Int. J. Anal. Appl. (2025), 23:27

International Journal of Analysis and Applications

Flow Representation of the Navier-Stokes Equations in Weighted Sobolev Spaces

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Abstract. Using Constantin-Iyer representation also known more generally as Euler-Lagrangian approach, we prove the local existence of the Navier-Stokes equations in weighted Sobolev spaces with external forcing on R^d, for any dimension d and p such that $p > d \geq 2$.

1. Introduction

The Navier-Stokes equations are widely regarded as one of the most important equations in fluid mechanics due to their broad science and engineering applications. While there is a huge amount of work dedicated to the analysis of velocity equations, the Lagrangian approach is less investigated. In Lagrangian settings, flow equations (positions of individual particles) are derived and the solutions naturally allow tracking of particles. In ([1], [2], [3]), the Euler coordinates were used to study motion of incompressible fluid on compact manifolds. Fluid flows were treated as intrinsically defined infinitely dimensional systems. In particular, Ebin and Marsden have shown in [2] the local well-posedness of the deterministic Euler equations by solving ODEs in the space of Sobolev volume-preserving diffeomorphisms. In [4], the author followed ideas of ([1], [2], [3]) to study the Euler equations with a random forcing term $\dot{W}_t$. Flow equations were derived and solved for $d \geq 2$ as ODEs in weighted Hölderlder spaces. Without utilizing geometric tools, the author

Received: Nov. 20, 2024.

2020 *Mathematics Subject Classification.* 42B37, 60H30, 35R60, 35Q30.

Key words and phrases. Navier-Stokes equations; Euler-Lagragian approach; Sobolev spaces; Itô-Wentzell formula; harmonic analysis .

https://doi.org/10.28924/2291-8639-23-2025-27
ISSN: 2291-8639

dealt with spaces of diffeomorphisms analytically. More recently, the authors of [5] provided a stochastic framework extending the geometric approach of Ebin and Mardens. Their results can be applied to prove the local well-posedness of the stochastic Euler equations with a random forcing term $\dot{W}_t$ in the Sobolev spaces with $p = 2$ and $s > \frac{d}{2} + 1$.

The equivalence between the deterministic Navier-Stokes equations and corresponding flow formulation was shown analytically in [6]. Later, a self-contained proof for the local existence in Hölder spaces was provided in [7]. The main idea was to perturb the flow by a Wiener process. By averaging out random trajectories, the velocity can be recovered. While the results of [6] have motivated a plethora of research into the Euler-Lagrangian approach of the Navier-Stokes and Euler equations, only a handful has investigated it in Sobolev spaces. In [8], the Lagrangian formulation was used to prove the local existence for the deterministic Euler equations in standard Sobolev spaces H^s_p with $p = 2$, $d \geq 2$ and $s > \frac{d}{2} + 1$ on the torus domain $\mathbb{T}^d := \mathbb{R}^d / 2\pi\mathbb{Z}^d$. Under the same setting, their results were extended by [9] to cover the stochastic Euler equations with Stratonovich transport noise.

While the geometric approach is elegant, the analytical approach is arguably more accessible, especially for practitioners. For this reason, we follow the idea of ([4], [6], [7]) to study the Lagrangian formulation of the Navier-Stokes equations. Our approach mostly relies on fundamental results in harmonic analysis and can be easily understood by non-technical readers.

Our novel contribution can be summarized as follows:

- We study Constantin-Iyer representation for the Navier-Stokes equations with random forcing $G(t)\,dt$ in the full space domain instead of the Euler equations with/without Stratonovich transport noise on tori ([8], [9], [10]) or the Navier-Stokes equations without forcing term on periodic domains ([6], [7]). Note that in [6], the authors were well aware that $G(t)\,dt$ can be handled, however, only formal discussion was provided.
- To the best of our knowledge, this work is the first to investigate L_p-theory of Constantin-Iyer representation with general $p > d \geq 2$ for the Navier-Stokes equations. Our results are new even without the forcing term. We elaborate this further in the next point.
- Based on our Constantin-Iyer representation, we provide a self-contained proof for the local existence of the Navier-Stokes equations in weighted Sobolev spaces that can cover $p > d \geq 2$ instead of non-weighted Sobolev spaces with $p = 2$ in ([8], [9]) or Hölder spaces in ([6], [7], [10]). Our work entails some additional analytical results needed to handle the challenge of general p in weighted spaces compared to the case of $p = 2$ in non-weighted spaces or the Hölder spaces. We impose an assumption $l > 1 + \frac{d}{p}$ for the existence of the flow equations which is similar to $s > 1 + \frac{d}{2}$ found in ([8], [9]).
- We also emphasize that our proof of existence covers the case of the Euler equations namely $\epsilon = 0$ without any special treatment such as passing to the limit $\epsilon \to 0$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration of $\sigma-$algebras $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ satisfying the usual conditions and B_t, W_t be independent standard $d-$dimensional Wiener processes on

$(\Omega, \mathcal{F}, P)$. Let $\mathbb{F}^W = \left(\mathcal{F}_t^W, t \geq 0\right)$ be the standard sub-filtration of $\mathbb{F}$ generated by W_t. We will prove the local existence of the velocity $u : \Omega \times (0, \infty) \times R^d \to R^d$ which is $\mathbb{F}^W-$adapted and evolves with t in a weighted Sobolev space according to the following Navier-Stokes equation with $\epsilon \geq 0$ (see Theorem 5.2),

$$\begin{aligned} du\left(t\right) &= \left[\mathcal{S}\left(-u^k\left(t\right)\partial_k u\left(t\right)\right) + \frac{\epsilon^2}{2}\Delta u\left(t\right) + G\left(t\right)\right]dt \\ u\left(0\right) &= u_0, \quad \operatorname{div} u = 0, \quad t > 0. \end{aligned}$$

The symbol $\mathcal{S}$ stands for the solenoidal projection. We assume that the initial datum u_0 is $\mathcal{F}_0^W-$measurable, $\operatorname{div} u_0 = 0$ and the external forcing term $G : \Omega \times (0, \infty) \times R^d \to R^d$ is $\mathbb{F}^W-$adapted.

Similar to [7], we define the perturbed flow $\eta : \Omega \times (0, \infty) \times R^d \to R^d$ by

$$d\eta\left(t\right) = u\left(t, \eta\left(t\right)\right)dt + \epsilon dB_t,\ \eta\left(0\right) = e,\ t > 0.$$

Here, e denotes the identity map on R^d. We derive (see Lemma 5.1 and Remark 5.2) the following equation for the flow η, denoting $\kappa\left(t\right) = \eta^{-1}\left(t\right)$ the spatial inverse of $\eta\left(t\right)$,

$$\begin{aligned} d\eta\left(t\right) &= E\left[\mathcal{S}\left(\nabla\kappa\left(t,z\right)\right)^* g_\eta\left(t, \kappa\left(t,z\right)\right)|\mathcal{F}_t^W\right]\Big|_{z=\eta(t)} dt + \epsilon dB_t \\ \eta\left(0\right) &= e, \quad t > 0. \end{aligned}$$

where

$$g_\eta\left(t,x\right) = u_0 + \int_0^t \left(\nabla\eta\left(s,x\right)\right)^* G\left(s, \eta\left(s,x\right)\right)ds, \quad \left(\omega, t, x\right) \in \Omega \times (0,\infty) \times R^d.$$

Once we prove the local existence of η, the velocity can be recovered from the flow via the formula,

$$u\left(t\right) = E\left[\mathcal{S}\left(\nabla\kappa\left(t\right)\right)^* g_\eta\left(t, \kappa\left(t\right)\right)|\mathcal{F}_t^W\right].$$

Lastly, we note that G can be random, although this does not cause any additional difficulties compared to the case of deterministic G. The fact that G can be $\mathbb{F}^W-$adapted heuristically allows passing to the limit to the stochastic equations with a stochastic integral as a forcing term, i.e., $G\left(t\right)dW_t$ instead of $G\left(t\right)dt$.

2. Notation

We list some commonly used notations in this paper.

- $E_t^W\left(X\right)$ denotes the conditional expectation $E\left(X|\mathcal{F}_t^W\right)$.
- We assume throughout this paper that $d \geq 2$.
- N_0^d denotes the set of all $d-$dimensional multi-indices.
- e denotes an identity map from R^d to R^d. I denotes the $d \times d$ identity matrix.
- For any matrix or vector A, A^* denotes its transpose.
- $C_0^\infty = C_0^\infty\left(R^d\right)$ denotes the set of all indefinitely differentiable real-valued functions on R^d with compact support.

- $|\cdot|$ denotes standard Euclidean norms for both vectors and matrices, regardless of dimensions.
- For a real-valued function f on R^d, $\left|f\right|_\infty = \sup_{x\in R^d}\left|f(x)\right|$. It is generalized to vector-valued and matrix-valued functions by taking the maximum of $|\cdot|_\infty$ of all entries.
- For a real-valued function f on $[0,\infty)\times R^d$, its partial derivatives are denoted by $\partial_t f = \partial f/\partial t, \partial_i f = \partial f/\partial x_i$, $\partial^2_{ij} f = \partial^2 f/\partial x_i \partial x_j$, $Df = \nabla f = (\partial_1 f, ..., \partial_d f)$. Given a multi-index $\gamma \in N_0^d$, $D^\gamma f = \partial^\gamma f = \frac{\partial^{|\gamma|} f}{\partial x_1^{\gamma_1} ... \partial x_d^{\gamma_d}}$ and the same notations is used for weak derivatives.
- For $f = \left(f^1, f^2, ..., f^m\right)^* : R^d \to R^m$ and a multi-index $\gamma \in N_0^d$, $D^\gamma f = \left(D^\gamma f^1, ..., D^\gamma f^m\right)^*$ denotes its partial derivative and $Df = \nabla f = \left(\partial_j f^i\right)_{i,j}$ denotes its Jacobian matrix. The notation $D^\gamma f$ is also extended to a matrix-valued function by entry-wise differentiation. We will also write $\left\|f\right\| = \left|f(0)\right| + \left|\nabla f\right|_\infty$.
- If f is a real-valued, vector-valued or matrix-valued function on R^d, we denote $D^k f = (D^\gamma f)_{|\gamma|=k}, k = 1, 2, 3, ...$ the tensor of all derivatives of order k.
- $C^n\left(R^d\right) = C^n\left(R^d; R\right)$, $n \geq 0$ denotes the set of all n-times continuously differentiable functions on R^d endowed with the finite norm $\left|f\right|_{C^n} = \sum_{0\leq|\gamma|\leq n} \sup_x \left|D^\gamma f(x)\right| < \infty$.
- $C^\alpha\left(R^d\right) = C^\alpha\left(R^d; R\right)$, $\alpha > 0$ denotes the standard Hölder spaces on R^d endowed with the finite norm
$$\left|f\right|_{C^\alpha} = \left|f\right|_{C^n} + \sup_{x\neq y} \frac{\left|f(x) - f(y)\right|}{\left|x-y\right|^\beta}$$
where $\alpha = n + \beta$, n is an integer and $\beta \in (0, 1]$.
- C^n (resp. $\mathcal{C}^n$) is extended to the space vector-valued and matrix-valued functions whose all components belong to C^n (resp. $\mathcal{C}^n$.) It is endowed with the the maximum of C^n-norm (resp. $\mathcal{C}^n-$norm) of all entries. The same notation $\left|f\right|_{C^n}$ (resp. $\left|f\right|_{\mathcal{C}^n}$) are used. For the spaces, we will write $C^n\left(R^d; B\right)$ and $\mathcal{C}^n\left(R^d; B\right)$ with appropriate B, e.g., $B = R^d$.
- For a multi-linear continuous operator $F : E \to F$ where E, F are Banach spaces, $\|F\|$ denotes its operator norm.
- $C, N, M \in (0,\infty)$ with or without subscriptions denote constants which generally change from line to line or even within the same line.
- We will use the Einstein summation convention over repeated indices when there is no chance of confusion.
- When two functions are equal almost everywhere, we generally refer to them by the same notation. If some properties of a function hold after a modification on a set of measure zero, then we simply say that a function satisfies such properties. In particular, if a function is Hölder continuous or continuously differentiable after a modification on a set of measure zero due to Sobolev embedding theorem, then we simply say that it has such properties.

3. Function Spaces and Decomposition of Vector Fields

3.1. **Function Spaces.** We will use Sobolev function spaces with asymptotic conditions originally introduced by Cantor in [11] and were used in [12] to prove the local well-posedness of the deterministic Euler equations.

