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A New Derivation of Extended Watson Summation Theorem Due to Kim et al. with an Application

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Abstract. Confluent representations of hypergeometric functions in one and two variables are firmly established across a range of fields, including applied mathematics, statistics, operations research, physics, and engineering mathematics. Their broad applicability is indisputable. In this article, we will derive the expanded Watson summation theorem for the series ${}_4F_3$, as introduced by Kim et al., using a novel approach. Additionally, we will evaluate four compelling integrals that involve the generalized hypergeometric function. This note will conclude with a discussion of several specific cases, clearly highlighting the natural emergence of symmetry in the results.

1. Introduction

C. F. Gauss [2] provided the following definition of his well-known infinite series in 1812:

$$1+\frac{\nu\beta}{\eta}\frac{z}{1!}+\frac{\nu(\nu+1)\beta(\beta+1)}{\eta(\eta+1)}\frac{z^2}{2!}+\dots \tag{1.1}$$

The series (1.1) is denoted by the symbol

$${}_2F_1\left[\begin{matrix}\nu, & \beta \\ & & ; & z \\ \eta\end{matrix}\right] \text{or } {}_2F_1\left[\begin{matrix}\nu, & \beta; \\ & & ; & z \\ \eta\end{matrix}\right] \text{or } {}_2F_1\left[\nu,\beta;\eta;z\right]$$

It is commonly referred to as the Gauss's function or the Gauss's hypergeometric function, or just F. In this case, z represents the series' argument, while ν, β, and η are the series' parameters. All of ν, β, η, and z can be real or complex, with the exception that γ shouldn't be 0 or a negative integer.

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It is called a "hypergeometric series" because, for either $\nu = 1$ and $\beta = \eta$ or $\beta = 1$ and $\nu = \eta$, the series (1.1) simplifies to the well-known Geometric series.

The Pochhammer's symbol $(\mu)_r$, is represented by, [2]:

$$(\mu)_r = \begin{cases} 1; & (r = 0,\ \mu \in \mathbb{C} \setminus \{0\}) \\ \mu(\mu+1)\ldots(\mu+r-1); & (r \in \mathbb{N}, \mu \in \mathbb{C}) \end{cases} \tag{1.2}$$

Therefore, the hypergeometric series (1.1) is represented by Pochhammer's symbol

$${}_2F_1\left[\begin{matrix} \nu, & \beta \\ & \\ \eta & \end{matrix} ;\ z\right] = \sum_{r=0}^{\infty} \frac{(\nu)_r(\beta)_r}{(\eta)_r} \frac{z^r}{r!}$$

It should be noted that almost all fundamental functions in mathematical physics and mathematics are special or limiting cases of ${}_2F_1$ or ${}_1F_1$.
Therefore, we can use Pochhammer's symbol (1.2) to define generalized hypergeometric functions [2] as follows:

$${}_pF_q\left[\begin{matrix} \nu_1, & \ldots, & \nu_p \\ & & \\ \beta_1, & \ldots, & \beta_q \end{matrix} ;\ z\right] = \sum_{r=0}^{\infty} \frac{(\nu_1)_r \ldots (\nu_p)_r}{(\beta_1)_r \ldots (\beta_q)_r} \frac{z^r}{r!} \tag{1.3}$$

It is substantial to note that the generalized hypergeometric function exhibits symmetry in its numerator $\nu_1, \ldots, \nu_p$ and denominator $\beta_1, \ldots, \beta_q$ parameters. This means that altering the order of each numerator parameter yields the same function, and the same is true for the denominator parameters. The relationship between the parameters of the numerator and denominator determines the convergent of the series (1.3), for more information see [2].

The function ${}_pF_q$ is implemented as HypergeometricPFQ in MATHEMATICA and can be used to calculate both symbolic and numerical data.

The theory of generalized hypergeometric series is fundamentally rooted in the classical summation theorems established by Gauss (both the first and second), Kummer, and Bailey for the series denoted as ${}_2F_1$, as well as by Watson, Dixon, Whipple, and Saalschütz for the series ${}_3F_2$, among others. Given the significant popularity and broad applicability of hypergeometric functions, many researchers have been inspired to explore and develop hypergeometric functions involving two or more variables.

