(\delta_1-\delta_2 t)dt \\ + \int_{t_2}^{t_3} y(\gamma-1)(\delta_1-\delta_2 t)dt + \int_{t_3}^{T}(-\delta_1+\delta_2 t)dt \end{array}\right],$$

$$= \frac{C_p}{2}\left[\begin{array}{c} (\gamma-1)\left[\begin{array}{c} x\delta_2\mu_2^2 + 2x\delta_2\mu_1\mu_2 - x\delta_2\mu_1^2 + ((1-x)(2\delta_1 t_1 + \delta_2 t_1^2)) \\ +((x-y)(2\delta_1 t_2 - \delta_2 t_2^2)) + y(2\delta_1 t_3 - \delta_2 t_3^2) \end{array}\right] \\ +(T(-2\delta_1+\delta_2 T) + t_3(2\delta_1 - \delta_2 t_3)) \end{array}\right]. \tag{6.17}$$

$$OC = \frac{C_0}{T}. \tag{6.18}$$

$$HC = \frac{C_h}{T}\begin{bmatrix} \int_0^{t_1}(\lambda_1+\lambda_2 t)I(t)\,dt + \int_{t_1}^{\mu_1}(\lambda_1+\lambda_2 t)I(t)\,dt + \int_{\mu_1}^{\mu_2}(\lambda_1+\lambda_2 t)I(t)\,dt \\ + \int_{\mu_2}^{t_2}(\lambda_1+\lambda_2 t)I(t)\,dt + \int_{t_2}^{t_3}(\lambda_1+\lambda_2 t)I(t)\,dt + \int_{t_3}^{T}(\lambda_1+\lambda_2 t)I(t)\,dt \end{bmatrix},$$

$$= \frac{C_h}{T}\begin{bmatrix} (\gamma-1)\begin{bmatrix} \begin{bmatrix} \frac{(3\lambda_1\delta_1 t_1^2+2\lambda_2\delta_1 t_1^3)}{6} \end{bmatrix} \\ +\begin{bmatrix} \frac{1}{6\theta^2}\begin{bmatrix} x((\mu_1-t_1)(((\theta\delta_1-\delta_2)(6\lambda_1+3\lambda_2(\mu_1+t_1))) \\ +3\lambda_1\theta\delta_2(\mu_1+t_1)+2\lambda_2\theta\delta_2(\mu_1+t_1)^2 \end{bmatrix}\end{bmatrix} \\ +\begin{bmatrix} \frac{1}{2\theta}\begin{bmatrix} ((1+x)(\delta_1+\delta_2\mu_1))((\mu_2-\mu_1) \\ (2\lambda_1+\lambda_2(\mu_2+\mu_1))) \end{bmatrix}\end{bmatrix} \\ +\begin{bmatrix} \frac{1}{6\theta^2}\begin{bmatrix} x((t_2-\mu_2)(((\theta\delta_1+\delta_2)(6\lambda_1+3\lambda_2(t_2+\mu_2))) \\ -3\theta\lambda_1\delta_2(t_2+\mu_2)-2\theta\lambda_2\delta_2(t_2+\mu_2)^2)) \end{bmatrix}\end{bmatrix} \\ +\begin{bmatrix} \frac{1}{6\theta^2}\begin{bmatrix} y((t_3-t_2)(((\theta\delta_1+\delta_2)(6\lambda_1+3\lambda_2(t_3+t_2))) \\ -3\theta\lambda_1\delta_2(t_3+t_2)-2\theta\lambda_2\delta_2(t_3+t_2)^2)) \end{bmatrix}\end{bmatrix} \end{bmatrix} \\ +\begin{bmatrix} \frac{1}{2\theta^2}\begin{bmatrix} (\theta\delta_1-\theta\delta_2 T+\delta_2)((T-t_3)(2\lambda_1+\lambda_2(T+t_3))) \end{bmatrix}\end{bmatrix} \end{bmatrix}. \tag{6.19}$$

$$DC = \frac{\theta C_d}{T}\begin{bmatrix} \int_0^{t_1} I(t)\,dt + \int_{t_1}^{\mu_1} I(t)\,dt + \int_{\mu_1}^{\mu_2} I(t)\,dt + \int_{\mu_2}^{t_2} I(t)\,dt + \int_{t_2}^{t_3} I(t)\,dt + \int_{t_3}^{T} I(t)\,dt \end{bmatrix},$$

$$= \frac{\theta C_d}{T}\begin{bmatrix} (\gamma-1)\begin{bmatrix} \begin{bmatrix} \frac{\delta_1 t_1^2}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2\theta^2}\begin{bmatrix} x((\mu_1-t_1)(2(\theta\delta_1-\delta_2)+\theta\delta_2(\mu_1+t_1) \end{bmatrix}\end{bmatrix} \\ +\begin{bmatrix} \frac{1}{\theta}\begin{bmatrix} ((1+x)(\delta_1+\delta_2\mu_1)(\mu_2-\mu_1)) \end{bmatrix}\end{bmatrix} \\ +\begin{bmatrix} \frac{1}{2\theta^2}\begin{bmatrix} x((t_2-\mu_2)(2(\theta\delta_1+\delta_2)-\theta\delta_2(t_2+\mu_2))) \end{bmatrix}\end{bmatrix} \\ +\begin{bmatrix} \frac{1}{2\theta^2}\begin{bmatrix} y((t_3-t_2)(2(\theta\delta_1+\delta_2)-\theta\delta_2(t_3+t_2))) \end{bmatrix}\end{bmatrix} \end{bmatrix} \\ +\begin{bmatrix} \frac{(\theta\delta_1-\theta\delta_2 T+\delta_2)(T-t_3)}{\theta^2} \end{bmatrix} \end{bmatrix}. \tag{6.20}$$

7. Computational Procedure

This section describes a method for evaluating the inventory strategy that reduces the overall cost per unit of time.

- The item's lifespan is assumed to be comprised of trapezoidal demand, time-dependent deterioration, and a preservation factor,

$$(i.e.,)\ \frac{dI(t)}{dt} + \theta I(t) = (\gamma-1)(\delta_1+\delta_2 t), \quad 0 \le t \le t_1.$$

- To calculate inventory level by using differential equations and boundary conditions. The stock level at any time t, equations (4.7) to (4.12) represent $I(t)$.
- At time $t = t_1, t = t_2$ and $t = t_3$, the inventory level reaches its maximum say, Q_1, Q_2 and Q_3.
- From equations (4.17), (4.18), (4.19) and (4.20) the estimated costs such as production, ordering, holding, and deteriorating costs are obtained.
- In this model, the total cost is associated with production, ordering, holding, and deteriorating costs.

- From equations (4.17), (4.18), (4.19) and (4.20) to get the values of t_1, t_2, t_3, T, and TC by using partial differentiation of the equation (4.16) with respect to t_1, t_2, t_3, T.
- Finally, t_1, t_2, t_3 and T give optimal total cost proven by positive definite of the Hessian Matrix.

8. Computational Algorithm

Step 1: Consider the values of following parameters $C_p, C_o, C_d, C_h, x, y, \theta, \delta_1, \delta_2, \lambda_1, \lambda_2, \mu_1, \mu_2$ and β.

Step 2: From the equation(4.16), to find

$$\frac{\partial TC}{\partial t_1}, \frac{\partial TC}{\partial t_2}, \frac{\partial TC}{\partial t_3} \quad \text{and} \quad \frac{\partial TC}{\partial T}.$$

Step 3: Using steps 1 and 2, to determine the values of t_1, t_2, t_3 and T with the help of equations (4.17), (4.18), (4.19), (4.20).