For $p > 1$, $l \geq 0$, $\delta \in R$, we denote by $H^l_{\delta,p}\left(R^d\right) = H^l_{\delta,p}\left(R^d; R\right)$ the space of real-valued functions $f : R^d \to R$ whose weak derivatives have the finite norm

$$\left|f\right|_{H^l_{\delta,p}} = \sum_{k=0}^{l} \sum_{|\gamma|=k} \left(\int_{R^d} w^{p(\delta-l+k-d/p)}\left(x\right)\left|D^\gamma f\left(x\right)\right|^p dx\right)^{1/p}$$

where $w\left(x\right) = \left(1 + |x|^2\right)^{1/2}$. By interpolation inequalities, the norm $\left|f\right|_{H^l_{\delta,p}}$ is equivalent to the norm,

$$\left(\int_{R^d} w^{p(\delta-l-d/p)}\left(x\right)\left|f\left(x\right)\right|^p dx\right)^{1/p} + \sum_{|\gamma|=l}\left(\int_{R^d} w^{p(\delta-d/p)}\left(x\right)\left|D^\gamma f\left(x\right)\right|^p dx\right)^{1/p}.$$

It is easy to show that $C_0^\infty\left(R^d\right)$ is dense in $H^l_{p,\delta}\left(R^d\right)$ (e.g., [13, Proposition 2.3.1].) If v is a vector, a matrix or even a multi-dimensional tensor, the norm $|v|_{H^l_{\delta,p}}$ is similarly defined by intrepreting $|\cdot|$ as the Euclidean norm. The corresponding spaces are denoted with respect to the dimensions of the range spaces, for example, $H^l_{\delta,p}\left(R^d; R^m\right)$ or $H^l_{\delta,p}\left(R^d; R^m \times R^n\right)$.

For $0 < T \leq \infty$, we denote by $\sum^l_{\delta,p}\left(T\right)$ the space of $H^l_{\delta,p}$−valued functions v on $[0, T]$ with the finite norm $\sup_{t\in[0,T]}\left|v\left(t\right)\right|_{H^l_{\delta,p}}$. We use the same notation $\sum^l_{\delta,p}\left(T\right)$ regardless of dimensions as there is no chance of confusion.

If $\delta = 0$, we write $H^l_p\left(R^d; B\right) = H^l_{0,p}\left(R^d; B\right)$ and if $\delta = l = 0$, we write $L_p\left(R^d; B\right) = H^0_{0,p}\left(R^d; B\right)$ with an appropriate B.

We use $|\cdot|_p$ to denote the L_p norms regardless of dimensions. The estimates of types $\left|fg\right|_p \leq C\left|f\right|_\infty\left|g\right|_p$ and $\left|fg\right|_\infty \leq C\left|f\right|_\infty\left|g\right|_\infty$ with some generic C will be used abundantly in this paper. We will not be pedantic about the value of C even when $C = 1$, since it depends on exact dimensions of f and g.

Whenever $p > d$, we will interpret all derivatives of $f \in H^l_p\left(R^d\right)$ and thus of $f \in H^l_{\delta,p}\left(R^d\right)$ when $\delta \geq l + \frac{d}{p}$ as classical derivatives (cf. [14, Theorem 5 of Chapter 5].) Consequently, we may rely on results in fundamental calculus such as chain rules, product rules, Taylor's expansion, and integration by-parts.

3.2. **Decomposition of Vector Fields.** Recall the definition of the Newtonian potential,

$$\Gamma\left(x\right) = \Gamma^d\left(|x|\right) = \begin{cases} \frac{|x|^{2-d}}{d(2-d)\omega_d}, & d > 2 \\ \frac{1}{2\pi}\ln|x|, & d = 2 \end{cases}$$

and

$$\Gamma_i\left(x\right) = \frac{\partial}{\partial x_i}\Gamma\left(x\right) = \frac{1}{d\omega_d}\frac{x_i}{|x|}|x|^{1-d}, \; x \neq 0, \quad d \geq 2$$

where ω_d is the volume of the unit ball in R^d.

Define the operators

$$T_i f(x) = \int_{R^d} \Gamma_i(x-y) f(y)\, dy,\ f \in C_0^\infty\left(R^d\right),\ i = 1, \ldots, d.$$

We will use $T_i, i = 1, ..., d$ to define gradient and solenoidal projections for $v \in H^l_{\theta+l,p}\left(R^d; R^d\right)$ respectively. First, we set for $v = \left(v^1, ..., v^d\right)^* \in C_0^\infty\left(R^d; R^d\right)$,

$$\begin{aligned} \mathcal{G}(v) &= \nabla T_i\left(v^i\right)^*, \\ \mathcal{S}(v) &= v - \mathcal{G}(v) \end{aligned}$$

where the standard summation convention over repeated indices is assumed. It is well-known that

$$\mathcal{G}(v) = \nabla T_i\left(v^i\right)^* = -RR_j v^j,$$

where $R_j f = -i\mathcal{F}^{-1}\left(\frac{\xi_j}{|\xi|}\mathcal{F} f\right)$ is the Riesz transform of f and $R = (R_1, ..., R_d)^*$. Usually, $\mathcal{G}(v)$ and $\mathcal{S}(v)$ are referred to as gradient and solenoidal projections of the vector field v respectively, and

$$\int \mathcal{G}(v) \cdot \mathcal{S}(f)\, dx = 0,\ v, f \in C_0^\infty\left(R^d; R^d\right).$$

In fact, $\mathcal{G}(v) = \nabla T_i\left(v^i\right)^*$ and $\mathcal{S}(v) = v - \mathcal{G}(v)$ are continuous in $L_p\left(R^d; R^d\right)$ i.e. $\left|\mathcal{G}(v)\right|_p \leq C|v|_p$ and $\left|\mathcal{S}(v)\right|_p \leq C|v|_p$ (see [15, Remark 3.5].) For more detailed discussion on projections in non-weighted Sobolev spaces see for instance [16, Section 3.1.2], [15, Section 3.2] and references therein. The following Helmholtz decomposition for weighted Sobolev spaces is a key result for our main proof.

Lemma 3.1. *Let $p > 1$, $l \geq 0$, $\theta \in (1, d)$. The operators*

$$T_i : H^l_{\theta+l,p}\left(R^d\right) \to H^{l+1}_{\theta+l,p}\left(R^d\right), \quad i = 1, ..., d$$

are bounded. That is for any $i = 1, \ldots, d$, there exists $C > 0$ such that

$$\left|T_i f\right|_{H^{l+1}_{\theta+l,p}} \leq C\left|f\right|_{H^l_{\theta+l,p}},\ f \in H^l_{\theta+l,p}\left(R^d\right).$$

Consequently,

$$\mathcal{G} : H^l_{\theta+l,p}\left(R^d; R^d\right) \to H^l_{\theta+l,p}\left(R^d; R^d\right),\ \mathcal{S} : H^l_{\theta+l,p}\left(R^d; R^d\right) \to H^l_{\theta+l,p}\left(R^d; R^d\right)$$

are linear continuous. Moreover,

$$H^l_{\theta+l,p}\left(R^d; R^d\right) = \mathcal{G}\left(H^l_{\theta+l,p}\left(R^d; R^d\right)\right) \oplus \mathcal{S}\left(H^l_{\theta+l,p}\left(R^d; R^d\right)\right)$$

and $\mathcal{S}\left(H^l_{\theta+l,p}\left(R^d; R^d\right)\right) = \left\{v \in H^l_{\theta+l,p}\left(R^d; R^d\right) \mid \quad div\ v = 0\right\}$.

Proof. The first estimate is an immediate consequence of Lemma 8.5 whose proof is provided fully in the Appendix. Regarding the direct sum and divergence-free vector fields, we refer to [15, Lemma 3.7] and its proof which clearly carry over to weighted Sobolev spaces. □

Remark 3.1. *The solonoidal projection has a convenient formula in weighted Sobolev spaces. Indeed, if* $f \in H^{l}_{\theta+l,p}\left(R^{d};R^{d}\right)$ *for some* $p>1$, $l\geq 0$, $\theta\in(1,d)$ *and* $u^{ij}=\partial_{i}N\left(f^{j}\right)$, *then by Lemma 8.5,* $u\in H^{l+1}_{\theta+l,p}\left(R^{d};R^{d}\times R^{d}\right)$. *Following [4, Proposition 2], we let*

$$\mathcal{G}f=\sum_{i}\nabla T_{i}\left(f^{i}\right)^{*}=\sum_{i}\nabla\left(u^{ii}\right)^{*},\ \left(\tilde{\mathcal{S}}f\right)^{j}=\sum_{i}\frac{\partial}{\partial x_{i}}\left(u^{ij}-u^{ji}\right),\ 1\leq j\leq d.$$

By passing to the limit, $f^{j}=\Delta N\left(f^{j}\right)=\sum_{i}\partial_{i}u^{ij}=\left(\tilde{\mathcal{S}}f\right)^{j}+(\mathcal{G}f)^{j}$. *Therefore,* $\mathcal{S}f=\tilde{\mathcal{S}}f$ *and*

$$(\mathcal{S}f)^{j}=\sum_{i}\frac{\partial}{\partial x_{i}}\left(\partial_{i}\left(N\left(f^{j}\right)\right)-\partial_{j}\left(N\left(f^{i}\right)\right)\right).$$

In particular, if $f=\nabla p$ *where* $p\in H^{l+1}_{\theta+l,p}\left(R^{d}\right)$ *is a scalar, then* $\mathcal{S}f=0$.

4. Schauder Ring Properties

We now establish slightly modified Schauder ring properties. All parameters are assumed to be non-negative.

Lemma 4.1. *(cf. [17, Theorem 4.39]) Let* $p>1$, $l>\frac{d}{p}$, $m+m'\leq l$. *Then there exists* $C>0$ *such that for all* $u\in H^{l-m}_{p}\left(R^{d}\right), v\in H^{l-m'}_{p}\left(R^{d}\right)$,

$$\int\left|u\left(x\right)v\left(x\right)\right|^{p}dx\leq C\left|u\right|^{p}_{H^{l-m}_{p}}\left|v\right|^{p}_{H^{l-m'}_{p}}.$$

Proof. We follow the proof of [17, Theorem 4.39]. According to which, we have the following embedding.

(i) Let $lp\leq d$ and $p\leq r\leq\frac{dp}{d-lp}$ (or $p\leq r<\infty$ if $lp=d$.) Then there is a constant C such that for all $g\in H^{l}_{p}\left(R^{d}\right)$,

$$\int\left|g\left(x\right)\right|^{r}dx\leq C\left|g\right|^{r}_{H^{l}_{p}}.$$

(ii) Let $lp>d$. Then there is a constant C such that for all $g\in H^{l}_{p}\left(R^{d}\right)$ and dx-a.s.,

$$\left|g\left(x\right)\right|\leq C\left|g\right|_{H^{l}_{p}}.$$

Hence, if $(l-m)\,p>d$, then

$$\int\left|u\left(x\right)v\left(x\right)\right|^{p}dx\leq C\left|u\right|^{p}_{H^{l-m}_{p}}\left|v\right|^{p}_{p}.$$

Similarly, if $(l-m')\,p>d$, then

$$\int\left|u\left(x\right)v\left(x\right)\right|^{p}dx\leq C\left|u\right|^{p}_{p}\left|v\right|^{p}_{H^{l-m'}_{p}}.$$

If both $(l-m)\,p\leq d$ and $(l-m')\,p\leq d$, noting that we always have $(l-m+l-m')\,p>d$, then

$$\frac{d-(l-m)\,p}{d}+\frac{d-(l-m')\,p}{d}<1,$$

and there exist positive numbers $r, r' > 1$ such that $\frac{1}{r} + \frac{1}{r'} = 1$,

$$p \leq rp < \frac{dp}{d - (l-m)\, p},$$
$$p \leq r'p < \frac{dp}{d - (l-m')\, p}.$$

Finally, we obtain by applying (i),

$$\begin{aligned}\int \left| u(x)\, v(x) \right|^{p} dx &\leq \left(\int \left| u(x) \right|^{rp} dx \right)^{1/r} \left(\int \left| v(x) \right|^{r'p} dx \right)^{1/r'} \\ &\leq C \left| u \right|^{p}_{H_p^{l-m}} \left| v \right|^{p}_{H_p^{l-m'}}.\end{aligned}$$

The proof is complete. □

Corollary 4.1. *Let $p > 1$, $l > \frac{d}{p}$, $k = k_1 + \ldots + k_N \leq l$. Then there exists $C > 0$ such that for all $u_i \in H_p^{l-k_i}\left(R^d\right), i = 1, ..., N$, the product $\prod_{i=i}^{N} u_i \in H_p^{l-k}\left(R^d\right)$ and*

$$\left| \prod_{i=i}^{N} u_i \right|_{H_p^{l-k}} \leq C \prod_{i=1}^{N} \left| u_i \right|_{H_p^{l-k_i}}.$$

Proof. Let $N = 2$, and $\mu, \mu' \in N_0^d$ be multi-indices so that $\left|\mu\right| + \left|\mu'\right| \leq l - k$. Since $D^{\mu} u_1 \in H_p^{l-k_1-|\mu|}\left(R^d\right), D^{\mu'} u_2 \in H_p^{l-k_2-|\mu'|}\left(R^d\right)$ and $k_1 + \left|\mu\right| + k_2 + |\mu'| \leq l$, we have by Lemma 4.1,

$$\left| D^{\mu} u_1 D^{\mu'} u_2 \right|_p \leq C \left| D^{\mu} u_1 \right|_{H_p^{l-k_1-|\mu|}} \left| D^{\mu'} u_2 \right|_{H_p^{l-k_2-|\mu'|}} \leq C \left| u_1 \right|_{H_p^{l-k_1}} \left| u_2 \right|_{H_p^{l-k_2}}.$$

The statement follows by induction. □

Corollary 4.1 can be generalized to weighted Sobolev spaces.