From 1992 to 1996, Lavoie et al. published a series of three influential research papers in which they established generalizations of these classical summation theorems. They also identified a number of special cases and limiting scenarios that arose from their findings. Following this, both Lewanowicz and Vidunas further advanced the field by generalizing Watson's and Kummer's summation theorems, respectively.

The development and expansion of these classical summation theorems continued in 2010 and 2011, with significant contributions from Kim et al. and Rakha and Rathie. In addition, computational tools such as MATHEMATICA and MAPLE were employed to derive and validate these results.

The ${}_3F_2$ hypergeometric function plays a particularly significant role in the theory of hypergeometric and generalized hypergeometric series. Despite this, the ${}_3F_2$ hypergeometric function has a number of mathematical applications; for further details, see [9, 10, 21, 23]. Additionally, it has numerous applications in statistics and physics, including Random Walks, which can be further explained at [11]. For other uses, refer to [12–15, 22]. The well-known Watson summation theorem [2] will be obtained in the Generalized hypergeometric function (1.3) with $p = 3,\ q = 2,\ \nu_1 = a,\ \nu_2 = b,\ \nu_3 = c,\ \beta_1 = \frac{1}{2}(a + b + 1)$ and $\beta_2 = 2c$ with argument $z = 1$.

$$
{}_3F_2\left[\begin{matrix} \nu, & \beta, & \eta \\ & & & ;\ 1 \\ \frac{1}{2}(\nu+\beta+1), & 2\eta \end{matrix}\right]
= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\eta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\eta-\frac{1}{2}\nu-\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\eta-\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\eta-\frac{1}{2}\beta+\frac{1}{2}\right)}, \tag{1.4}
$$

provided $Re(2\eta - \nu - \beta) > -1$.

Bailey [1], in his paper mentioned several interesting applications by using the aforementioned classical summation theorems. In 2010, these classical summation theorems have been extended by Kim et al. [5]. Despite this, we would want to highlight a few of the extended summation theorems that will be necessary for our current research.

- Extension of Gauss second summation theorem:

$$
{}_3F_2\left[\begin{matrix} \nu, & \beta, & \delta+1 \\ & & & ;\frac{1}{2} \\ \frac{1}{2}(\nu+\beta+3), & \delta \end{matrix}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\beta-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\beta+\frac{3}{2}\right)}
\times\left\{\frac{\left(\frac{1}{2}(\nu+\beta-1)-\frac{\nu\beta}{\delta}\right)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)} + \frac{\left(\frac{\nu+\beta+1}{\delta}-2\right)}{\Gamma\left(\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\beta\right)}\right\}, \tag{1.5}
$$

for $\delta = \frac{1}{2}(\nu + \beta + 1)$, the result (1.5) reduces to the following well-known Gauss second summation theorem [2, 8] viz.

$$
{}_2F_1\left[\begin{matrix} \nu, & \beta \\ & & ;\frac{1}{2} \\ \frac{1}{2}(\nu+\beta+1) \end{matrix}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}. \tag{1.6}
$$

- Extension of Watson summation theorem [5]:

$$
{}_4F_3\left[\begin{matrix} \nu, & \beta, & \eta, & \delta+1 \\ & & & & ;\ 1 \\ \frac{1}{2}(\nu+\beta+3), & 2\eta, & \delta \end{matrix}\right]
$$

$$= \frac{2^{\nu+\beta-2}\Gamma\left(\eta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{3}{2}\right)\Gamma\left(\eta-\frac{1}{2}\nu-\frac{1}{2}\beta-\frac{1}{2}\right)}{(\nu-\beta-1)(\nu-\beta+1)\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)\Gamma(\beta)}$$

$$\times\left\{\eta_1\frac{\Gamma\left(\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\beta\right)}{\Gamma\left(\eta-\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\eta-\frac{1}{2}\beta+\frac{1}{2}\right)}+\eta_2\frac{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\eta-\frac{1}{2}\nu\right)\Gamma\left(\eta-\frac{1}{2}\beta\right)}\right\}=\Omega, \tag{1.7}$$

provided $Re(2\eta-\nu-\beta)>1$. Also, the constant η_1 and η_2 are given by

$$\eta_1=\alpha(2\eta-\nu)+\beta(2\eta-\beta)-2\eta+1-\frac{\nu\beta}{\delta}(4\eta-\nu-\beta-1),$$

and

$$\eta_2=\frac{4}{\delta}(\nu+\beta+1)-8.$$

For $\delta=\frac{1}{2}(\nu+\beta+1)$.