Step 4: Using step 3, finally substituting the values in equation (4.16), to get the total cost value.

Step 5: Evaluate different principal minor of the Hessian matrix. Here, consider TC=F,

$$H = \begin{bmatrix} \frac{\partial^2(F)}{\partial t_1^2} & \frac{\partial^2(F)}{\partial t_1 \partial t_2} & \frac{\partial^2(F)}{\partial t_1 \partial t_3} & \frac{\partial^2(F)}{\partial t_1 \partial T} \\ \frac{\partial^2(F)}{\partial t_2 \partial t_1} & \frac{\partial^2(F)}{\partial t_2^2} & \frac{\partial^2(F)}{\partial t_2 \partial t_3} & \frac{\partial^2(F)}{\partial t_2 \partial T} \\ \frac{\partial^2(F)}{\partial t_3 \partial t_1} & \frac{\partial^2(F)}{\partial t_3 \partial t_2} & \frac{\partial^2(F)}{\partial t_3^2} & \frac{\partial^2(F)}{\partial t_3 \partial T} \\ \frac{\partial^2(F)}{\partial T \partial t_1} & \frac{\partial^2(F)}{\partial T \partial t_2} & \frac{\partial^2(F)}{\partial T \partial t_3} & \frac{\partial^2(F)}{\partial T^2} \end{bmatrix}$$

at the point (t_1, t_2, t_3 and T).

Step 6: Thus, the t_1, t_2, t_3 and T gives optimal total cost proven by positive definite of the Hessian matrix,

$$H_{11} > 0;\ H_{22} > 0;\ H_{33} > 0 \quad \text{and} \quad H_{44} > 0.$$

If the Hessian matrix is positive definite at the point (t_1, t_2, t_3, and T) gives a global optimal solution.

Step 7: Find TC(t_1, t_2, t_3 and T) which is the optimal total cost for the system. (See Appendix)

9. Numerical Analysis

Numerical examples to validate the proposed model are presented in this section.

9.1. **Case 1:** Consider, $C_p = Rs.200,\ C_0 = Rs.200,\ C_d = Rs.200,\ C_h = Rs.20,\ x = 10,\ y = 12,\ \beta = 20,\ \theta = 0.02,\ \delta_1 = 2,\ \delta_2 = 3,\ \mu_1 = 0.07,\ \mu_2 = 1.01,\ \lambda_1 = 0.02,\ \lambda_2 = 0.03.$

The optimum values of $t_1 = 0.0423,\ t_2 = 1.0514,\ t_3 = 1.0835,\ T = 1.1207,\ PC = Rs.78,829,$ $OC = Rs.178, HC = Rs.400, DC = Rs.48,144$ and the optimum total cost is $TC = Rs.1,27,551$ and the optimum of maximum order quantities are, $Q_1 = 14,14,000\ units,\ Q_2 = 16,07,400\ units$ and $Q_3 = 16,95,700\ units.$

From the instance, an analysis of the ideal quantity, cycle time, and rate of deteriorative goods may be deduced that when the rate of deterioration of the goods increases, while the overall cost increases, the cycle time and ideal quantity decrease.

9.2. **Case 2:** In this case, the trapezoidal demand becomes triangular demand when $\mu_1 = \mu_2$.

Consider, $C_p = Rs.200,\ C_0 = Rs.200,\ C_d = Rs.200,\ C_h = Rs.20,\ x = 10,\ y = 12,\ \beta = 20,\ \theta = 0.02,\ \delta_1 = 2,\ \delta_2 = 3,\ \mu_1 = \mu_2 = 0.02,\ \lambda_1 = 0.02,\ \lambda_2 = 0.03.$

The optimum values of $t_1 = 0.0104,\ t_2 = 0.0392,\ t_3 = 0.0534,\ T = 1.7851,\ PC = Rs.37,445,$ $OC = Rs.112,\ HC = Rs.184,\ DC = Rs.6,27,520$ and the optimum total cost is $TC = Rs.6,65,261$ and the optimum of maximum order quantities are, $Q_1 = 13,98,200\ units,\ Q_2 = 14,42,900\ units$ and $Q_3 = 17,31,000\ units.$

9.3. **Observations from Case 1 and Case 2:**

- The above example explains that the ordering cost, holding cost and production cost in case 1 is 52.49%, 58.92%, and 54.25% lower than in case 2.
- But, on the comparison between the trapezoidal and triangular demand by deteriorating cost and total cost were found to be 92.27% and 80.83% higher than the trapezoidal method value from the triangular method.
- As the above examples illustrate, this trapezoidal demand is more profitable at a lower cost than the triangular demand.

9.4. **Comparison of trapezoidal and triangular demand in graphical representation:**

Observations from graphic representation:

- From Figure 2, the triangular method clearly describes that decrease in production cost, optimum cost, and holding cost at the beginning but an increase in deteriorated cost and total cost finally. It concludes that the triangular method is more expensive than the trapezoidal method.
- From Figure 3, compared to the triangular, the trapezoidal is much less expensive in total cost.

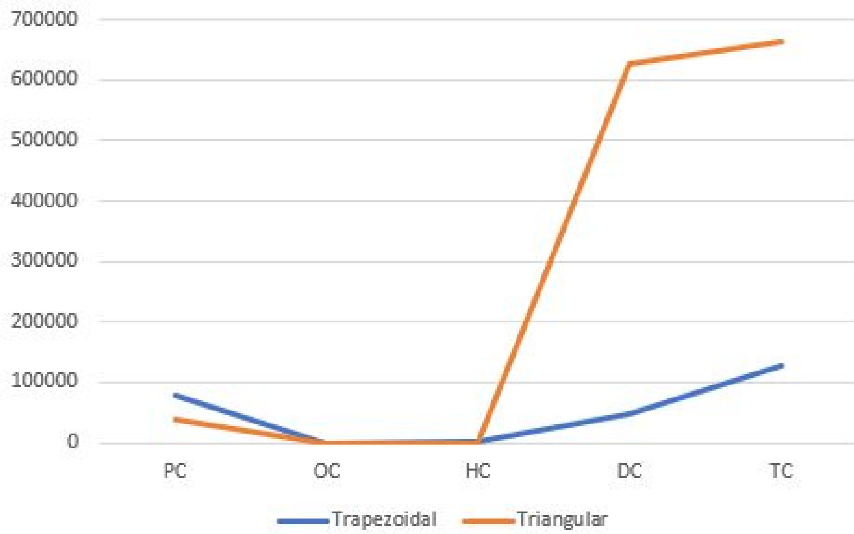

Figure 3. Graphic Representation of Case 1 and Case 2.

Figure 4. Graphic Representation of Case 1 and Case 2 in percentage level.

10. Sensitivity Analysis

10.1. **Sensitivity Analysis for Case 1:**

Observations from Table 2:

- In Table 2, the parameter x shows that the values of cycle time, holding cost, cost of production and deteriorating cost are piling up, but the value of ordering cost is decreased.
- Next, when looking at the parameter of y, its values of cycle time, total cost, and deteriorating cost are increased, but its values of holding cost, production cost and ordering cost remain low in Table 2.