Corollary 4.2. *Let $p > 1$, $l > \frac{d}{p}$, $k = k_1 + \ldots + k_N \leq l$ and $\delta \leq \delta_1 + \ldots + \delta_N - (N-1)\frac{d}{p}$. Then there exists $C > 0$ such that for all $u_i \in H_{\delta_i+l,p}^{l-k_i}\left(R^d\right), i = 1, \ldots, N$, the product $\prod_{i=i}^{N} u_i \in H_{\delta+l,p}^{l-k}\left(R^d\right)$ and*

$$\left| \prod_{i=i}^{N} u_i \right|_{H_{\delta+l,p}^{l-k}} \leq C \prod_{i=1}^{N} \left| u_i \right|_{H_{\delta_i+l,p}^{l-k_i}}.$$

In particular, if $\delta_1 = \ldots = \delta_N = \delta \geq \frac{d}{p}$, then $\delta_1 + \ldots + \delta_N - (N-1)\frac{d}{p} = \delta + (N-1)\left(\delta - \frac{d}{p}\right) \geq \delta$ and hence,

$$\left| \prod_{i=i}^{N} u_i \right|_{H_{\delta+l,p}^{l-k}} \leq C \prod_{i=1}^{N} \left| u_i \right|_{H_{\delta+l,p}^{l-k_i}}.$$

If, in addition, $k_i = 0$ for all i, then

$$\left| \prod_{i=i}^{N} u_i \right|_{H_{\delta+l,p}^{l}} \leq C \prod_{i=1}^{N} \left| u_i \right|_{H_{\delta+l,p}^{l}}.$$

Proof. Indeed, for any multi-index $\gamma \in N_0^d$ such that $\left|\gamma\right| \leq l-k$,

$$w^{\delta+k+|\gamma|-d/p} D^{\gamma}\left(\prod_{i=1}^{N} u_i\right) = w^{\delta-(\sum_i \delta_i-(N-1)d/p)} \sum_{\mu_1+\ldots+\mu_N=\gamma} \prod_{i=1}^{N} w^{\delta_i+k_i+|\mu_i|-d/p} D^{\mu_i} u_i.$$

Since $D^{\mu_i} u_i \in H^{l-k_i-|\mu_i|}_{\delta_i+l,p}\left(R^d\right)$, we have $w^{\delta_i+k_i+|\mu_i|-d/p} D^{\mu_i} u_i \in H_p^{l-k_i-|\mu_i|}\left(R^d\right)$ and

$$\left|w^{\delta_i+k_i+|\mu_i|-d/p} D^{\mu_i} u_i\right|_{H_p^{l-k_i-\mu_i}} \leq C \left|D^{\mu_i} u_i\right|_{H^{l-k_i-|\mu_i|}_{\delta_i+l,p}} \leq C \left|u_i\right|_{H^{l-k_i}_{\delta_i+l,p}}.$$

The statement follows by Corollary 4.1. □

Clearly, Corollary 4.1 and Corollary 4.2 can be extended to multiplication between matrices. We have the following estimate of function composition. We note that all derivatives are interpreted as classical since $p > d$.

Lemma 4.2. *Let $p > d$, $l \geq 0, \delta, \delta' \geq \frac{d}{p}$, $N > 0$. Then there exists $C > 0$ such that for all $f \in H^l_{\delta+l,p}\left(R^d\right)$; and $g : R^d \to R^d$ a diffeomorphism with $\left|\det \nabla\left(g^{-1}\right)\right|_{\infty} + \left\|g^{-1}\right\| + \chi_{l\geq 1}\left|\nabla g\right|_{\infty} + \chi_{l\geq 2}\left|D^2 g\right|_{H^{l-2}_{\delta'+l-1,p}} \leq N$,*

$$\begin{aligned} &\left|f \circ g\right|_{H^l_{\delta+l,p}} \\ &\leq C\left(1+\left\|g^{-1}\right\|\right)^{\delta-d/p+l} \left|f\right|_{H^l_{\delta+l,p}} \left(1+\chi_{l\geq 1}\left|\nabla g\right|_{\infty} + \chi_{l\geq 2}\left|D^2 g\right|_{H^{l-2}_{\delta'+l-1,p}}\right)^l. \end{aligned}$$

Proof. We first mention a simple estimate,

$$\frac{w\left(g^{-1}(x)\right)}{w(x)} = \frac{\left(1+\left|g^{-1}(x)\right|^2\right)^{1/2}}{\left(1+|x|^2\right)^{1/2}} \leq 1+\left\|g^{-1}\right\|,\ \left\|g^{-1}\right\| = \left|g^{-1}(0)\right| + \left|\nabla g^{-1}\right|_{\infty}.$$

If $l = 0$, by changing the variable of integration,

$$\left|f \circ g\right|_{H^0_{\delta,p}} = \left|w^{\delta-d/p} f(g)\right|_p \leq \left(1+\left\|g^{-1}\right\|\right)^{\delta-d/p} \left|w^{\delta-d/p} f\right|_p.$$

For a multi-index $\gamma \in N_0^d$ with $\left|\gamma\right| = l \geq 1$, $w^{\delta-d/p+l} D^{\gamma}(f \circ g)$ is a summation of terms in the form of

$$\mathcal{A} = w^{\delta-d/p+m} D^{\mu} f(g) \prod_{i=1}^{m} w^{|\mu_i|-1} D^{\mu_i} g^{a_i}$$

where $\mu_1 + \ldots + \mu_m = \gamma, \left|\mu_i\right| \geq 1$ for $i = 1, ..., m$, $\left|\mu\right| = m, 1 \leq m \leq l$, and $1 \leq a_i \leq d$ are component indices.
If $\left|\mu_i\right| = 1$ for all $i = 1, ..., m$ then

$$\left|\mathcal{A}\right|_p \leq C\left(1+\left\|g^{-1}\right\|\right)^{\delta-d/p+m} \left|w^{\delta-d/p+m} D^{\mu} f\right|_p \left|\nabla g\right|_{\infty}^m.$$

We may now assume that $l \geq 2$ and $\left|\mu_i\right| \geq 2$ for some i. Since $(l-m)p > d$, by Sobolev embedding theorem for f and Corollary 4.1,

$$|\mathcal{A}|_p \le C\left(1+\left\|g^{-1}\right\|\right)^{\delta-d/p+m}\left|w^{\delta-d/p+m}D^{\mu}f\right|_{\infty}\prod_{i=1,|\mu_i|=1}^{m}\left|\nabla g^{a_i}\right|_{\infty}\left|\prod_{i=1,|\mu_i|\ge 2}^{m}w^{|\mu_i|-1}D^{\mu_i}g^{a_i}\right|_p$$
$$\le C\left(1+\left\|g^{-1}\right\|\right)^{\delta-d/p+m}|f|_{H^l_{\delta+l,p}}\prod_{i=1,|\mu_i|=1}^{m}\left|\nabla g^{a_i}\right|_{\infty}\prod_{i=1,|\mu_i|\ge 2}^{m}\left|D^{\mu_i}g^{a_i}\right|_{H^{l-|\mu_i|}_{d/p+l-1,p}}.$$

The proof is complete. □

Examining the proof above the restriction $\delta \ge \frac{d}{p}$ can be easily relaxed as follows:

Corollary 4.3. *Let $p > d$, $l \ge 0, \delta \ge 0$, $\delta' \ge \frac{d}{p}$, $N > 0$. Then there exists $C > 0$ such that for all $f \in H^l_{\delta+l,p}\left(R^d\right)$; and $g : R^d \to R^d$ a diffeomorphism with $\left|\det\nabla\left(g^{-1}\right)\right|_{\infty} + \|g\| + \left\|g^{-1}\right\| + \chi_{l\ge1}\left|\nabla g\right|_{\infty} + \chi_{l\ge2}\left|D^2 g\right|_{H^{l-2}_{\delta'+l-1,p}} \le N$,*

$$\left|f\circ g\right|_{H^l_{\delta+l,p}}$$
$$\le C\left(1+\|g\|+\left\|g^{-1}\right\|\right)^{|\delta-d/p|+l}|f|_{H^l_{\delta+l,p}}\left(1+\chi_{l\ge1}\left|\nabla g\right|_{\infty}+\chi_{l\ge2}\left|D^2 g\right|_{H^{l-2}_{\delta'+l-1,p}}\right)^l.$$

Clearly, Lemma 4.2 and Corollary 4.3 also hold if f is a vector or a matrix.

5. Flow Representation and Main Results

In this section, we discuss flow representation of the Navier-Stokes equations and state our main results. We will assume that the prescribed fields u_0 and G satisfy the following assumption with $l \ge 2$.

Assumption $F(l)$.

(i) u_0 is $\mathcal{F}^W_0-$measurable. For all $\omega\in\Omega$, u_0 is divergence-free, and $u_0 \in C^2\left(R^d;R^d\right)\cap H^{l-2}_{\theta+l-2,p}\left(R^d;R^d\right)$.

(ii) G is $\mathbb{F}^W-$adapted. For all $(\omega,t)\in\Omega\times[0,\infty)$, $G(t)$ is divergence-free and $G(t)\in H^{l-2}_{\theta+l-2,p}\left(R^d;R^d\right)$. For all $\omega\in\Omega$, $G\in C\left([0,\infty), C^2\left(R^d;R^d\right)\right)$.

5.1. **Flow Representation.** Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a filtration $\mathbb{F}$ of $\sigma-$ algebras $(\mathcal{F}_t, t\ge 0)$ satisfying the usual conditions and B_t, W_t be independent standard $d-$dimensional Wiener processes on $(\Omega,\mathcal{F},P)$. Let $\mathbb{F}^W = \left(\mathcal{F}^W_t, t\ge 0\right)$ be the standard sub-filtration of $\mathbb{F}$ generated by W_t. We consider the following Navier-Stokes equations with a random forcing term G,

$$du(t) = \left[\mathcal{S}\left(-u^k(t)\,\partial_k u(t)\right) + \frac{\varepsilon^2}{2}\Delta u(t) + G(t)\right]dt \tag{5.1}$$
$$u(0) = u_0, \quad \operatorname{div} u(t) = 0, \quad t > 0.$$

We now formulate the flow representation of (5.1). Note that u_0 and G have sufficient regularity thanks to Assumption $F(l)$ to make the computation rigorous. To make the statement less cumbersome, we define some notations for the next Lemma. For a smooth vector field u, we let the perturbed flow $\eta(t) : R^d \to R^d$, $t > 0$ be given by

$$d\eta(t) = u(t, \eta(t))\, dt + \varepsilon B_t,\ \eta(0) = e,\ t > 0.$$

and let $\kappa(t) = \eta^{-1}(t)$ be its spatial inverse whenever it is well-defined. Also, we let

$$g_\eta(t,x) = u_0 + \int_0^t (\nabla\eta(s,x))^* G(s, \eta(s,x))\, ds,\ (\omega, t, x) \in \Omega \times (0,\infty) \times R^d.$$

Lemma 5.1. *Let $p > d$, $l \geq 2$, $\theta \in (1,d)$, $\alpha \in (0,1]$, $N_1, N_2 > 0$ and $F(l)$ holds. Let $u : \Omega \times [0,T] \times R^d \to R^d$ be $\mathbb{F}^W-$adapted such that*

$$\sup_{(\omega,t)\in\Omega\times[0,T]} \left|u(t)\right|_{C^{3+\alpha}} \leq N_1.$$

If (i) $(\nabla\kappa(t))^* g_\eta(t, \kappa(t)) \in H^l_{\theta+l,p}\left(R^d; R^d\right)$ for all $(\omega, t) \in \Omega \times [0,T]$ and
(ii) $E\left|\mathcal{S}\left[(\nabla\kappa(t))^* g_\eta(t, \kappa(t))\right]\right|_{C^2} \leq N_2$ for all $t \in [0,T]$,

then for $P-$a.s., $y(t) = E^W_t \mathcal{S}\left[(\nabla\kappa(t))^* g_\eta(t, \kappa(t))\right]$ solves

$$\begin{aligned} dy(t) =& \mathcal{S}\left[-u^k(t)\,\partial_k y(t) - (\nabla u(t))^* y(t)\right] dt + \left[\frac{\varepsilon^2}{2}\Delta y(t) + G(t)\right] dt \\ y(0) =& u_0, \quad div\, y(t) = 0, \quad t \in (0,T] \end{aligned} \tag{5.2}$$

as an equality in $H^{l-2}_{\theta+l-2,p}\left(R^d; R^d\right)$.