For $\delta=\frac{1}{2}(\nu+\beta+1)$, the result (1.7) reduces to the classical Watson summation theorem (1.4).

- Gauss summation theorem [2,8]:

$${}_2F_1\left[\begin{matrix}\nu, & \beta \\ & \\ \eta & \end{matrix};1\right]=\frac{\Gamma(\eta)\Gamma(\eta-\nu-\beta)}{\Gamma(\eta-\nu)\Gamma(\eta-\beta)}, \tag{1.8}$$

provided $Re(\eta-\nu-\beta)>0$.

- Special case of (1.8) [[8],p.49]:

$${}_2F_1\left[\begin{matrix}-\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2} \\ & \\ \eta+\frac{1}{2} & \end{matrix};1\right]=\frac{2^n(\eta)_n}{(2\eta)_n}. \tag{1.9}$$

- A definite integral due to MacRobert [7]:

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}[1+ax+b(1-x)]^{-\lambda-\mu}dx=\frac{1}{(1+a)^{\lambda}(1+b)^{\mu}}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}, \tag{1.10}$$

provided $Re(\lambda)>0, Re(\mu)>0$ and the constant a and b are such that none of the expressions $1+a, 1+b$ and $[1+ax+b(1-x)], 0\le x\le 1$ is not zero.

- Relation between Pochhammer symbol and Gamma function:

$$(d)_n=\frac{\Gamma(d+n)}{\Gamma(d)}. \tag{1.11}$$

- Elementary identities:

$$(-n)_{2m}=2^{2m}\left(-\frac{1}{2}n\right)_m\left(-\frac{1}{2}n+\frac{1}{2}\right)_m=\frac{n!}{(n-2m)!}. \tag{1.12}$$

$$(\beta)_{n+2m}=(\beta)_{2m}(\beta+2m)_n. \tag{1.13}$$

$$(\nu)_{2n}=2^{2n}\left(\frac{1}{2}\nu\right)_n\left(\frac{1}{2}\nu+\frac{1}{2}\right)_n. \tag{1.14}$$

- A result recorded in Rainville [[8], Equ. 8, p.57]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\left[\frac{n}{2}\right]}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n+2m). \tag{1.15}$$

The following is how the paper is set up. We will present a new derivation of the extended Watson summation theorem (1.7) in section 1. The extended Watson summation theorem (1.7) will be used in section 2 as an application for obtaining integrals involving generalized hypergeometric function ${}_3F_2$, and section 3 will discuss some of the remarkable particular cases of our major results.

2. A New Derivation of the Result (1.7)

We are going to demonstrate an updated derivation of the extended Watson summation theorem (1.7) in this section.

We follow these steps to obtain the result (1.7).

Consider the integral [[24], equ. 29a, p.7]. Thus, For $Re(d) > 0$:

$$I=\int_0^{\infty}e^{-t}t^{d-1}\,{}_4F_4\left[\begin{matrix}\nu, & \beta, & \eta, & \delta+1 \\ & & & & ; & t\\ \frac{1}{2}(\nu+\beta+3), & 2\eta, & \delta, & d\end{matrix}\right]dt.$$

By describing ${}_4F_4$ as a series and changing the integration and series order, which is simply supported by the series' uniform convergence, we have

$$I=\sum_{n=0}^{\infty}\frac{(\nu)_n(\beta)_n(\eta)_n(\delta+1)_n}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n(2\nu)_n(\delta)_n(d)_n n!}\int_0^{\infty}e^{-t}t^{d+n-1}dt.$$

After some simplification, we obtain the following by evaluating the well-known gamma integral and applying the result (1.11):

$$I=\Gamma(d)\sum_{n=0}^{\infty}\frac{(\nu)_n(\beta)_n(\eta)_n(\delta+1)_n}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n(2\eta)_n(\delta)_n n!}. \tag{2.1}$$