Table 2. Sensitivity analysis for case 1 parameters

Par.	% value	t_1	t_2	t_3	T	PC	OC	HC	DC	TC
x	-50%	0.0197	0.0812	1.0527	1.0721	48,807	187	220	36,581	85,795
	-25%	0.0276	1.0194	1.0743	1.0934	66,914	183	307	46,380	1,13,783
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0614	1.0683	1.0989	1.1384	86,426	176	463	60,466	1,47,530
	+50%	0.0798	1.0782	1.1024	1.1455	91,387	175	483	85,205	1,77,250
y	-50%	0.0089	0.1231	1.0081	1.0623	82,537	188	497	21,260	1,04,482
	-25%	0.0172	0.1869	1.0314	1.1022	80,502	181	432	36,076	1,17,191
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0523	1.0528	1.0921	1.1298	77,166	177	378	51,706	1,29,427
	+50%	0.0620	1.0603	1.0996	1.1300	75,618	177	362	65,801	1,41,957
γ	-50%	0.0657	1.0751	1.0999	1.1395	36,299	176	224	25,837	62,536
	-25%	0.0518	1.0605	1.0904	1.1274	57,410	177	353	32,291	90,231
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0310	1.0422	1.0802	1.1198	80,063	179	414	57,245	1,37,901
	+50%	0.0293	1.0386	1.0795	1.1009	82,177	182	474	62,934	1,45,767
θ	-50%	0.0286	1.0902	1.1031	1.1433	79,427	175	171	65,630	1,45,403
	-25%	0.0394	1.0731	1.0798	1.1378	78,931	176	291	58,509	1,37,907
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0492	1.0457	1.0921	1.1192	77,842	179	413	37,401	1,15,856
	+50%	0.0518	1.0244	1.0909	1.1074	77,095	181	429	37,223	1,14,943
δ_1	-50%	0.0807	1.0321	1.0274	1.1620	11,981	172	127	26,129	38,409
	-25%	0.0612	1.0422	1.0483	1.1593	37,101	173	248	39,870	77,392
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0377	1.0573	1.0921	1.1181	80,616	179	423	51,765	1,32,983
	+50%	0.0301	1.0608	1.0996	1.1092	98,914	180	458	55,667	1,55,219
δ_2	-50%	0.0568	0.0611	1.0902	1.1374	78,504	176	472	26,920	1,06,072
	-25%	0.0491	1.0288	1.0890	1.1251	78,572	178	420	33,317	1,12,487
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0385	1.0690	1.0821	1.1182	79,453	179	368	55,817	1,35,817
	+50%	0.0300	1.0707	1.0800	1.1055	81,175	181	342	66,771	1,48,469
μ_1	-50%	0.0281	1.0748	1.0912	1.1528	75,285	173	288	48,543	1,24,289
	-25%	0.0378	1.0611	1.0861	1.1463	76,925	174	342	48,310	1,25,751
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0485	1.0480	1.0764	1.1192	80,979	179	408	26,817	1,08,382
	+50%	0.0539	1.0428	1.0700	1.1061	83,154	181	415	17,291	1,01,041
μ_2	-50%	0.0140	1.0698	1.0989	1.1081	32,308	180	474	28,334	61,297
	-25%	0.0211	1.0600	1.0923	1.1114	52,321	180	429	46,935	99,865
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0574	1.0521	1.0801	1.1246	81,280	178	381	52,963	1,34,802
	+50%	0.0601	1.0492	1.0784	1.1305	85,469	177	271	60,495	1,46,412
λ_1	-50%	0.0861	1.0249	1.0372	1.0942	78,925	183	213	34,482	1,13,802
	-25%	0.0598	1.0431	1.0589	1.1125	78,849	180	390	36,338	1,15,757
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0374	1.0692	1.0914	1.1281	78,395	177	427	51,854	1,30,853
	+50%	0.0317	1.0711	1.0967	1.1304	78,017	177	479	56,229	1,34,902
λ_2	-50%	0.0166	1.0231	1.0478	1.0888	82,219	184	488	56,166	1,39,057
	-25%	0.0257	1.0379	1.0654	1.1053	80,800	181	432	49,040	1,30,453
	100%	0.0423	1.0514	1.0835	1.1207	78,829	178	400	48,144	1,27,551
	+25%	0.0641	1.0680	1.0901	1.1281	77,227	177	324	47,575	1,25,303
	+50%	0.0714	1.0722	1.0972	1.1304	76,401	177	301	38,549	1,15,428

- Thirdly, the deteriorating items θ define that t_1, holding cost, and ordering cost values are proliferated, but the values of t_2, t_3, T, production cost, cost of deteriorating and total cost are becoming smaller.
- Parameter of γ concludes that the values of ordering cost, deteriorating cost, total cost, production cost and holding cost are made high and the periodicity t_1, t_2, t_3, and T values are decreased.
- In the λ_1 parameter, the values of cycle time(t_1, t_3, T), holding cost, deteriorating cost and total cost become greater, while the values of production cost, ordering cost and t_2 are very low.
- All of the cost values have been lowered and yet the cycle time values have been increased in the λ_2 parameter by its value gained.
- The parameters μ_1 and μ_2 maintain the same amount of inventory level which means t_1, total cost, production cost, and deteriorating cost are increased but t_2, t_3, ordering cost and holding cost are decreased.
- An increase in the value of parameter δ_1 leads to an increase in the value of all costs, t_2 and t_3, otherwise the value of t_1 and T are declined.
- In the values of δ_2 implies when its value is raised with t_2 and cost values are also raised in its value but the values of t_1, t_3, T and holding cost are dropped.
- In the parameter θ, describes how the increase of holding cost and ordering cost leads to a decrease in the production cost and total cost (including deterioration cost) given maximum profit to the manufacturer with minimum cost.
- Also, the λ_2 parameter seems to support the above concept, but all the costs are completely reduced here.
- Especially in the parameters θ, β, and δ_1, the holding cost and ordering cost are increased simultaneously. It conveys that the holding of things is one of the liabilities of the manufacturer, but in addition fact that holding things creates demand in the market, increasing the value of ordering cost seems profitable.

10.2. ***Sensitivity Analysis for Case 2:***

Observations from Table 3:

- In Table 3, the parameter of x denotes that if the cyclic time increases, all of the cost values decrease.
- If the cycle time and production cost decrease, holding cost, deterioration cost, ordering cost, and total cost will increase in the parameter y.
- The parameter of γ concludes that if production cost and periodicity have increased their values, the deteriorating cost and total cost also while the ordering cost and holding cost values become high to low.