Remark 5.1. *We will justify in the proof that κ is well-defined. For clarity, we note that u_0 satisfies both the above conditions and assumption $F(l)$.*

Proof. We consider the perturbed flow $\eta(t) : R^d \to R^d$ given by

$$\begin{aligned} d\eta(t) &= u(t, \eta(t))\, dt + \varepsilon B_t \\ \eta(0) &= e, \quad t \in (0,T]. \end{aligned} \tag{5.3}$$

According to [18, Theorem 2.1 and 2.4], the classical solution $\eta(t)$ of (5.3) and its spatial inverse $\kappa(t) = \eta^{-1}(t)$ belongs to $C\left([0,T], C^3\left(R^d; R^d\right)\right)$ and for $P-$a.s., $\kappa(t)$ is the classical solution of the following equation,

$$\begin{aligned} d\kappa(t) &= \left[-\nabla\kappa(t)\, u(t) + \frac{\varepsilon^2}{2}\Delta\kappa(t)\right] dt - \varepsilon\partial_k\kappa(t)\, dB^k_t \\ \kappa(0) &= e, \quad t \in (0,T]. \end{aligned}$$

Therefore, for any $m, l = 1, \ldots, d$,

$$
\begin{aligned}
& d\partial_l \kappa^m(t) \\
&= \left[-\nabla \partial_l \kappa^m(t)\, u(t) - \nabla \kappa^m(t)\, \partial_l u(t) + \frac{\varepsilon^2}{2} \Delta \partial_l \kappa^m(t)\right] dt - \varepsilon \partial_k \partial_l \kappa^m(t)\, dB_t^k.
\end{aligned}
$$

By writing the above equation in a matrix form with $A(t) = (\partial_l \kappa^m(t))_{\substack{1 \le m \le d, \\ 1 \le l \le d}}$ and multiplying by $g_\eta(t, \kappa(t))$ from the right, we obtain

$$
\begin{aligned}
& (dA(t)^*)\, g_\eta(t, \kappa(t)) \\
&= \left(-u^k(t)\, \partial_k A(t)^* - (\nabla u(t))^* A(t)^* + \frac{\varepsilon^2}{2} \Delta A(t)^*\right) g_\eta(t, \kappa(t))\, dt \\
& - \varepsilon \partial_k A(t)^*\, g_\eta(t, \kappa(t))\, dB_t^k.
\end{aligned}
$$

Next applying Itô-Wentzell formula for $g_\eta(t, \kappa(t))$ and multiplying by $A(t)^*$ from the left, we obtain

$$
\begin{aligned}
& A(t)^*\, dg_\eta(t, \kappa(t)) \\
&= \left[G(t) - A(t)^* \nabla g_\eta(t, \kappa(t))\, \nabla \kappa(t)\, u(t)\right] dt \\
&+ \frac{\varepsilon^2}{2} \left(A(t)^* \nabla g_\eta(t, \kappa(t))\, \Delta \kappa(t) + A(t)^* \partial_{ij}^2 g_\eta(t, \kappa(t)) \left(\nabla \kappa^i(t) \cdot \nabla \kappa^j(t)\right)\right) dt \\
&- \varepsilon A(t)^* \nabla g_\eta(t, \kappa(t))\, \partial_k \kappa(t)\, dB_t^k.
\end{aligned}
$$

Finally, the covariation term is $d\left[A(t)^*, g_\eta(t, \kappa(t))\right] = \epsilon^2 \partial_k A(t)^* \nabla g_\eta(t, \kappa(t))\, \partial_k \kappa(t)\, dt$.
By Itô product rule, summing the terms above, $z(t) = A(t)^* g_\eta(t, \kappa(t))$ must satisfy the following equation

$$
\begin{aligned}
dz(t) &= \left[G(t) - u^k(t)\, \partial_k z(t) - (\nabla u(t))^* z(t) + \frac{\varepsilon^2}{2} \Delta z(t)\right] dt \\
&- \varepsilon \partial_k z(t)\, dB_t^k \\
z(0) &= u_0.
\end{aligned}
$$

Due to (i), $z(t) \in H^l_{\theta+l,p}\left(R^d; R^d\right)$. According to Lemma 3.1, we may let $z(t) = \mathcal{S}z(t) + (\nabla p(t))^*$ be its Helmholtz decomposition in $H^l_{\theta+l,p}\left(R^d; R^d\right)$ where $p(t) \in H^{l+1}_{\theta+l}\left(R^d\right)$ is a scalar. By collecting all gradient terms, we derive

$$
\begin{aligned}
dz(t) &= \left[G(t) - u^k(t)\, \partial_k \mathcal{S}z(t) - (\nabla u(t))^* \mathcal{S}z(t) + \frac{\varepsilon^2}{2} \Delta \mathcal{S}z(t) + (\nabla q(t))^*\right] dt \\
&- \varepsilon \partial_k z(t)\, dB_t^k \\
z(0) &= u_0
\end{aligned}
$$

where $q(t) = -\nabla p(t)\, u(t) - \frac{\varepsilon^2}{2} \Delta p(t)$. Next, we take the optional projection E_t^Won both sides. Due to (ii), E_t^W can be interchanged with the integral with respect to dt and derivatives of $\mathcal{S}z(t)$.

Finally, by taking a solenoidal projection in $H^{l-2}_{\theta+l-2,p}\left(R^d;R^d\right)$ and applying Remark 3.1, it is easily verified that

$$y(t) \quad = E^W_t \mathcal{S} z(t) = \quad E^W_t \mathcal{S}\left[(\nabla\kappa(t))^* g_\eta(t,\kappa(t))\right]$$

is a solution of (5.2) as an equality in $H^{l-2}_{\theta+l-2,p}\left(R^d;R^d\right)$. □

In the next Lemma, we derive a simplified form of $E^W_t \mathcal{S}\left[(\nabla\kappa(t))^* g_\eta(t,\kappa(t))\right]$.

Lemma 5.2. *Let $p>d$, $l\geq 2$, $\theta\in(1,d)$ and $F(l)$ holds. Let $\eta(t): R^d\to R^d$, $t\in[0,T]$ be a diffeomorphism and $\kappa(t): R^d\to R^d$, $t\in[0,T]$ be its spatial inverse such that*

(i) $(\nabla\kappa(t))^* g_\eta(t,\kappa(t))\in H^l_{\theta+l,p}\left(R^d;R^d\right)$ for all $(\omega,t)\in\Omega\times[0,T]$,
(ii) $\eta(t)$, $\kappa(t)$ are spatially twice differentiable for all $(\omega,t)\in\Omega\times[0,T]$,
(iii) for any multi-index $\gamma\in N^d_0$ with $0\leq|\gamma|\leq 2$, $D^\gamma\eta(t,x)$ is continuous in t for all $(\omega,x)\in\Omega\times R^d$.
Then for all $(\omega,t)\in\Omega\times[0,T]$,

$$\mathcal{S}\left((\nabla\kappa(t))^* g_\eta(t,\kappa(t))\right) = K_{\eta,h_\eta}(t) \tag{5.4}$$

where for $j=1,...,d$,

$$K^j_{\eta,h_\eta}(t,x) = \int \Gamma_i(x-z)\left[\phi^{ji}_{\eta,h_\eta}(t,z)-\phi^{ij}_{\eta,h_\eta}(t,z)\right]dz, \tag{5.5}$$
$$\phi_{\eta,h}(t,x) = (\nabla\kappa(t,x))^* h(t,\kappa(t,x))\,\nabla\kappa(t,x)$$

and

$$h_\eta(t,x) = \nabla u_0(x) + \int_0^t (\nabla\eta(s,x))^*\nabla G(s,\eta(s,x))\,(\nabla\eta(s,x))\,ds,$$

for all $(\omega,t,x)\in\Omega\times[0,T]\times R^d$.

Proof. In fact, if $f\in H^l_{\theta+l,p}\left(R^d;R^d\right)$ then by Remark 3.1,

$$(\mathcal{S}f)^j = \int \Gamma_i(\cdot-z)\left(\partial_i f^j(z)-\partial_j f^i(z)\right)dz. \tag{5.6}$$

Applying (5.6) for $(\nabla\kappa(t))^* g_\eta(t,\kappa(t))\in H^l_{\theta+l,p}\left(R^d;R^d\right)$, we obtain

$$\begin{aligned}
&\left(\mathcal{S}\left((\nabla\kappa(t))^* g_\eta(t,\kappa(t))\right)\right)^j\\
&= \int \Gamma_i(\cdot-z)\left[\partial_i\left(\partial_j\kappa^k(t,z)\,g^k_\eta(t,\kappa(t,z))\right)-\partial_j\left(\partial_i\kappa^k(t,z)\,g^k_\eta(t,\kappa(t,z))\right)\right]dz\\
&= \int \Gamma_i(\cdot-z)\left[\partial_j\kappa^k(t,z)\,\nabla g^k_\eta(t,\kappa(t,z))\,(\partial_i\kappa(t,z))^* - \partial_i\kappa^k(t,z)\,\nabla g^k_\eta(t,\kappa(t,z))\left(\partial_j\kappa(t,z)\right)^*\right]dz\\
&= \int \Gamma_i(\cdot-z)\left[\left(\nabla\kappa(t,z)^*\nabla g_\eta(t,\kappa(t,z))\,\nabla\kappa(t,z)\right)^{ji}-\left(\nabla\kappa(t,z)^*\nabla g_\eta(t,\kappa(t,z))\,\nabla\kappa(t,z)\right)^{ij}\right]dz
\end{aligned}$$

and by (iii),

$$\nabla g_\eta(t) = \nabla u_0 + \int_0^t (\nabla\eta(s))^*\nabla G(s,\eta(s))\,\nabla\eta(s)\,ds + \int_0^t A(s)\,ds$$

where $(A(s))_{ij} = \partial^2_{ij}\eta^k(s)\,G^k(s,\eta(s))$ is symmetric. The statement follows. □

Remark 5.2. *For the time being, we provide a formal argument to derive the form of the flow equations. If a diffeomorphism $\eta(t)$ and $\kappa(t) = \eta^{-1}(t)$ its spatial inverse satisfy*

$$\begin{aligned} d\eta(t) &= E^W_t \mathcal{S}\left[(\nabla\kappa(t,z))^* g_\eta(t,\kappa(t,z))\right]|_{z=\eta(t)}dt + \varepsilon dB_t \\ &= E^W_t K_{\eta,h_\eta}(t,\eta(t)) + \epsilon dB_t \\ \eta(0) &= e, \quad t \in (0,T], \end{aligned} \tag{5.7}$$

then by letting $u(t) = E^W_t\mathcal{S}\left[(\nabla\kappa(t))^* g_\eta(t,\kappa(t))\right]$ in Lemma 5.1, (5.2) becomes (5.1) where the term $(\nabla u(t))^* u(t) = \frac{1}{2}\nabla\left|u(t)\right|^2$ disappears under the solenoidal projection.

Our strategy is to find a solution of (5.7) in appropriate weighted Sobolev spaces and then return to Lemma 5.1 and show that indeed the velocity

$$u(t) = E^W_t\mathcal{S}\left[(\nabla\kappa(t))^* g_\eta(t,\kappa(t))\right]$$

is a $H^{l+1}_{\theta,p}$–solution of (5.1) where the equality is understood in $H^{l-2}_{\theta+l-2,p}\left(R^d;R^d\right)$ (see Theorem 5.2 for the complete statement.)

5.2. **Main Results.** We are now ready to state main results of this paper. Intuitively, $\eta(t)$ remains close to the identity mapping for a short time. Due to the lack of Sobolev regularity of constants and the identity map e, it is convenient to consider for $t \geq 0$ the displacement $\zeta(t)$ defined by

$$\zeta(t) = \eta(t) - e - \epsilon B_t.$$

Therefore, from (5.7), $\zeta(t)$ must satisfy the equation

$$\begin{aligned} d\zeta(t) &= E^W_t K_{\eta,h_\eta}(t,\eta(t))\,dt \\ \zeta(0) &= 0. \end{aligned}$$

We start with the existence of the flow equation.