In summary, the series has

$$I=\Gamma(d)\,{}_4F_3\left[\begin{matrix}\nu, & \beta, & \eta, & \delta+1 \\ & & & & ; & 1\\ \frac{1}{2}(\nu+\beta+3), & 2\eta, & \delta, & \end{matrix}\right]. \tag{2.2}$$

On the other hand, writing (2.1) in the following form:

$$I=\Gamma(d)\sum_{n=0}^{\infty}\frac{(\nu)_n(\beta)_n(\delta+1)_n}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n(\delta)_n 2^n n!}\left\{\frac{2^n(\eta)_n}{(2\eta)_n}\right\}.$$

Applying the result (1.9), we now obtain

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(\nu)_n (\beta)_n (\delta+1)_n}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n (\delta)_n 2^n n!} \; {}_2F_1 \left[\begin{array}{c} -\frac{1}{2}n, \quad -\frac{1}{2}n+\frac{1}{2} \\ \\ \eta+\frac{1}{2} \end{array} ;1 \right].$$

Further, expressing ${}_2F_1$ as a series, following some simplification,

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(\nu)_n (\beta)_n (\delta+1)_n \left(-\frac{1}{2}n\right)_m \left(-\frac{1}{2}n+\frac{1}{2}\right)_m}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n (\delta)_n 2^n \left(\eta+\frac{1}{2}\right)_m m! n!}.$$

with the help of the identity (1.12),

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(\nu)_n (\beta)_n (\delta+1)_n}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n (\delta)_n \left(\eta+\frac{1}{2}\right)_m (\delta)_n 2^{2m+n} m! (n-2m)!}.$$

Next, replacing n by $n+2m$ and making use of the result (1.15), we have

$$I = \Gamma(d) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\nu)_{n+2m} (\beta)_{n+2m} (\delta+1)_{n+2m}}{\left(\frac{1}{2}(\nu+\beta+3)\right)_{n+2m} (\delta)_{n+2m} \left(\nu+\frac{1}{2}\right)_m 2^{n+4m} m! n!}.$$

Now, by empolying the identity (1.13) and with some simplification

$$I = \Gamma(d) \sum_{m=0}^{\infty} \frac{(\nu)_{2m} (\beta)_{2m} (\delta+1)_{2m}}{\left(\frac{1}{2}(\nu+\beta+3)\right)_{2m} \left(\eta+\frac{1}{2}\right)_m (\delta)_{2m} 2^{4m} m!} \sum_{n=0}^{\infty} \frac{(\nu+2m)_n (\beta+2m)_n (\delta+1+2m)_n}{\left(\frac{1}{2}(\nu+\beta+3)+2m\right)_n (\delta+2m)_n 2^n n!}.$$

Summing up the inner series, we have

$$I = \Gamma(d) \sum_{m=0}^{\infty} \frac{(\nu)_{2m} (\beta)_{2m} (\delta+1)_{2m}}{\left(\frac{1}{2}(\nu+\beta+3)\right)_{2m} \left(\eta+\frac{1}{2}\right)_m (\delta)_{2m} 2^{4m} m!}$$
$$\times \; {}_3F_2 \left[\begin{array}{ccc} \nu+2m, & \beta+2m, & \delta+1+2m \\ \\ \frac{1}{2}(\nu+\beta+3)+2m, & \delta+2m & \end{array} ; \; \frac{1}{2} \right].$$

We now observe that the ${}_3F_2$ appearing can be expressed with the help of the result (1.5) and once it has been simplified and using the result (1.14) separating into four parts then summarizing the series and employing the result(1.8), we obtain