Table 3. Sensitivity analysis for case 2 parameters

Par.	% value	t_1	t_2	t_3	T	PC	OC	HC	DC	TC
x	-50%	0.0034	0.0104	0.0533	1.3218	43,849	151	379	6,52,110	6,96,489
	-25%	0.0071	0.0173	0.0533	1.5633	41,436	128	341	6,43,830	6,85,735
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.1498	0.2130	0.2915	1.8338	34,148	109	153	5,76,730	6,11,140
	+50%	0.2190	0.4651	0.5482	1.8612	32,832	107	127	4,67,340	5,00,406
y	-50%	0.1814	0.2753	0.3172	1.8778	85,009	107	129	2,88,140	3,73,385
	-25%	0.1098	0.1573	0.2242	1.8012	84,268	108	137	3,46,370	4,30,883
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0064	0.0245	0.0333	1.6960	24,486	115	357	7,53,480	7,78,438
	+50%	0.0055	0.0216	0.0299	1.5143	21,532	118	376	8,39,510	8,61,536
γ	-50%	0.0766	0.0882	0.2564	1.8183	43,135	110	117	7,50,860	7,94,222
	-25%	0.0520	0.0625	0.1778	1.8056	40,138	111	146	7,17,920	7,58,315
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0068	0.0213	0.0421	1.6139	31,803	124	265	5,92,550	6,24,742
	+50%	0.0031	0.0091	0.0201	1.4188	25,921	141	293	4,95,860	5,22,215
θ	-50%	0.0347	0.0613	0.1826	1.8961	41,646	105	263	7,11,410	7,53,424
	-25%	0.0260	0.0438	0.0963	1.8184	40,522	110	211	6,31,320	6,72,163
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0076	0.0268	0.0431	1.6923	31,777	118	175	5,92,800	6,24,870
	+50%	0.0057	0.0179	0.0334	1.6359	25,659	122	116	5,56,620	5,82,517
δ_1	-50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	-25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
δ_2	-50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	-25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
μ_1	-50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	-25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
μ_2	-50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	-25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
λ_1	-50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	-25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+50%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
λ_2	-50%	0.0042	0.0156	0.0532	1.4446	42,974	138	238	6,41,670	6,85,020
	-25%	0.0085	0.0296	0.0533	1.6780	39,342	119	199	6,33,040	6,72,700
	100%	0.0104	0.0392	0.0534	1.7851	37,445	112	184	6,27,520	6,65,261
	+25%	0.0450	0.0710	0.0840	1.8247	35,740	109	169	6,18,770	6,54,788
	+50%	0.0462	0.1742	0.1921	1.9323	30,147	103	153	6,10,060	6,40,463

- The parameter of θ denotes the cyclic time, production cost, holding cost and deteriorating cost is decreasing, but its order cost will increase. Mention, the ordering cost value will rise based on the above things as possible.
- In Table 3 implies that, the parameters of $\delta_1, \delta_2, \mu_1, \mu_2$ and λ_1 have the same total cost in their respective rows.
- The above parameters also have the same holding cost, ordering cost, deteriorating cost and periodicity values as shown here.
- In the parameter of λ_2 given the observation that, if the production cost decreases, all of the remaining costs of the total cost, deteriorating cost, ordering cost and holding cost will fall while the T, t_1, t_2 and t_3 values will be raised.

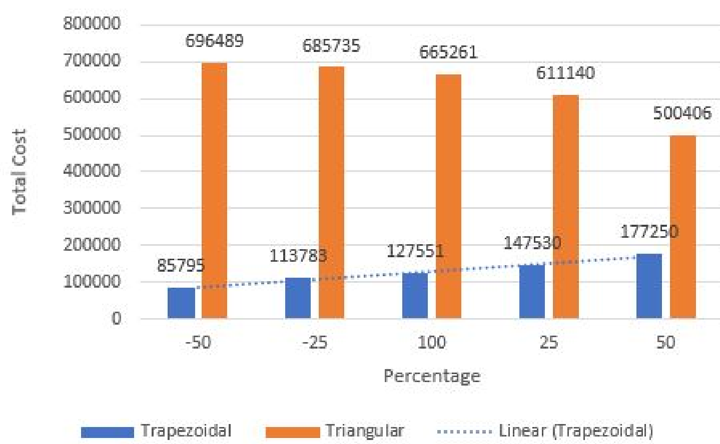

FIGURE 5. Comparison between the demands in the parameter x.

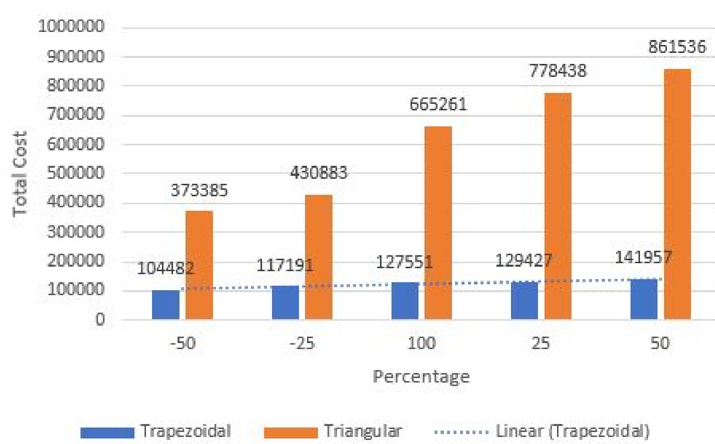

FIGURE 6. Comparison between the demands in parameter y.

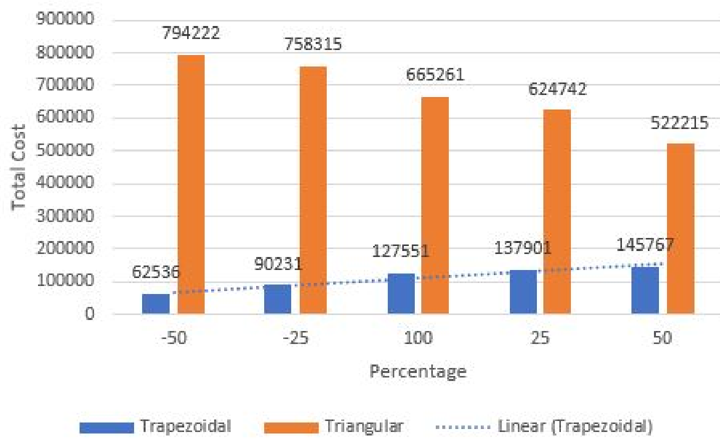

Figure 7. Comparison between the demands in parameter γ.

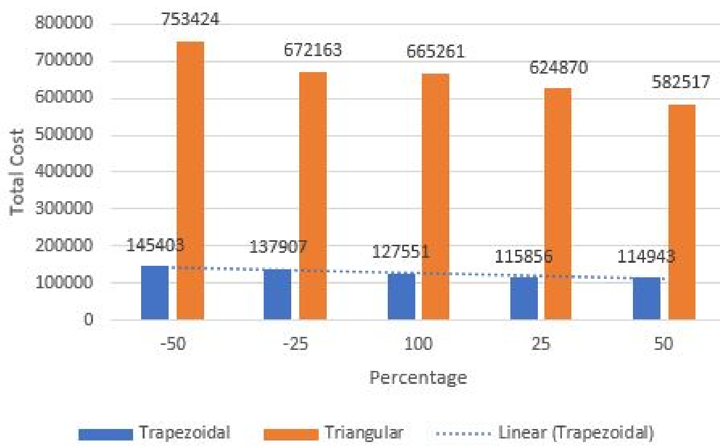

Figure 8. Comparison between the demands in parameter θ.

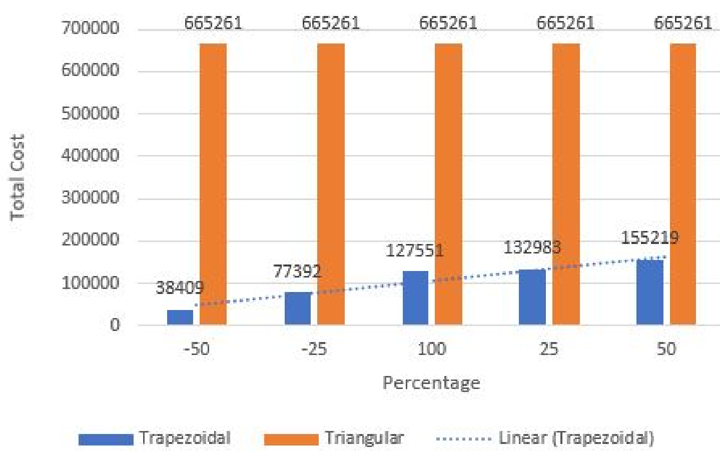

Figure 9. Comparison between the demands in parameter δ_1.