Theorem 5.1. *Let $p > d$, $l > 1 + \frac{d}{p}$, $\theta \in \left[1 + \frac{d}{p}, d\right)$, $N > 0$ and*

$$\sup_{\omega\in\Omega}|u_0|_{H^{l+1}_{\theta+l,p}} + \sup_{(\omega,t)\in\Omega\times[0,\infty)}\left|G(t)\right|_{H^{l+1}_{\theta+l,p}} \leq N.$$

Then for some deterministic $T > 0$, there exists $\zeta \in C\left([0,T], H^{l+1}_{\theta+l,p}\left(R^d;R^d\right)\right)$ such that

$$\zeta(t) = \int_0^t E^W_s K_{\eta,h_\eta}(s,\eta(s))\,ds,\ (\omega,t) \in \Omega\times[0,T] \tag{5.8}$$

holds in $H^l_{\theta+l-1,p}\left(R^d;R^d\right)$. Moreover, there exists $M > 0$ such that

$$\sup_{(\omega,t)\in\Omega\times[0,T]}|\zeta|_{H^{l+1}_{\theta+l,p}} \leq M.$$

Next, we state the existence of the velocity equation.

Theorem 5.2. *Let $p > d$, $l > 2 + \frac{d}{p}$, $\theta \in \left[1 + \frac{d}{p}, d\right)$, $N > 0$, $F(l)$ holds and*

$$\sup_{\omega\in\Omega} |u_0|_{H^{l+1}_{\theta+l,p}} + \sup_{(\omega,t)\in\Omega\times[0,\infty)} \left|G(t)\right|_{H^{l+1}_{\theta+l,p}} \leq N.$$

Suppose that ζ is the solution of (5.8) as given in Theorem 5.1. Then for $P-$a.s., $u(t) := E^W_t \mathcal{S}(\nabla\kappa(t))^* g_\eta(t,\kappa(t))$, $(\omega,t) \in \Omega\times[0,T]$ solves

$$\begin{aligned} du(t) =& \left[\mathcal{S}\left(-u^k(t)\,\partial_k u(t)\right) + \frac{\varepsilon^2}{2}\Delta u(t) + G(t)\right] dt \\ u(0) =& u_0, \quad div\, u(t) = 0, \quad t \in (0,T] \end{aligned} \tag{5.9}$$

as an equality in $H^{l-2}_{\theta+l-2,p}\left(R^d; R^d\right)$. Moreover, there exists $M > 0$ such that

$$\sup_{(\omega,t)\in\Omega\times[0,T]} \left|u(t)\right|_{H^{l+1}_{\theta+l,p}} \leq M.$$

6. Estimates of Diffeomorphisms

We collect some basic estimates regarding diffeomorphisms. First, for any differentiable function $f : R^d \to R^d$,

$$\frac{w(f(x))}{w(x)} = \frac{\left(1+\left|f(x)\right|^2\right)^{1/2}}{\left(1+|x|^2\right)^{1/2}} \leq 1 + \left\|f\right\|, \ \left\|f\right\| = \left|f(0)\right| + \left|\nabla f\right|_\infty.$$

If $\eta = e + b + \zeta$, for some $b \in R^d$ and a continuously differentiable $\zeta : R^d \to R^d$ with $|\nabla\zeta|_\infty \leq \frac{1}{2d^2}$, then η is a diffeomorphism with spatial inverse κ, $\nabla\eta = I + \nabla\zeta$, $(\nabla\eta)^{-1} = \sum_{n=0}^\infty (-1)^n (\nabla\zeta)^n$, and

$$\begin{aligned} |\nabla\kappa|_\infty &= \left|(\nabla\eta)^{-1}\right|_\infty \leq 1 + \sum_{n=1}^\infty d^{n-1} |\nabla\zeta|^n_\infty \leq 1 + \sum_{n=1}^\infty \frac{1}{2^n d^{n+1}} \\ &\leq 1 + \frac{1}{d(2d-1)} \leq 2. \end{aligned} \tag{6.1}$$

Moreover,

$$|\nabla\kappa - I|_\infty = \left|(\nabla\eta)^{-1} - I\right|_\infty \leq \frac{1}{d(2d-1)} \leq \frac{1}{2d}. \tag{6.2}$$

For any $x, y \in R^d$,

$$\left|x-y\right| \leq |\nabla\kappa|_\infty \left|\eta(x) - \eta(y)\right|$$

If we take $y = \kappa(0)$, $x = 0$, then

$$\left|\kappa(0)\right| \leq |\nabla\kappa|_\infty \left|\eta(0)\right|. \tag{6.3}$$

Therefore, by (6.1),

$$\begin{aligned} ||\kappa|| &= \left|\kappa(0)\right| + |\nabla\kappa|_\infty \leq \left(1 + \left|\eta(0)\right|\right)|\nabla\kappa|_\infty \\ &\leq 2\left(1 + |b| + \left|\zeta(0)\right|\right). \end{aligned} \tag{6.4}$$

For determinants of Jacobian matrices, by (6.1) for all $x \in R^d$,

$$\left|\det \nabla \kappa (x)\right| \leq C |\nabla \kappa|_\infty^d \leq C, \tag{6.5}$$
$$\left|\det \nabla \eta (x)\right| \leq C \left(1 + |\nabla \zeta|_\infty\right)^d \leq C. \tag{6.6}$$

We now discuss linear combination of $\eta = e + b + \zeta$ and $\bar{\eta} = e + \bar{b} + \bar{\zeta}$ where $|\nabla \zeta|_\infty, \left|\nabla \bar{\zeta}\right|_\infty \leq \frac{1}{2d^2}$. Considering for $s \in [0,1]$, $\eta_s = (1-s)\eta + s\bar{\eta}$, $b_s = (1-s) b + s\bar{b}$, $\zeta_s = (1-s)\zeta + s\bar{\zeta}$, we have

$$\eta_s = e + b_s + \zeta_s,$$
$$\nabla \eta_s = I + \nabla \zeta_s,$$

where $|\nabla \zeta_s|_\infty \leq \frac{1}{2d^2}$. Hence, by (6.1), $\left|\nabla \left(\eta_s^{-1}\right)\right|_\infty = \left|(\nabla \eta_s)^{-1}\right|_\infty \leq 2$. Denoting $b_0 = \max\left\{|b|, \left|\bar{b}\right|\right\}$, $l_0 = \max\left\{\left|\zeta(0)\right|, \left|\bar{\zeta}(0)\right|\right\}$, it follows from (6.4) that

$$\left\|\eta_s^{-1}\right\| \leq 2\left(1 + b_0 + l_0\right). \tag{6.7}$$

Obviously, for all $x \in R^d$,

$$\left|\det \nabla \left(\eta_s^{-1}\right)(x)\right| \leq C, \left|\det \nabla \eta_s (x)\right| \leq C. \tag{6.8}$$

Definition 6.1. *For $M > 0$, $T > 0$, $p > 1$, $l \geq 0$, $\theta \in (1, d)$, we say that $\zeta \in \mathcal{A}_{M,T}^{p,l,\theta}$ if for all $(\omega, t) \in \Omega \times [0, T]$, $\zeta(t)$ is continuously differentiable and*
(i)$\left|\nabla \zeta(t)\right|_\infty \leq \frac{1}{2d^2}$, $\left|\zeta(t, 0)\right| \leq M$,
(ii) $\left|\zeta(t)\right|_{H_{\theta+l,p}^{l+1}} \leq M$.

We will consistently denote $\eta(t) = e + \epsilon B_t + \zeta(t)$ and $\kappa(t)$ to be the spatial inverse of $\eta(t)$. Assumptions (i), (ii) will be used without explicit mention. Clearly, if $\zeta \in \mathcal{A}_{M,T}^{p,l,\theta}$ then $\eta(t)$ is a diffeomorphism. We start with a simple Lemma which facilitates later computations. To ease notation, we will write $\lambda_t = \lambda_t^\epsilon = 1 + \epsilon \sup_{s\in[0,t]} |B_s|$, $t \geq 0$.

Lemma 6.1. *There exists $C > 0$ such that for all $\zeta \in \mathcal{A}_{M,T}^{p,l,\theta}$ and $(\omega, t, x) \in \Omega \times [0, T] \times R^d$,*

$$\frac{w(\eta(t))}{w} \leq C\lambda_t, \quad \frac{w(\kappa(t))}{w} \leq C\lambda_t.$$

Moreover,

$$\left|\det \nabla \eta(t)\right|_\infty \leq C, \quad \left|\det \nabla \kappa(t)\right|_\infty \leq C.$$

All estimates also hold for a linear interpolation $\eta_a = a\eta_1 + (1-a)\eta_2$, $a \in [0,1]$ of $\eta_i \in \mathcal{A}_{M,T}^{p,l,\theta}$, $i = 1, 2$ with constants independent of a.

Proof. Trivially,

$$\begin{aligned}\frac{w(\eta(t))}{w} &\leq 1 + \left\|\eta(t)\right\| = 1 + \left|\eta(t, 0)\right| + \left|\nabla \eta(t)\right|_\infty \\ &\leq 1 + \left|\zeta(t, 0)\right| + \epsilon |B_t| + 1 + \left|\nabla \zeta(t)\right|_\infty \\ &\leq C\lambda_t.\end{aligned}$$

The estimates for $\kappa(t) = \eta^{-1}(t)$ follows directly from (6.4). The estimates for Jacobian matrices follows immediately from (6.6) and (6.5) respectively. □

6.1. **Growth Estimates of the Spatial Inverse.**

Lemma 6.2. *Let $p > d$, $l \geq 1$, $\theta \in (1, d)$, $\sigma = \theta - \frac{d}{p}$.*
(i) There exists $C > 0$ such that for all $\zeta \in \mathcal{A}^{p,l,\theta}_{M,T}$ and $(\omega, t) \in \Omega \times [0, T]$,

$$\left|\nabla\kappa(t) - I\right|_\infty \leq \frac{1}{2d},$$
$$\left|\kappa(t, 0)\right| \leq C\lambda_t.$$

(ii) There exists $C > 0$ such that for all $\zeta \in \mathcal{A}^{p,l,\theta}_{M,T}$ and $(\omega, t) \in \Omega \times [0, T]$,

$$\left|w^{\sigma-1}(\nabla\kappa(t) - I)\right|_p \leq C\lambda_t^{|\sigma-1|}.$$

(iii) There exists $C > 0$ such that for all $\zeta \in \mathcal{A}^{p,l,\theta}_{M,T}$, $(\omega, t) \in \Omega \times [0, T]$ and $1 \leq k \leq l$,

$$\begin{aligned}\left|w^k D^k \nabla\kappa(t)\right|_p &\leq C\lambda_t^{p_k},\\ \left|w^{\sigma+k} D^k \nabla\kappa(t)\right|_p &\leq C\lambda_t^{\sigma+p_k},\end{aligned}$$

where p_k is defined recursively as $p_k = k(1 + p_{k-1})$, $k \geq 2$, $p_1 = 1$.

Proof. The variable t is mostly dropped throughout the proof since all estimates simply holds for each t. We resort to tools in differential calculus on norm vector spaces. For a primer on the subject, an interested reader may consult [19].
(i) The first estimate is simply (6.2). For the second estimate, by (6.3),

$$\left|\kappa(t, 0)\right| \leq C\left|\eta(t, 0)\right| \leq C\lambda_t.$$

(ii) By changing the variable of integration and Lemma 6.1,

$$\begin{aligned}&\left|w^{\sigma-1}(\nabla\kappa - I)\right|_p\\ &\leq \sum_{n=1}^{\infty}\left|w^{\sigma-1}(\nabla\zeta(\kappa))^n\right|_p \leq C\lambda_t^{|\sigma-1|}\sum_{n=1}^{\infty}\left|(\nabla\zeta)\right|_\infty^{n-1}\left|w^{\sigma-1}\nabla\zeta\right|_p\\ &\leq C\lambda_t^{|\sigma-1|}.\end{aligned}$$