$$I = \Gamma(d) \frac{2^{\nu+\beta-2} \Gamma\left(\eta+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{3}{2}\right) \Gamma\left(\eta-\frac{1}{2}\nu-\frac{1}{2}\beta-\frac{1}{2}\right)}{(\nu-\beta-1)(\nu-\beta+1) \Gamma\left(\frac{1}{2}\right) \Gamma(\nu) \Gamma(\beta)}$$
$$\times \left\{ \eta_1 \frac{\Gamma\left(\frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}\beta\right)}{\Gamma\left(\eta-\frac{1}{2}\nu+\frac{1}{2}\right) \Gamma\left(\eta-\frac{1}{2}\beta+\frac{1}{2}\right)} + \eta_2 \frac{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}{\Gamma\left(\eta-\frac{1}{2}\nu\right) \Gamma\left(\eta-\frac{1}{2}\beta\right)} \right\}, \tag{2.3}$$

where the constant η_1 and η_2 are the same as given in equation (1.7). Finally, equating the results (2.2) and (2.3), we arrive at our desired result (1.7). This completes the derivation of the result (1.7).

3. Application

In this section, by employing extended Watson summation theorem (1.7), the following integrals will be determined by applying the generalized hypergeometric function ${}_3F_2$. With the constants a and b are such that none of the expressions $1+a,\ 1+b$ and $[1+ax+b(1-x)],\ 0 \le x \le 1$ is not a zero, these integrals are

Theorem 3.1.

$$\int_0^1 x^{\gamma-1}(1-x)^{\gamma-1}[1+ax+b(1-x)]^{-2\gamma} \times {}_3F_2\left[\begin{matrix} \nu, & \beta, & \delta+1 \\ \frac{1}{2}(\nu+\beta+3), & \delta & \end{matrix} ; \frac{(1+a)x}{1+ax+b(1-x)}\right] dx = \frac{1}{(1+a)^{\eta}(1+b)^{\eta}} \frac{\Gamma(\eta)\Gamma(\eta)}{\Gamma(2\eta)}\Omega, \tag{3.1}$$

provided $Re(\eta) > 0,\ Re(2\eta-\nu-\beta) > 1$. *Also* Ω *is the same as given in (1.7).*

Theorem 3.2.

$$\int_0^1 x^{\beta-1}(1-x)^{2\eta-\beta-1}[1+ax+b(1-x)]^{-2\eta} \times {}_3F_2\left[\begin{matrix} \nu, & \eta, & \delta+1 \\ \frac{1}{2}(\nu+\beta+3), & \delta & \end{matrix} ; \frac{(1+a)x}{1+ax+b(1-x)}\right] dx = \frac{1}{(1+a)^{\beta}(1+b)^{2\eta-\beta}} \frac{\Gamma(\beta)\Gamma(2\eta-\beta)}{\Gamma(2\eta)}\Omega, \tag{3.2}$$

provided $Re(\beta) > 0,\ Re(2\eta-\beta) > 0,\ Re(2\eta-\nu-\beta) > 0$.
Also Ω *is the same as given in (1.7).*

Theorem 3.3.

$$\int_0^1 x^{\beta-1}(1-x)^{\frac{1}{2}(\nu-\beta+3)-1}[1+ax+b(1-x)]^{\frac{1}{2}(\nu+\beta+3)} \times {}_3F_2\left[\begin{matrix} \nu, & \eta, & \delta+1 \\ 2\eta, & \delta & \end{matrix} ; \frac{(1+a)x}{1+ax+b(1-x)}\right] dx = \frac{1}{(1+a)^{\beta}(1+b)^{\frac{1}{2}(\nu-\beta+3)}} \frac{\Gamma(\beta)\Gamma\left(\frac{1}{2}(\nu-\beta+3)\right)}{\Gamma\left(\frac{1}{2}(\nu+\beta+3)\right)}\Omega, \tag{3.3}$$

provided $Re(\beta) > 0,\ Re(\nu-\beta) > -3,\ Re(2\eta-\nu-\beta) > 0$.
Also Ω *is the same as given in (1.7).*

Theorem 3.4.