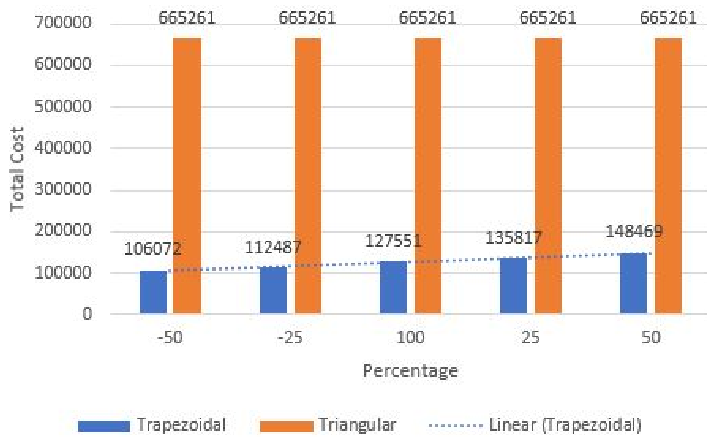

Figure 10. Comparison between the demands in parameter δ_2.

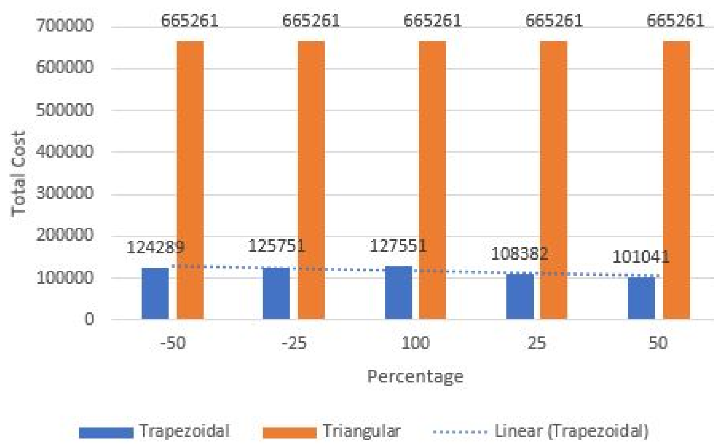

Figure 11. Comparison between the demands in parameter μ_1.

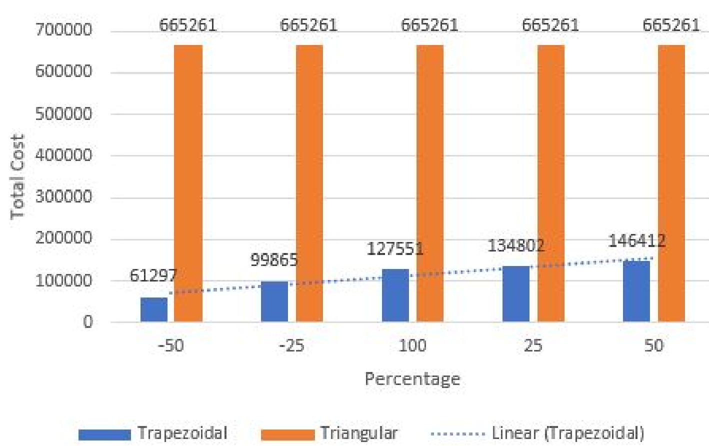

Figure 12. Comparison between the demands in parameter μ_2.

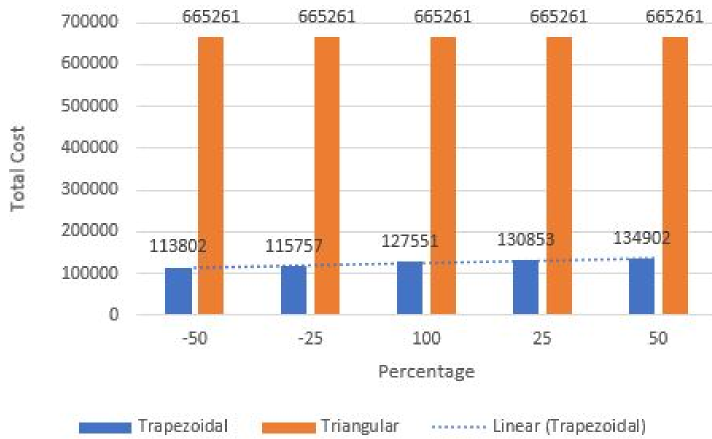

FIGURE 13. Comparison between the demands in parameter λ_1.

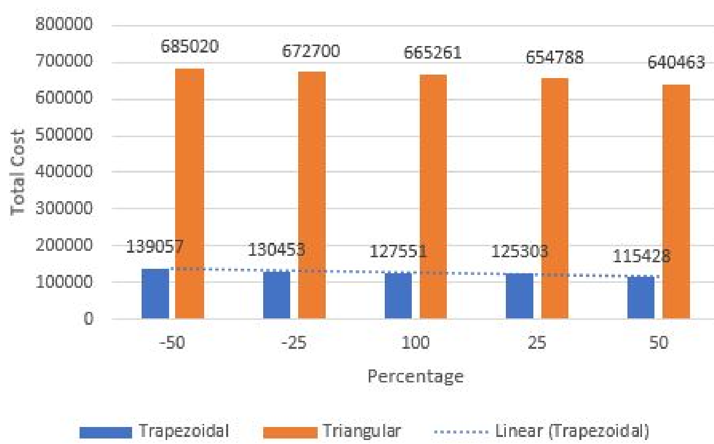

FIGURE 14. Comparison between the demands in parameter λ_2.

10.3. **Graphical representations based on sensitivity analysis and parameters:**

Observations from graphical representations:

- Comparing both trapezoidal and triangular demand, trapezoidal demand is found to be lower than triangular demand based on cost values.
- In trapezoidal demand, the parameters (λ_2, μ_1 and θ) cost values are gradually decreased. However, the remaining parameter's cost values are increased.
- In triangular demand, the parameters $\mu_1, \mu_2, \delta_1, \delta_2$ and λ_1 have the same level of cost values in all percentages there are no changes between them.
- The parameters x, θ, γ and λ_2 cost values are decreased but the y parameter has increased from its low-cost value to high-cost value in triangular demand.

11. Managerial Insights:

- ✠ After considering all the parameters in this paper, it has been found that the cost of triangular demand is significantly higher than the cost of trapezoidal demand. The only exception to this is during the time period between μ_1 and μ_2, where both demands show the same rate of demand and cost values.
- ✠ Additionally, as the deteriorating rate decreases, there is a decrease in holding and production costs. This leads to the conclusion that the total cost of production is dependent on production, ordering and the deteriorating rate. Upon comparing all the cost values, it is evident that the holding cost is the highest. While this may seem like an advantage, it results in a reduction of all other cost values, especially the total cost. This highlights the fact that holding inventory is a liability for manufacturers, but it also creates demand in the market. Therefore, increasing the value of ordering costs can be profitable.
- ✠ Overall, the optimal three-stage EPQ model with time-dependent triangular and trapezoidal demand in the operating method is a significant improvement over the normal time or triangular-dependent model.