(iii) Our goal now is to obtain the form of derivatives of κ. Let $M_{d\times d}$ the set of all $d \times d$ matrices and U be the set of invertible $d \times d$ matrices. Then $F(A) = A^{-1}, A \in U$, is smooth and its $n-$th Frechet derivative is a continuous multilinear mapping defined as

$$F^{(n)}(A) \cdot (x_1, ..., x_n) = (-1)^n \sum_{\sigma} A^{-1}x_{\sigma(1)}A^{-1}...A^{-1}x_{\sigma(n)}A^{-1}.$$

where the summation is taken over all possible permutations. Clearly, the operator norm of $F^{(n)}$ satisfies

$$\left\|F^{(n)}(A)\right\| \leq C\left|A^{-1}\right|^{n+1}, A \in U, n \geq 0$$

for some $C > 0$ (for more details see [19, Theorem 5.4.3 and relevant exercises].)
Let $a \in R^d$, $b = \nabla\eta(a) \in U$, $1 \le n \le l$, we write the order n Taylor's expansion of $\nabla\eta$ at a and F at b as follows:

$$\nabla\eta(a+x) = \nabla\eta(a) + \sum_{i=1}^{n} \varphi_i(x) + r(x)$$

where $\varphi_i(x) = \frac{1}{i!}\sum_{|\gamma|=i} D^\gamma \nabla\eta(a)\, x^\gamma$, $\left|r(x)\right| = o\left(|x|^n\right), x \in R^d$, and

$$F(b+y) = F(b) + \sum_{j=1}^{n} \psi_j(y) + s(y)$$

where $\psi_j(y) = \psi_j(y, y, ..., y) = \frac{1}{j!}F^{(j)}(b)\cdot(y, y, ..., y)$, $s(y) = o\left(\left|y\right|^n\right), y \in M_{d\times d}$.
We denote $\tilde{\psi}_j$ the multi-linear symmetrical mapping associated with ψ_j that is $\tilde{\psi}_j\left(y_1, ..., y_j\right) = \frac{1}{j!}\sum_\sigma \psi_j\left(y_{\sigma(1)}, ..., y_{\sigma(j)}\right)$ where the summation is taken over all possible permutations. Next, by the method outlined in [19, Section 7.5], the homogeneous component of order n in the finite expansion of $h = F \circ \nabla\eta$ at a is given by

$$\sum_{j=1}^{n} \sum_{i_1+i_2+...+i_j=n} \tilde{\psi}_j\left(\varphi_{i_1}(x), \varphi_{i_2}(x), ..., \varphi_{i_j}(x)\right)$$

$$= \sum_{j=1}^{n} \sum_{i_1+i_2+...+i_j=n} \frac{1}{j!}\sum_\sigma \psi_j\left(\varphi_{i_{\sigma(1)}}(x), \varphi_{i_{\sigma(2)}}(x), ..., \varphi_{i_{\sigma(j)}}(x)\right)$$
$$= \sum_{j=1}^{n} \frac{1}{(j!)^2} \sum_{i_1+i_2+...+i_j=n} \sum_\sigma F^{(j)}(b)\cdot\left(\varphi_{i_{\sigma(1)}}(x), \varphi_{i_{\sigma(2)}}(x), ..., \varphi_{i_{\sigma(j)}}(x)\right)$$

and it is equal to $\frac{1}{n!}h^{(n)}(a)\cdot(x, ..., x) = \frac{1}{n!}\sum_{|\mu|=n} D^\mu h(a)\, x^\mu$. Due to uniqueness of the coefficient of x^μ for each $\mu \in N_0^d$ such that $\left|\mu\right| = n$ and the explicit form of $F^{(j)}$ as multiplication of matrices, $D^\mu h(a)$ must be a linear combination of terms in the form of

$$F^{(j)}(\nabla\eta(a))\cdot\left(D^{\mu_1}\nabla\eta(a), ..., D^{\mu_j}\nabla\eta(a)\right)$$

where $j = 1, ..., n$ and $\mu_1 + \mu_2 + ... + \mu_j = \mu$.
Let $\gamma \in N_0^d$ such that $1 \le \left|\gamma\right| = k \le l$, then $w^k D^\gamma \nabla\kappa = w^k D^\gamma(h(\kappa))$ is a linear combination of terms in the form of

$$F^{(j)}(\nabla\eta(\kappa))\cdot\left(w^{|\mu_1|}D^{\mu_1}\nabla\eta(\kappa), ..., w^{|\mu_j|}D^{\mu_j}\nabla\eta(\kappa)\right)\prod_{i=1}^{n} w^{|\alpha_i|-1}D^{\alpha_i}\kappa^{a_i}$$

where $\alpha_1 + \ldots + \alpha_n = \gamma$, $|\alpha_i| \ge 1$ for $i = 1, ..., n$, $\left|\mu\right| = n$, $1 \le n \le k \le l$, and $1 \le a_i \le d$ are component indices.
We note that by (i),

$$\left|(\nabla\eta(\kappa))^{-1}\right|_\infty = |\nabla\kappa|_\infty \le 2.$$

If $k = 1$, then by Lemma 6.1,

$$\begin{aligned}|wD^{\gamma}\nabla\kappa|_p &\leq C\left|wD^2\eta\left(\kappa\right)\right|_p|\nabla\kappa|_\infty\\&\leq C\lambda_t.\end{aligned}$$

Now for $2 \leq k \leq l$, we proceed with a strong induction- assuming that the first estimate in (iii) holds up to $1 \leq k-1 \leq l-1$. We estimate each term in the summation denoting

$$\mathcal{A} = F^{(j)}\left(\nabla\eta\left(\kappa\right)\right)\cdot\left(w^{|\mu_1|}D^{\mu_1}\nabla\eta\left(\kappa\right),...,w^{|\mu_j|}D^{\mu_j}\nabla\eta\left(\kappa\right)\right)\prod_{i=1}^{n}w^{|\alpha_i|-1}D^{\alpha_i}\kappa^{a_i}.$$

If $n < l$ then $l - \left|\mu_i\right| \geq l - n > \frac{d}{p}$. By Lemma 6.1 and Sobolev embedding theorem,

$$\left|F^{(j)}\left(\nabla\eta\left(\kappa\right)\right)\cdot\left(w^{|\mu_1|}D^{\mu_1}\nabla\eta\left(\kappa\right),...,w^{|\mu_j|}D^{\mu_j}\nabla\eta\left(\kappa\right)\right)\right|_\infty \leq C\lambda_t^n.$$

Therefore, by Corollary 4.1 with $k > \frac{d}{p}$ and the induction hypothesis,

$$\begin{aligned}|\mathcal{A}|_p &\leq C\lambda_t^n\prod_{i=1,|\alpha_i|\geq 2}^{n}\left|w^{|\alpha_i|-1}D^{\alpha_i}\kappa^{a_i}\right|_{H_p^{k-|\alpha_i|}}\\&\leq C\lambda_t^n\prod_{i=1,|\alpha_i|\geq 2}^{n}|D^{\alpha_i}\kappa^{a_i}|_{H^{k-|\alpha_i|}_{d/p+k-1,p}} \leq C\lambda_t^n\left|D^2\kappa^{a_i}\right|^n_{H^{k-2}_{d/p+k-1,p}}\\&\leq C\lambda_t^k\lambda_t^{kp_{k-1}} = C\lambda_t^{p_k}.\end{aligned}$$

If $n = l$ then $|\alpha_i| = 1$, $i = 1, ..., l$ and thus $\prod_{i=1}^{n}\left|w^{|\alpha_i|-1}D^{\alpha_i}\kappa^{a_i}\right|_\infty \leq C$.
Hence, due to Lemma 6.1 and Corollary 4.1 ,

$$\begin{aligned}|\mathcal{A}|_p \leq& C\left|F^{(j)}\left(\nabla\eta\left(\kappa\right)\right)\cdot\left(w^{|\mu_1|}D^{\mu_1}\nabla\eta\left(\kappa\right),...,w^{|\mu_j|}D^{\mu_j}\nabla\eta\left(\kappa\right)\right)\right|_p\\\leq& C\lambda_t^k\left|\prod_{i=1}^{j}w^{|\mu_i|}D^{\mu_i}\nabla\eta\right|_p \leq C\lambda_t^{p_k}.\end{aligned}$$

Therefore, the first estimate in (iii) holds for k completing the induction. The second estimate follows from multiplying $\mathcal{A}$ by w^{σ}, replacing $w^{|\mu_1|}D^{\mu_1}\nabla\eta\left(\kappa\right)$ with $w^{\sigma+|\mu_1|}D^{\mu_1}\nabla\eta\left(\kappa\right)$, and applying the first estimate. □

Next, we derive a growth estimate for $K_{\eta,h}$ given by (5.5) with a general h in place of h_η.

Lemma 6.3. *Let $p > d$, $l > \frac{d}{p}$, $\theta \in (1, d)$. Then there exist $C, r > 0$ so that for all $\zeta \in \mathcal{A}^{p,l,\theta}_{M,T}$ and $h : \Omega\times[0,T] \to H^l_{\theta+l,p}\left(R^d; R^d\times R^d\right)$,*

$$\left|K_{\eta,h}\left(t\right)\right|_{H^{l+1}_{\theta+l,p}} \leq C\lambda_t^r\left|h\left(t\right)\right|_{H^l_{\theta+l,p}},\ (\omega,t)\in\Omega\times[0,T].$$

Proof. The variable t is omitted. Now by using Lemma 3.1, Corollary 4.2, Lemma 4.2 and Lemma 6.2 (i, ii, iii) in order,

$$\left|K_{\eta,h}\right|_{H^{l+1}_{\theta+l,p}} \leq C\left|\phi_{\eta,h}\right|_{H^l_{\theta+l,p}}$$

$$\begin{aligned} &\leq C\left(1+|\nabla\kappa-I|_{H^l_{\theta+l,p}}\right)^2\left|h\left(\kappa\right)\right|_{H^l_{\theta+l,p}} \\ &\leq C\lambda_t^r\,|h|_{H^l_{\theta+l,p}}\,. \end{aligned}$$

The proof is complete. □

6.2. **Difference Estimates.** In this section, we estimate the difference of spatial inverses. We write for $i=1,2$, $\eta_i(t)=\zeta_i(t)+e+\epsilon B_t$ and denote their spatial inverses by $\kappa_i(t)$. The exponent of λ_t will be generically denoted by r and allowed to grow as needed. Due to its simplicity, Lemma 6.1 will from now be applied without further reference.

Lemma 6.4. *Let $p>d$, $l>\frac{d}{p}$, $\theta\in(1,d)$. Then there exist $C,r>0$ such that for all $\zeta_1,\zeta_2\in\mathcal{A}^{p,l,\theta}_{M,T}$,*

$$\left|\kappa_1(t)-\kappa_2(t)\right|_{H^l_{\theta+l-1,p}}\leq C\lambda_t^r\left|\zeta_1(t)-\zeta_2(t)\right|_{H^l_{\theta+l-1,p}},\ (\omega,t)\in\Omega\times[0,T]\,.$$

Proof. The variable t is omitted. We observe the following identity,

$$\begin{aligned} \kappa_1-\kappa_2 &= (\kappa_1\circ\eta_2-\kappa_1\circ\eta_1)\circ\kappa_2 \\ &= \left[\int_0^1\nabla\kappa_1\left(a\eta_2+(1-a)\,\eta_1\right)(\eta_2-\eta_1)\,da\right]\circ\kappa_2. \end{aligned}$$

By changing the variable of integration and Lemma 6.2 (i),

$$\begin{aligned} &\left|w^{\sigma-1}(\kappa_1-\kappa_2)\right|_p \\ &\leq C\lambda_t^{|\sigma-1|}\,|\nabla\kappa_1|_\infty\left|w^{\sigma-1}(\eta_2-\eta_1)\right|_p\leq C\lambda_t^{|\sigma-1|}\left|w^{\sigma-1}(\eta_2-\eta_1)\right|_p. \end{aligned}$$

Now, we easily check the condition in Corollary 4.3. Owing to Lemma 6.2 (i, iii),

$$\left|\det\nabla\eta_2\right|_\infty+\left\|\eta_2\right\|+\|\kappa_2\|+|\nabla\kappa_2|_\infty+\chi_{l\geq 2}\left|D^2\kappa_2\right|_{H^{l-2}_{\theta+l-1,p}}\leq C\lambda_t^r.$$

Therefore, applying Corollary 4.3 with $\delta=\theta-1$ and $\delta'=\theta$,

$$\begin{aligned} |\kappa_1-\kappa_2|_{H^l_{\theta+l-1,p}} &\leq C\lambda_t^r\int_0^1\left|\nabla\kappa_1\left(a\eta_2+(1-a)\,\eta_1\right)(\eta_2-\eta_1)\right|_{H^l_{\theta+l-1,p}}da. \\ &\leq C\lambda_t^r\int_0^1\left|\left(\nabla\kappa_1\left(a\eta_2+(1-a)\,\eta_1\right)-I\right)(\eta_2-\eta_1)\right|_{H^l_{\theta+l-1,p}}da \\ &\quad+C\lambda_t^r\left|\eta_2-\eta_1\right|_{H^l_{\theta+l-1,p}} \\ &=\mathcal{I}_1+\mathcal{I}_2. \end{aligned}$$

Next, we apply Corollary 4.2 with $l>\frac{d}{p}$, $\delta=\theta-1$, $\delta_1=\theta$, $\delta_2=\theta-1$,

$$\mathcal{I}_1\leq C\lambda_t^r\int_0^1\left|\nabla\kappa_1\left(a\eta_2+(1-a)\,\eta_1\right)-I\right|_{H^l_{\theta+l,p}}\left|\eta_2-\eta_1\right|_{H^l_{\theta+l-1,p}}da.$$

Recalling (6.7), we have $\left\|\left(a\eta_2+(1-a)\,\eta_1\right)^{-1}\right\|\leq C\lambda_t,\ a\in[0,1]$. In addition, from (6.8), we have $\left|\det\nabla\left(a\eta_2+(1-a)\,\eta_1\right)^{-1}\right|_\infty\leq C$. Thus, applying Lemma 4.2 with $\delta=\delta'=\theta$,

$$\mathcal{I}_1\leq C\lambda_t^r\left|\nabla\kappa_1-I\right|_{H^l_{\theta+l,p}}\left|\eta_2-\eta_1\right|_{H^l_{\theta+l-1,p}}.$$

The statement now follows from Lemma 6.2 (ii, iii). □

Remark 6.1. *The estimates* $\left\|\left(a\eta_2+(1-a)\,\eta_1\right)^{-1}\right\|\leq C\lambda_t$ *and* $\left|\det\nabla\left(a\eta_2+(1-a)\,\eta_1\right)^{-1}\right|_\infty\leq C$ *will be needed later.*

We now provide the Lipschitz continuity of $K_{\eta,h}$ with respect to η defined by (5.5) with a general h in place of h_η.