$$\int_0^1 x^{\delta}(1-x)^{2\eta-\delta-2}[1+ax+b(1-x)]^{-2\eta}$$
$$\times\ {}_3F_2\left[\begin{matrix} \nu, & \beta, & \eta \\ & & & ; & \frac{(1+a)x}{1+ax+b(1-x)} \\ \frac{1}{2}(\nu+\beta+3), & \delta & \end{matrix}\right]dx$$
$$=\frac{1}{(1+a)^{\delta}(1+b)^{2\eta-\delta-1}}\frac{\Gamma(\delta+1)\Gamma(2\eta-\delta-1)}{\Gamma(2\eta)}\Omega, \tag{3.4}$$

provided $Re(\delta)>0,\ Re(2\eta-\delta)>1,\ Re(2\eta-\alpha-\beta)>1.$

Proof. In order to establish the theorems (3.1) to (3.4) we proceed as follows:
By using I to represent the left side of (3.1), we have

$$I=\int_0^1 x^{\eta-1}(1-x)^{\eta-1}[1+ax+b(1-x)]^{-2\eta}$$
$$\times\ {}_3F_2\left[\begin{matrix} \nu, & \beta, & \delta+1 \\ & & & ; & \frac{(1+a)x}{1+ax+b(1-x)} \\ \frac{1}{2}(\nu+\beta+3), & \delta & \end{matrix}\right]dx.$$

Expressing ${}_3F_2$ as series, change the order of integration and summation, we achieve after some algebra.

$$I=\sum_{n=0}^{\infty}\frac{(\nu)_n(\beta)_n(\delta+1)_n(1+a)^n}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n(\delta)_n n!}\int_0^{\infty} x^{\eta+n-1}(1-x)^{\eta-1}[1+ax+b(1-x)]^{-2\eta-n}dx.$$

Now, evaluating the integral with the help of the known integral (1.10) due to MacRobert, we obtain

$$I=\frac{\Gamma(\eta)}{(1+a)^{\eta}(1+b)^{\eta}}\sum_{n=0}^{\infty}\frac{(\nu)_n(\beta)_n(\delta+1)_n}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n(\delta)_n n!}\frac{\Gamma(\eta+n)}{\Gamma(2\eta+n)}.$$

Using the relation (1.11), we accomplish

$$I=\frac{\Gamma(\eta)\Gamma(\eta)}{(1+a)^{\eta}(1+b)^{\eta}}\sum_{n=0}^{\infty}\frac{(\nu)_n(\beta)_n(\nu)_n(\delta+1)_n}{\left(\frac{1}{2}(\nu+\beta+3)\right)_n(2\eta)_n(\delta)_n n!}.$$

Summing up the series, we have

$$I=\frac{\Gamma(\eta)\Gamma(\eta)}{(1+a)^{\eta}(1+b)^{\eta}\Gamma(2\eta)}\ {}_4F_3\left[\begin{matrix} \nu, & \beta, & \eta, & \delta+1 \\ & & & & ; & 1 \\ \frac{1}{2}(\nu+\beta+3), & 2\eta, & \delta & \end{matrix}\right]$$

Using the result (1.7), it is now clear that the ${}_4F_3$ that appears may be evaluated, and we can easily get our first integral (3.1). The other integrals (3.2) to (3.4) can be established in precisely the same way. Therefore, we left this as an exercise for the interested reader. □

Remark 3.1. *For a similar proof, we refer a recent paper by Jun and Kilicman [4].*

4. Special Cases

We will discuss a few of the remarkable specific cases of our major results in this section. For this, it's easily seen that in the integrals (3.1) to (3.4), if n is zero or a positive integer and $\beta = -2n$ and ν is replaced by $\nu + 2n$ or $\beta = -2n - 1$, ν is replaced by $\nu + 2n + 1$. In both cases, one of the two terms appearing in the right-hand side of the integrals (3.1) to (3.4) vanish and we can easily obtain eight new and interesting results. But due to lack of space we are not given here.

5. Concluding Remark

It should be noted that the result may be highly significant from both a theoretical and practical standpoint whenever the generalized hypergeometric function ${}_pF_q$ (including ${}_2F_1$) with its specified argument (e.g., unit argument or $\frac{1}{2}$ argument) can be summed to be expressed in terms of the Gamma functions. In this note, we have given a new derivation of the extended Watson theorem due to Kim et al.. As an application, we evaluated four interesting integrals involving generalized hypergeometric functions. In the end, we mention outlines of the special cases. To wrap up this note, we would like to say that the evaluation of finite single and double integrals is now being studied as an application and will be covered in our upcoming article.

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