12. Conclusion

The development of world economies is heavily reliant on their manufacturing industries. The manufacturing sector plays a crucial role in contributing to the global economy. This study proposes a three-stage production inventory model that aims to reduce total costs and degradation for degrading items, while considering trapezoidal demand. Trapezoidal demand refers to the fluctuation of a product's production over time, which is influenced by factors such as market price, demand, quality, durability, societal importance and value.

This type of demand is commonly observed in various manufacturing sectors, including electronics, sports, beverages, pharmaceuticals, and dairy and meat products. By incorporating this type of demand, manufacturing industries can stabilize their expenses and reduce fluctuations. In conclusion, this paper suggests that using the trapezoidal demand method can reduce total costs and maximize profits in the manufacturing industry.

Real life applications:

This application method has been implemented to become a practical element even in our daily life, we can know with the following examples:

(i) In the agriculture industry, machinery equipment such as harvesters experience trapezoidal demand. During the harvesting period, there is a higher demand for these machines, but after the season ends, the demand returns to normal levels. This shows that agricultural machinery is a prime example of trapezoidal demand.

(ii) Similarly, in the biomedical field, advanced machines are constantly in demand at every stage of progress. For instance, implantable pacemakers and defibrillators, biomedical imaging, drug delivery systems, joint replacement implants, and tissue-engineered skin are all examples of trapezoidal demand in the medical world. As new technology advances, the demand for these machines peaks and then gradually decreases as newer, more advanced machines are introduced.

(iii) A strong illustration of trapezoidal demand can be seen in the market for stationary items. This demand experiences a spike at the start of the academic year but does not decrease significantly throughout the year. Instead, it maintains a consistent demand index until the end of the school year, when it begins to decline. This pattern repeats itself each year, with demand rising again at the start of the next academic year.

(iv) Another perfect example of trapezoidal demand is new and updated mobile and computer software. Whenever new software is introduced to the market, even if they have an interest in the initial stage, there is a situation where the old software becomes incompatible with the latest mobile phones or computers is the main motto of buying activities. As a result, users are forced to switch to the new software. For instance, Apple software applications designed for the iPhone4s will not function on the iPhone 11 Pro. Similarly, older Microsoft software applications that are compatible with Windows 7 will not work on the newer Windows 10. Thus, with the introduction of new software, there is a surge in demand as people make the switch and this demand stabilizes until the next new software variant is released. After the release of the new one, it will turn to high.

For future research directions, this production inventory model can be extended in a fuzzy environment, time-dependent demand, stochastic demand, inflation, and applying various deterioration rates among others.

Appendix A. Appendix

$$H = \begin{bmatrix} \frac{\partial^2(F)}{\partial t_1^2} & \frac{\partial^2(F)}{\partial t_1 \partial t_2} & \frac{\partial^2(F)}{\partial t_1 \partial t_3} & \frac{\partial^2(F)}{\partial t_1 \partial T} \\ \frac{\partial^2(F)}{\partial t_2 \partial t_1} & \frac{\partial^2(F)}{\partial t_2^2} & \frac{\partial^2(F)}{\partial t_2 \partial t_3} & \frac{\partial^2(F)}{\partial t_2 \partial T} \\ \frac{\partial^2(F)}{\partial t_3 \partial t_1} & \frac{\partial^2(F)}{\partial t_3 \partial t_2} & \frac{\partial^2(F)}{\partial t_3^2} & \frac{\partial^2(F)}{\partial t_3 \partial T} \\ \frac{\partial^2(F)}{\partial T \partial t_1} & \frac{\partial^2(F)}{\partial T \partial t_2} & \frac{\partial^2(F)}{\partial T \partial t_3} & \frac{\partial^2(F)}{\partial T^2} \end{bmatrix}$$

We get the values of all partial derivatives by using Matlab R2021b.

$$\frac{\partial^2(F)}{\partial t_1 \partial t_2} = 0; \frac{\partial^2(F)}{\partial t_1 \partial t_3} = 0; \frac{\partial^2(F)}{\partial t_2 \partial t_1} = 0; \frac{\partial^2(F)}{\partial t_2 \partial t_3} = 0; \frac{\partial^2(F)}{\partial t_3 \partial t_1} = 0;$$

$$\frac{\partial^2(F)}{\partial t_3 \partial t_2} = 0; \frac{\partial^2(F)}{\partial T \partial t_1} = 0; \frac{\partial^2(F)}{\partial T \partial t_2} = 0; \frac{\partial^2(F)}{\partial T \partial t_3} = 0.$$

$$\frac{\partial^2(F)}{\partial t_1^2}=\begin{bmatrix} \left[\ \frac{C_p}{2}\left[\ 2\delta_2\gamma-2x\delta_2\gamma-2\delta_2+2x\delta_2\ \right]\ \right] \\ +\frac{C_h}{T}(\gamma-1)\left[\ \lambda_1\delta_1+2\lambda_2\delta_1 t_1+\frac{x}{\theta^2}\left[\ (\theta\delta_1-\delta_2)(-\lambda_2)-\theta\lambda_1\delta_2-2\theta\lambda_2\delta_2 t_1\ \right]\ \right] \\ +\frac{\theta C_d}{T}(\gamma-1)\left[\ \delta_1+\frac{x}{\theta^2}\left[\ -\theta\delta_1+\delta_2-\theta\delta_2\ \right]\ \right] \end{bmatrix};$$

$$\frac{\partial^2(F)}{\partial t_1\partial T}=\begin{bmatrix} \frac{-C_h}{T^2}(\gamma-1)\left[\ \lambda_1\delta_1 t_1+\lambda_2\delta_1 t_1^2+\frac{x}{\theta^2}\left[\ (\theta\delta_1-\delta_2)(-\lambda_1-\lambda_2 t_1)-\theta\lambda_1\delta_2 t_1-\theta\lambda_2\delta_2 t_1^2\ \right]\ \right] \\ -\frac{\theta C_d}{T^2}(\gamma-1)\left[\ \delta_1 t_1+\frac{x}{\theta^2}\left[\ -\theta\delta_1+\delta_2-\theta\delta_2 t_1\ \right]\ \right] \end{bmatrix};$$

$$\frac{\partial^2(F)}{\partial t_2^2}=\begin{bmatrix} \left[\ C_p(\gamma-1)\left[\ (x-y)(-\delta_2)\ \right]\ \right] \\ +\frac{C_h}{T}(\gamma-1)\begin{bmatrix} \frac{x}{\theta^2}\left[\ (\theta\delta_1+\delta_2)\lambda_2-\theta\lambda_1\delta_2-2\theta\lambda_2\delta_2 t_2\ \right] \\ +\frac{y}{\theta^2}\left[\ (\theta\delta_1+\delta_2)(-\lambda_2)+\theta\lambda_1\delta_2+2\theta\lambda_2\delta_2 t_2\ \right] \end{bmatrix} \\ +\frac{C_d}{T}(\gamma-1)\left[\ (y-x)(\delta_2)\ \right] \end{bmatrix};$$