Lemma 6.5. *Let* $p>d$, $l>1+\frac{d}{p}$, $\theta\in(1,d)$. *Then there exist* $C,r>0$ *such that for all* $\zeta_1,\zeta_2\in\mathcal{A}^{p,l,\theta}_{M,T}$ *and* $h:\Omega\times[0,T]\to H^l_{\theta+l,p}\left(R^d;R^d\times R^d\right)$,

$$\begin{aligned}&\left|K_{\eta_1,h}(t)-K_{\eta_2,h}(t)\right|_{H^l_{\theta+l-1,p}}\\&\leq C\lambda_t^r\left|h(t)\right|_{H^l_{\theta+l,p}}\left|\zeta_1(t)-\zeta_2(t)\right|_{H^l_{\theta+l-1,p}},\ (\omega,t)\in\Omega\times[0,T].\end{aligned}$$

Proof. The variable t is omitted. Applying Lemma 3.1,

$$\begin{aligned}\left|K_{\eta_1,h}-K_{\eta_2,h}\right|_{H^l_{\theta+l-1,p}}&\leq C\left|\phi_{\eta_1,h}-\phi_{\eta_2,h}\right|_{H^{l-1}_{\theta+l-1,p}}\\&\leq C\left|(\nabla\kappa_1)^*h(\kappa_1)\left(\nabla\kappa_1-\nabla\kappa_2\right)\right|_{H^{l-1}_{\theta+l-1,p}}\\&+C\left|(\nabla\kappa_1)^*\left(h(\kappa_1)-h(\kappa_2)\right)\nabla\kappa_2\right|_{H^{l-1}_{\theta+l-1,p}}\\&+C\left|\left((\nabla\kappa_1)^*-(\nabla\kappa_2)^*\right)h(\kappa_2)\nabla\kappa_2\right|_{H^{l-1}_{\theta+l-1,p}}\\&=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned}$$

We estimate $\mathcal{I}_1$ and similarly $\mathcal{I}_3$. Applying Corollary 4.2 with $l-1>\frac{d}{p}$, Lemma 4.2 with $l-1\geq 0$ and $\delta=\delta'=\theta$, and finally Lemma 6.2 (i, ii, iii), we derive

$$\begin{aligned}\mathcal{I}_1&\leq C\left(\left|(\nabla\kappa_1)^*-I\right|_{H^{l-1}_{\theta+l-1,p}}+1\right)\left|h(\kappa_1)\right|_{H^{l-1}_{\theta+l-1,p}}\left|\nabla\kappa_1-\nabla\kappa_2\right|_{H^{l-1}_{\theta+l-1,p}}\\&\leq C\lambda_t^r\left|h\right|_{H^{l-1}_{\theta+l-1,p}}\left|\nabla\kappa_1-\nabla\kappa_2\right|_{H^{l-1}_{\theta+l-1,p}}.\end{aligned}$$

We now proceed to estimate $\mathcal{I}_2$. Applying Corollary 4.2 with $l-1>\frac{d}{p}$ followed by Lemma 6.2 (ii, iii),

$$\begin{aligned}\mathcal{I}_2&\leq C\left(1+\left|I-(\nabla\kappa_1)^*\right|_{H^{l-1}_{\theta+l-1,p}}\right)\left(1+\left|I-\nabla\kappa_2\right|_{H^{l-1}_{\theta+l-1,p}}\right)\\&\times\int_0^1\left|\nabla h\left(a\kappa_1+(1-a)\,\kappa_2\right)(\kappa_1-\kappa_2)\right|_{H^{l-1}_{\theta+l-1,p}}da\\&\leq C\lambda_t^r\int_0^1\left|\nabla h\left(a\kappa_1+(1-a)\,\kappa_2\right)(\kappa_1-\kappa_2)\right|_{H^{l-1}_{\theta+l-1,p}}da.\end{aligned}$$

Applying Corollary 4.2 with $l-1>\frac{d}{p}$, $\delta=\theta$, $\delta_1=\theta+1$, $\delta_2=\theta-1$,

$$\mathcal{I}_2 \le C\lambda_t^r \int_0^1 \left|\nabla h\left(a\kappa_1+(1-a)\,\kappa_2\right)\right|_{H^{l-1}_{\theta+l,p}} |\kappa_1-\kappa_2|_{H^{l-1}_{\theta+l-2,p}}.$$

We observe that (6.2) ensures that $a\nabla\kappa_1+(1-a)\nabla\kappa_2$ is close to the identity matrix. Indeed,

$$\left|a\nabla\kappa_1+(1-a)\nabla\kappa_2-I\right|_\infty \le a\,|\nabla\kappa_1-I|_\infty+(1-a)\,|\nabla\kappa_2-I|_\infty \le \frac{1}{2d}.$$

Due to Ostrowski's lower bound for determinants, $\det\nabla\left(a\kappa_1+(1-a)\,\kappa_2\right)\ge c>0$. By the inverse function theorem, $a\kappa_1+(1-a)\kappa_2$ has the spatial inverse denoted by $(a\kappa_1+(1-a)\kappa_2)^{-1}$. By the same calculation as (6.4) and Lemma 6.2 (i),

$$\begin{aligned}&\left\|\left(a\kappa_1+(1-a)\,\kappa_2\right)^{-1}\right\|\\ &\le \left(1+\left(a\kappa_1(0)+(1-a)\,\kappa_2(0)\right)\right)\left|\nabla\left(a\kappa_1+(1-a)\,\kappa_2\right)^{-1}\right|_\infty\\ &\le C\lambda_t,\ a\in[0,1].\end{aligned}$$

Finally, applying Lemma 4.2 with $l-1\ge 0$, $\delta=\theta+1$, $\delta'=\theta$, and Lemma 6.2 (i, iii)

$$\mathcal{I}_2 \le C\lambda_t^r\,|\nabla h|_{H^{l-1}_{\theta+l,p}}\,|\kappa_1-\kappa_2|_{H^{l-1}_{\theta+l-2,p}}.$$

The proof is completed by Lemma 6.4. □

The following Lipschitz continuity will also be needed.

Lemma 6.6. *Let $p>d$, $l>1+\frac{d}{p}$, $\theta\in(1,d)$, $N>0$ and $\sup_{(\omega,t)\in\Omega\times[0,T]}\left|G(t)\right|_{H^{l+1}_{\theta+l}}\le N$. Denote for $i=1,2$,*

$$h_{\eta_i}(t)=\int_0^t\left(\nabla\eta_i(s)\right)^*\nabla G\left(s,\eta_i(s)\right)\left(\nabla\eta_i(s)\right)ds,\ (\omega,t)\in\Omega\times[0,T].$$

Then there exist $C,r>0$ such that for all $\zeta_1,\zeta_2\in\mathcal{A}^{p,l,\theta}_{M,T}$,

$$\left|h_{\eta_1}(t)-h_{\eta_2}(t)\right|_{H^{l-1}_{\theta+l-1}}\le Ct\lambda_t^r\,|\zeta_1-\zeta_2|_{\Sigma^l_{\theta+l-1,p}(t)},\ (\omega,t)\in\Omega\times[0,T].$$

Proof. We split the difference as follows:

$$\begin{aligned}&\left|h_{\eta_1}(t)-h_{\eta_2}(t)\right|_{H^{l-1}_{\theta+l-1}}\\ &\le\left|\int_0^t\left(\nabla\eta_1(s)-\nabla\eta_2(s)\right)^*\nabla G\left(s,\eta_1(s)\right)\left(\nabla\eta_1(s)\right)ds\right|_{H^{l-1}_{\theta+l-1}}\\ &+\left|\int_0^t\left(\nabla\eta_2(s)\right)^*\left(\nabla G\left(s,\eta_1(s)\right)-\nabla G\left(s,\eta_2(s)\right)\right)\left(\nabla\eta_1(s)\right)ds\right|_{H^{l-1}_{\theta+l-1}}\\ &+\left|\int_0^t\left(\nabla\eta_2(s)\right)^*\nabla G\left(s,\eta_2(s)\right)\left(\nabla\eta_1(s)-\nabla\eta_2(s)\right)ds\right|_{H^{l-1}_{\theta+l-1}}\\ &=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned}$$

We start with $\mathcal{I}_2$. Applying Corollary 4.2 with $l-1 > \frac{d}{p}$,

$$\mathcal{I}_2 \leq C \prod_{i=1,2} \left(1 + \left|I - \nabla \eta_i\right|_{\Sigma^{l-1}_{\theta+l-1,p}(t)}\right) \left|\int_0^t \left(\nabla G\left(s, \eta_1\left(s\right)\right) - \nabla G\left(s, \eta_2\left(s\right)\right)\right) ds\right|_{H^{l-1}_{\theta+l-1,p}}$$
$$\leq C \left|\int_0^t \left(\nabla G\left(s, \eta_1\left(s\right)\right) - \nabla G\left(s, \eta_2\left(s\right)\right)\right) ds\right|_{H^{l-1}_{\theta+l-1,p}}.$$

By the fundamental theorem of calculus followed by Corollary 4.2 with $l-1 > \frac{d}{p}$, $\delta = \theta$, $\delta_1 = \theta + 1$, $\delta_2 = \theta - 1$,

$$\mathcal{I}_2 \leq C \int_0^t \left|D^2 G\left(s, a\eta_1\left(s\right) + \left(1-a\right)\eta_2\right)\right|_{H^{l-1}_{\theta+l,p}} ds \left|\zeta_1 - \zeta_2\right|_{\Sigma^{l-1}_{\theta+l-2,p}(t)}.$$

Finally, using Lemma 4.2 with $l-1 \geq 0$, $\delta = \theta+1$, $\delta' = \theta$, and Remark 6.1,

$$\mathcal{I}_2 \leq Ct\lambda_t^r \left|D^2 G\right|_{\Sigma^{l-1}_{\theta+l,p}(t)} \left|\zeta_1 - \zeta_2\right|_{\Sigma^{l-1}_{\theta+l-2,p}(t)}.$$

Now, we now estimate $\mathcal{I}_1$ and similarly $\mathcal{I}_3$. Applying Corollary 4.2 with $l-1 > \frac{d}{p}$,

$$\mathcal{I}_1 \leq C \left|\nabla\zeta_1 - \nabla\zeta_2\right|_{\Sigma^{l-1}_{\theta+l-1,p}(t)} \int_0^t \left|\nabla G\left(s, \eta_1\left(s\right)\right)\right|_{H^{l-1}_{\theta+l-1,p}} ds.$$

Finally, using Lemma 4.2 with $l-1 \geq 0$, $\delta = \theta$, $\delta' = \theta$,

$$\mathcal{I}_1 \leq Ct\lambda_t^r \left|\nabla\zeta_1 - \nabla\zeta_2\right|_{\Sigma^{l-1}_{\theta+l-1,p}(t)} \left|\nabla G\right|_{\Sigma^{l-1}_{\theta+l-1,p}(t)}.$$

The proof is complete. □

7. Proof of the Main Theorem

In this section, we construct a solution of the flow equation via iteration. Specifically, we will show a contraction in an appropriate function space of the mapping $\eta \to K_{\eta,h_\eta}$ given by (5.5).

7.1. **Proof of Theorem 5.1.** We now prove Theorem 5.1.