$$\frac{\partial^2(F)}{\partial t_2\partial T}=\begin{bmatrix} \frac{-C_h}{T^2}(\gamma-1)\begin{bmatrix} \frac{x}{\theta^2}\left[\ (\theta\delta_1+\delta_2)(\lambda_1+\lambda_2 t_2)-\theta\lambda_1\delta_2 t_2-\theta\lambda_2\delta_2 t_2^2\ \right] \\ +\frac{y}{\theta^2}\left[\ (\theta\delta_1+\delta_2)(-\lambda_1-\lambda_2 t_2)+\theta\lambda_1\delta_2 t_2+\theta\lambda_2\delta_2 t_2^2\ \right] \end{bmatrix} \\ -\frac{\theta C_d}{T^2}(\gamma-1)\left[\ \frac{x}{\theta^2}\left[\ \theta\delta_1+\delta_2-\theta\delta_2 t_2\ \right]+\frac{y}{\theta^2}\left[\ -\theta\delta_1-\delta_2+\theta\delta_2 t_2\ \right]\ \right] \end{bmatrix};$$

$$\frac{\partial^2(F)}{\partial t_3^2}=\begin{bmatrix} \left[\ C_p(\gamma-1)(-\delta_2)-\delta_2\ \right]+\frac{C_h}{T\theta^2}\begin{bmatrix} \left[\ y(\gamma-1)(\theta\delta_1+\delta_2)\lambda_2-\theta\lambda_1\delta_2-2\theta\lambda_2\delta_2 t_3\ \right] \\ +\frac{1}{\theta^2}\left[\ (-\lambda_2)(\theta\delta_1-\theta\delta_2 T+\delta_2)\ \right] \end{bmatrix} \\ +\frac{C_d}{\theta T}\left[\ y(\gamma-1)(-\theta\delta_2)\ \right] \end{bmatrix};$$

$$\frac{\partial^2(F)}{\partial t_3\partial T}=\begin{bmatrix} \frac{-C_h}{T^2\theta^2}\begin{bmatrix} \left[\ y(\gamma-1)(\theta\delta_1+\delta_2)(\lambda_1+\lambda_2 t_3)-\theta\lambda_1\delta_2 t_3-\theta\lambda_2\delta_2 t_3^2\ \right] \\ +\frac{1}{\theta^2}\left[\ (-\lambda_1-\lambda_2 t_3)(\theta\delta_1-\theta\delta_2 T+\delta_2)\ \right] \end{bmatrix} \\ +\frac{-C_d}{\theta T^2}\left[\ \left[\ y(\gamma-1)(\theta\delta_1+\delta_2)-\theta\delta_2 t_3\ \right]-\theta\delta_1+\theta\delta_2 T-\delta_2\ \right] \end{bmatrix};$$

$$\frac{\partial^2(F)}{\partial T^2}=\left[\ C_p\delta_2+\frac{2C_0}{T^3}+\frac{2C_h}{\theta^2 T^3}\left[\ (\theta\delta_1-\theta\delta_2+\delta_2)(\lambda_1+\lambda_2)\ \right]+\frac{2C_d}{\theta T^3}\left[\ \theta\delta_1-\theta\delta_2+\delta_2\ \right]\ \right].$$

$$H=\begin{bmatrix} 56,52,700 & 0 & 0 & -39,98,200 \\ 0 & 1,03,020 & 0 & 9,35,680 \\ 0 & 0 & 7,93,190 & -63,45,900 \\ 0 & 0 & 0 & 53,813 \end{bmatrix}$$

$$H_{11} = 56,52,700 > 0.$$

$$H_{22} = \begin{bmatrix} 56,52,700 & 0 \\ 0 & 1,03,020 \end{bmatrix},$$

$$= 5.82341154E + 11 > 0.$$

$$H_{33} = \begin{bmatrix} 56,52,700 & 0 & 0 \\ 0 & 1,03,020 & 0 \\ 0 & 0 & 7,93,190 \end{bmatrix},$$

$$= 4.61907180E + 17 > 0.$$

$$H_{44} = \begin{bmatrix} 56,52,700 & 0 & 0 & -39,98,200 \\ 0 & 1,03,020 & 0 & 9,35,680 \\ 0 & 0 & 7,93,190 & -63,45,900 \\ 0 & 0 & 0 & 53,813 \end{bmatrix},$$

$$= 2.48566111E + 21 > 0.$$

As per the principles of the Hessian matrix,

$$H_{11} > 0;\ H_{22} > 0;\ H_{33} > 0 \quad \text{and} \quad H_{44} > 0.$$

Therefore, t_1, t_2, t_3 and T gives optimal total cost.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] P.L. Abad, Optimal Price and Order Size for a Reseller under Partial Backordering, Comput. Oper. Res. 28 (2001), 53–65. https://doi.org/10.1016/S0305-0548(99)00086-6.

[2] P.L. Abad, Optimal Pricing and Lot-Sizing Under Conditions of Perishability and Partial Backordering, Manag. Sci. 42 (1996), 1093–1104. https://doi.org/10.1287/mnsc.42.8.1093.

[3] A. Kumar, P.K. Santra, G.S. Mahapatra, Fractional Order Inventory System for Time-Dependent Demand Influenced by Reliability and Memory Effect of Promotional Efforts, Comput. Ind. Eng. 179 (2023), 109191. https://doi.org/10.1016/j.cie.2023.109191.

[4] D. Chakraborty, D.K. Jana, T.K. Roy, Two-Warehouse Partial Backlogging Inventory Model with Ramp Type Demand Rate, Three-Parameter Weibull Distribution Deterioration under Inflation and Permissible Delay in Payments, Comput. Ind. Eng. 123 (2018), 157–179. https://doi.org/10.1016/j.cie.2018.06.022.

[5] M. Cheng, G. Wang, A Note on the Inventory Model for Deteriorating Items with Trapezoidal Type Demand Rate, Comput. Ind. Eng. 56 (2009), 1296–1300. https://doi.org/10.1016/j.cie.2008.07.020.

[6] M. Cheng, B. Zhang, G. Wang, Optimal Policy for Deteriorating Items with Trapezoidal Type Demand and Partial Backlogging, Appl. Math. Model. 35 (2011), 3552–3560. https://doi.org/10.1016/j.apm.2011.01.001.

[7] K.-W. Chuang, C.-N. Lin, C.-H. Lan, Order Policy Analysis for Deteriorating Inventory Model with Trapezoidal Type Demand Rate, J. Netw. 8 (2013), 1838–1844. https://doi.org/10.4304/jnw.8.8.1838-1844.

[8] K.-J. Chung, C.-N. Lin, Optimal Inventory Replenishment Models for Deteriorating Items Taking Account of Time Discounting, Comput. Oper. Res. 28 (2001), 67–83. https://doi.org/10.1016/S0305-0548(99)00087-8.

[9] P.S. Deng, R.H.-J. Lin, P. Chu, A Note on the Inventory Models for Deteriorating Items with Ramp Type Demand Rate, Eur. J. Oper. Res. 178 (2007), 112–120. https://doi.org/10.1016/j.ejor.2006.01.028.

[10] P.M. Ghare, G.P. Schrader, A Model for an Exponentially Decaying Inventory, J. Ind. Eng. 14 (1963), 12-50.

[11] H. Barman, M. Pervin, S.K. Roy, G.-W. Weber, Back-Ordered Inventory Model with Inflation in a Cloudy-Fuzzy Environment, J. Ind. Manag. Optim. 17 (2021), 1913. https://doi.org/10.3934/jimo.2020052.

[12] R.J. Henery, Inventory Replenishment Policy for Increasing Demand, J. Oper. Res. Soc. 30 (1979), 611–617. https://doi.org/10.1057/jors.1979.153.

[13] R.M. Hill, Inventory Models for Increasing Demand Followed by Level Demand, Journal of the Operational Research Society 46 (1995), 1250–1259. https://doi.org/10.1057/jors.1995.172.

[14] K.-C. Hung, An Inventory Model with Generalized Type Demand, Deterioration and Backorder Rates, Eur. J. Oper. Res. 208 (2011), 239–242. https://doi.org/10.1016/j.ejor.2010.08.026.

[15] P. Kumar, Optimal Policies for Inventory Model with Shortages, Time-Varying Holding and Ordering Costs in Trapezoidal Fuzzy Environment, Indep. J. Manag. Prod. 12 (2021), 557–574. https://doi.org/10.14807/ijmp.v12i2.1212.

[16] K.-P. Lin, An Extended Inventory Models with Trapezoidal Type Demands, Appl. Math. Comput. 219 (2013), 11414–11419. https://doi.org/10.1016/j.amc.2013.05.056.

[17] B. Mandal, A.K. Pal, Order Level Inventory System with Ramp Type Demand Rate for Deteriorating Items, J. Interdiscip. Math. 1 (1998), 49–66. https://doi.org/10.1080/09720502.1998.10700243.

[18] U. Mishra, An Inventory Model for Deteriorating Items under Trapezoidal Type Demand and Controllable Deterioration Rate, Prod. Eng. 9 (2015), 351–365. https://doi.org/10.1007/s11740-015-0625-8.

[19] V.K. Mishra, L.S. Singh, R. Kumar, An Inventory Model for Deteriorating Items with Time-Dependent Demand and Time-Varying Holding Cost under Partial Backlogging, J. Ind. Eng. Int. 9 (2013), 4. https://doi.org/10.1186/2251-712X-9-4.

[20] H. Mohammadi, M. Ghazanfari, M.S. Pishvaee, E. Teimoury, Fresh-Product Supply Chain Coordination and Waste Reduction Using a Revenue-and-Preservation-Technology-Investment-Sharing Contract: A Real-Life Case Study, J. Clean. Prod. 213 (2019), 262–282. https://doi.org/10.1016/j.jclepro.2018.12.120.

[21] R. Mondal, A.A. Shaikh, A.K. Bhunia, I.M. Hezam, R.K. Chakrabortty, Impact of Trapezoidal Demand and Deteriorating Preventing Technology in an Inventory Model in Interval Uncertainty under Backlogging Situation, Mathematics 10 (2021), 78. https://doi.org/10.3390/math10010078.

[22] G. Padmanabhan, P. Vrat, Inventory Model with a Mixture of Back Orders and Lost Sales, Int. J. Syst. Sci. 21 (1990), 1721–1726. https://doi.org/10.1080/00207729008910488.

[23] M. Pervin, S. Kumar Roy, G. Wilhelm Weber, An Integrated Inventory Model with Variable Holding Cost under Two Levels of Trade-Credit Policy, Numer. Algebra Control Optim. 8 (2018), 169–191. https://doi.org/10.3934/naco.2018010.

[24] M. Pervin, S.K. Roy, G. Weber, An Integrated Vendor-Buyer Model with Quadratic Demand under Inspection Policy and Preservation Technology, Hacettepe J. Math. Stat. 49 (2020), 1168–1189. https://doi.org/10.15672/hujms.476056.

[25] M. Pervin, S.K. Roy, P. Sannyashi, G.W. Weber, Sustainable Inventory Model with Environmental Impact for Non-Instantaneous Deteriorating Items with Composite Demand, RAIRO-Oper. Res. 57 (2023), 237–261. https://doi.org/10.1051/ro/2023005.

[26] M. Pervin, S. Kumar Roy, G. Wilhelm Weber, Deteriorating Inventory with Preservation Technology under Price- and Stock-Sensitive Demand, J. Ind. Manag. Optim. 16 (2020), 1585–1612. https://doi.org/10.3934/jimo.2019019.

[27] P. Jawla, S.R. Singh, An Economic Production Quantity Model for Imperfect Items With Exponential Demand and Variable Holding Cost Under Learning and Fuzzy Environment, Int. J. Math. Sci. 12 (2014), 720-726.

[28] M. Rajendra Kumar, J.K. Dash, B. Bhaula, An Exponential Economic Piece-Wise Inventory Mathematical Model for Demand-Dependent Productions with Partial Back-Ordering, J. Interdiscip. Math. 25 (2022), 659–680. https://doi.org/10.1080/09720502.2021.2012893.

[29] M. Resh, M. Friedman, L.C. Barbosa, On a General Solution of the Deterministic Lot Size Problem with Time-Proportional Demand, Oper. Res. 24 (1976), 718–725. https://doi.org/10.1287/opre.24.4.718.

[30] S. Kumar Roy, M. Pervin, G. Wilhelm Weber, Imperfection with Inspection Policy and Variable Demand under Trade-Credit: A Deteriorating Inventory Model, Numer. Algebra Control Optim. 10 (2020), 45–74. https://doi.org/10.3934/naco.2019032.

[31] S. Kumar Roy, M. Pervin, G. Wilhelm Weber, A Two-Warehouse Probabilistic Model with Price Discount on Backorders under Two Levels of Trade-Credit Policy, J. Ind. Manag. Optim. 16 (2020), 553–578. https://doi.org/10.3934/jimo.2018167.

[32] T. Singh, H. Pattnayak, An EOQ Inventory Model for Deteriorating Items with Varying Trapezoidal Type Demand Rate and Weibull Distribution Deterioration, J. Inf. Optim. Sci. 34 (2013), 341–360. https://doi.org/10.1080/02522667.2013.838445.

[33] Vandana, H.M. Srivastava, An Inventory Model for Ameliorating/Deteriorating Items with Trapezoidal Demand and Complete Backlogging under Inflation and Time Discounting, Math. Methods Appl. Sci. 40 (2017), 2980–2993. https://doi.org/10.1002/mma.4214.

[34] H.-M. Wee, S.-T. Law, Economic Production Lot Size for Deteriorating Items Taking Account of the Time-Value of Money, Comput. Oper. Res. 26 (1999), 545–558. https://doi.org/10.1016/S0305-0548(98)00078-1.

[35] H.-M. Wee, Economic Production Lot Size Model for Deteriorating Items with Partial Back-Ordering, Comput. Ind. Eng. 24 (1993), 449–458. https://doi.org/10.1016/0360-8352(93)90040-5.

[36] T.M. Whitin, The Theory of Inventory Management, Princeton University Press, Princeton 1957.

[37] J. Wu, K. Skouri, J.-T. Teng, Y. Hu, Two Inventory Systems with Trapezoidal-Type Demand Rate and Time-Dependent Deterioration and Backlogging, Expert Syst. Appl. 46 (2016), 367–379. https://doi.org/10.1016/j.eswa.2015.10.048.

[38] C. Xu, M. Bai, Q. Wang, Y. Wang, An Inventory System Optimization for Solving Joint Pricing and Ordering Problem with Trapezoidal Demand and Partial Backlogged Shortages in a Limited Sales Period, in: D.-Z. Du, D.

Du, C. Wu, D. Xu (Eds.), Theory and Applications of Models of Computation, Springer, Cham, 2022: pp. 359–369. https://doi.org/10.1007/978-3-031-20350-3_29.

[39] C. Xu, D. Zhao, J. Min, J. Hao, An Inventory Model for Nonperishable Items with Warehouse Mode Selection and Partial Backlogging under Trapezoidal-Type Demand, J. Oper. Res. Soc. 72 (2021), 744–763. https://doi.org/10.1080/01605682.2019.1708822.