Proof. We will show the existence for all $\omega \in \Omega$ and all estimates will be independent of ω. Fixing $0 < T < \infty$, we consider the mapping $\mathcal{L} : \zeta \to \int_0^t E_s^W K_{\eta,h_\eta}\left(s, \eta\left(s\right)\right) ds = \tilde{\zeta}\left(t\right)$, $t \in [0,T]$ where

$$K^j_{\eta,h_\eta}\left(t,x\right) = \int \Gamma_i\left(x-z\right)\left[\phi^{ji}_{\eta,h_\eta}\left(t,z\right) - \phi^{ij}_{\eta,h_\eta}\left(t,z\right)\right] dz,$$
$$\phi_{\eta,h}\left(t,x\right) = \left(\nabla\kappa\left(t,x\right)\right)^* h\left(t,\kappa\left(t,x\right)\right) \nabla\kappa\left(t,x\right),$$

and

$$h_\eta\left(t,x\right) = \nabla u_0\left(x\right) + \int_0^t \left(\nabla\eta\left(s,x\right)\right)^* \nabla G\left(s,\eta\left(s,x\right)\right)\left(\nabla\eta\left(s,x\right)\right) ds,$$

for all $(\omega, t, x) \in \Omega \times [0,T] \times R^d$.

Recall that $\zeta \in \mathcal{A}^{p,l,\theta}_{M,T}$ if for all $(\omega, t) \in \Omega \times [0,T]$, $\zeta\left(t\right)$ is a continuously differentiable and
(i) $\left|\nabla\zeta\left(t\right)\right|_\infty \leq \frac{1}{2d^2}$, $\left|\zeta\left(t,0\right)\right| \leq M$,
(ii) $\left|\zeta\left(t\right)\right|_{H^{l+1}_{\theta+l,p}} \leq M$.

We mention that by Lemma 6.2 (i), $\left\|\kappa(s)\right\| \leq C\lambda_s$ which will be used several times. Applying Corollary 4.2 and Lemma 4.2 with $\delta = \delta' = \theta$,

$$\left|h_\eta(s)\right|_{H^l_{\theta+l,p}} \leq |\nabla u_0|_{H^l_{\theta+l,p}} + Cs\lambda_s^r |\nabla G|_{\Sigma^l_{\theta+l,p}(s)} \leq C\lambda_s^r. \tag{7.1}$$

Therefore, by Lemma 6.3,

$$\left|K_{\eta,h_\eta}(s)\right|_{H^{l+1}_{\theta+l,p}} \leq C\lambda_s^r \left|h_\eta(s)\right|_{H^l_{\theta+l,p}} \leq C\lambda_s^r, \tag{7.2}$$

and by Corollary 4.3 with $l+1 \geq 0$, $\delta = \theta - 1$, $\delta' = \theta$,

$$E_s^W \left|K_{\eta,h_\eta}(s,\eta(s))\right|_{H^{l+1}_{\theta+l,p}} \leq CE_s^W \lambda_s^r \leq C. \tag{7.3}$$

Hence,

$$\left|\tilde{\zeta}(t)\right|_{H^{l+1}_{\theta+l,p}} \leq Ct.$$

By Sobolev embedding theorem, we have $\left|\tilde{\zeta}(t,0)\right| \leq Ct$, $\left|\nabla\tilde{\zeta}(t)\right|_\infty \leq Ct$ and $\tilde{\zeta}$ is continuously differentiable. We emphasize that C depends on M and T but is independent of ζ. Therefore, fixing $M > 0$ and making T smaller i.e., $CT \leq M$, then $\mathcal{L} : \zeta \to \int_0^t E_s^W K_{\eta,h_\eta}(s,\eta(s))\,ds = \tilde{\zeta}(t)$ maps $\mathcal{A}^{p,l,\theta}_{M,T,l}$ into $\mathcal{A}^{p,l,\theta}_{M,T,l}$.

Next, we show the contraction in the weaker $\sum^l_{\theta+l-1,p}(T)$ norm. We assume that $\zeta, \tilde{\zeta} \in \mathcal{A}^{p,l,\theta}_{M,T}$ are $\mathbb{F}^W$−adapted.

$$\begin{aligned}
&\int_0^t \left|E_s^W K_{\eta,h_\eta}(s,\eta(s)) - E_s^W K_{\tilde{\eta},h_{\tilde{\eta}}}(s,\tilde{\eta}(s))\right|_{H^l_{\theta+l-1,p}} ds \\
\leq &\int_0^t E_s^W \left|K_{\eta,h_\eta}(s,\eta(s)) - K_{\eta,h_\eta}(s,\tilde{\eta}(s))\right|_{H^l_{\theta+l-1,p}} ds \\
+ &\int_0^t E_s^W \left|K_{\eta,h_\eta}(s,\tilde{\eta}(s)) - K_{\tilde{\eta},h_{\tilde{\eta}}}(s,\tilde{\eta}(s))\right|_{H^l_{\theta+l-1,p}} ds \\
= &\int_0^t \mathcal{I}_1(s)\,ds + \int_0^t \mathcal{I}_2(s)\,ds.
\end{aligned}$$

Estimate of $\mathcal{I}_1$: Applying Corollary 4.2 with $\delta = \theta - 1$, $\delta_1 = \theta$, $\delta_2 = \theta - 1$,

$$\begin{aligned}
\mathcal{I}_1(s) = &E_s^W \left|K_{\eta,h_\eta}(s,\eta(s)) - K_{\eta,h_\eta}(s,\tilde{\eta}(s))\right|_{H^l_{\theta+l-1,p}} \\
\leq &E_s^W \left|\int_0^1 \nabla K_{\eta,h_\eta}(s, a\eta(s) + (1-a)\tilde{\eta}(s))(\eta(s) - \tilde{\eta}(s))\,da\right|_{H^l_{\theta+l-1,p}} \\
\leq &C\int_0^1 E_s^W \left|\nabla K_{\eta,h_\eta}(s, a\eta(s) + (1-a)\tilde{\eta}(s))\right|_{H^l_{\theta+l,p}} da \left|\eta(s) - \tilde{\eta}(s)\right|_{H^l_{\theta+l-1,p}}.
\end{aligned}$$

By Lemma 4.2 with $\delta = \delta' = \theta$, (7.2), and Remark 6.1,

$$\mathcal{I}_1(s) \leq C\left|\zeta(s) - \tilde{\zeta}(s)\right|_{H^l_{\theta+l-1,p}}.$$

Estimate of $\mathcal{I}_2$: By Corollary 4.3 with $\delta = \theta - 1$, $\delta' = \theta$,

$$\begin{aligned}\mathcal{I}_2(s) &= E_s^W \left|\left(K_{\eta,h_\eta}(s) - K_{\tilde{\eta},h_{\tilde{\eta}}}(s)\right)\circ \tilde{\eta}(s)\right|_{H^l_{\theta+l-1,p}} \\ &\le E_s^W \left|K_{\eta,h_\eta - h_{\tilde{\eta}}}(s)\circ \tilde{\eta}(s)\right|_{H^l_{\theta+l-1,p}} + E_s^W \left|\left(K_{\eta,h_{\tilde{\eta}}}(s) - K_{\tilde{\eta},h_{\tilde{\eta}}}(s)\right)\circ \tilde{\eta}(s)\right|_{H^l_{\theta+l-1,p}} \\ &\le CE_s^W\left[\lambda_s^r \left|K_{\eta,h_\eta - h_{\tilde{\eta}}}(s)\right|_{H^l_{\theta+l-1,p}}\right] + CE_s^W\left[\lambda_s^r\left|K_{\eta,h_{\tilde{\eta}}}(s) - K_{\tilde{\eta},h_{\tilde{\eta}}}(s)\right|_{H^l_{\theta+l-1,p}}\right] \\ &= \mathcal{I}_{21}(s) + \mathcal{I}_{22}(s).\end{aligned}$$

Applying Lemma 6.3 with $l-1 > \frac{d}{p}$,

$$\mathcal{I}_{21}(s) \le CE_s^W\left[\lambda_s^r \left|h_\eta(s) - h_{\tilde{\eta}}(s)\right|_{H^{l-1}_{\theta+l-1,p}}\right].$$

Owing to Lemma 6.6, $\mathcal{I}_{21}(s) \le C\left|\zeta - \tilde{\zeta}\right|_{\Sigma^l_{\theta+l-1,p}(s)}$.

By Lemma 6.5 and (7.1),

$$\mathcal{I}_{22}(s) \le C\left|\zeta(s) - \tilde{\zeta}(s)\right|_{H^l_{\theta+l-1,p}}.$$

Combining estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$, we derive

$$\left|\mathcal{L}(\zeta) - \mathcal{L}\left(\tilde{\zeta}\right)\right|_{\Sigma^l_{\theta+l-1,p}(T)} \le C\left|\zeta - \tilde{\zeta}\right|_{\Sigma^l_{\theta+l-1,p}(T)}. \tag{7.4}$$

By a standard successive iteration starting with $\zeta^{(0)}(t) = 0$,

$$\zeta^{(n+1)}(t) = \int_0^t E_s^W K_{\eta^{(n)},h_{\eta^{(n)}}}\left(s, \eta^{(n)}(s)\right)ds,\ t\in[0,T],$$

there exists $\zeta^* \in \sum^l_{\theta+l-1,p}(T)$ such that

$$\zeta^*(t) = \int_0^t E_s^W K_{\eta^*,h_{\eta^*}}(s,\eta^*(s))\,ds$$

as an equality in $H^l_{\theta+l-1,p}\left(R^d;R^d\right)$. Owing to Lemma 8.1, $\zeta^* \in \sum^{l+1}_{\theta+l,p}(T)$. Finally, we show that $\zeta \in C\left([0,T], H^{l+1}_{\theta+l,p}\left(R^d;R^d\right)\right)$. Clearly, by passing to the limit, $\zeta^* \in \mathcal{A}^{p,l,\theta}_{M,T}$ possibly with a larger M. Hence, by (7.3),

$$\begin{aligned}\left|\zeta^*(t) - \zeta^*(t')\right|_{H^{l+1}_{\theta+l,p}} &\le \int_{t'}^t E_s^W\left|K_{\eta^*,h_{\eta^*}}(s,\eta^*(s))\right|_{H^{l+1}_{\theta+l,p}} ds \\ &\le C(t-t').\end{aligned}$$

The proof is complete. □

7.2. **Proof of Theorem 5.2.** We now prove Theorem 5.2.

Proof. We start by checking the assumptions of Lemma 5.2.

(i) By applying Corollary 4.2 and Lemma 4.2,

$$\left(\nabla\kappa(t)\right)^* g_\eta(t,\kappa(t)) \in H^l_{\theta+l,p}\left(R^d;R^d\right),\ (\omega,t)\in\Omega\times[0,T].$$

which also justifies condition (i) of Lemma 5.1.

(ii) follows from differentiability of $\zeta(t)$ and Lemma 6.2.
(iii) follows from continuity of $\left|\zeta(t)\right|_{H^{l+1}_{\theta+l,p}}$ and Sobolev embedding theorem.
Hence, by (5.4) in Lemma 5.2,

$$\mathcal{S}(\nabla\kappa(t))^{*} g_{\eta}(t,\kappa(t)) = K_{\eta,h_{\eta}}(t),\ (\omega,t)\in\Omega\times[0,T].$$

We let

$$u(t) = E^{W}_{t}\mathcal{S}(\nabla\kappa(t))^{*} g_{\eta}(t,\kappa(t)) = E^{W}_{t}K_{\eta,h_{\eta}}(t),\ (\omega,t)\in\Omega\times[0,T].$$

By (7.2),

$$\sup_{(\omega,t)\in\Omega\times[0,T]}\left|u(t)\right|_{H^{l+1}_{\theta+l,p}} \leq \sup_{(\omega,t)\in\Omega\times[0,T]} E^{W}_{t}\left|K_{\eta,h_{\eta}}(t)\right|_{H^{l+1}_{\theta+l,p}} \leq N < \infty.$$

Because $l+1 > 3+\frac{d}{p}$, we have by Sobolev embedding theorem that

$$\sup_{(\omega,t)\in\Omega\times[0,T]}\left|u(t)\right|_{C^{3+\alpha}} \leq N < \infty$$

for some $\alpha\in(0,1]$. Furthermore, the above inequality also justifies condition (ii) of Lemma 5.1. Therefore, applying Lemma 5.1, $u(t)$ solves (5.2) as an equality in $H^{l-2}_{\theta+l-2,p}\left(R^d;R^d\right)$.
By Corollary 4.2 with $\delta=\theta$, $\delta_1=\theta$, $\delta_2=\theta-1$,

$$(\nabla u(t))^{*}u(t) = \frac{1}{2}\nabla\left|u(t)\right|^{2}\in H^{l}_{\theta+l,p}\left(R^d;R^d\right).$$

Thus, $(\nabla u(t))^{*}u(t)$ disappears under the Solenoidal projection in $H^{l-2}_{\theta+l-2,p}\left(R^d;R^d\right)$. Therefore, $u(t)$ solves (5.9) as an equality in $H^{l-2}_{\theta+l-2,p}\left(R^d;R^d\right)$. □

8. Appendix

8.1. **Weaker Norm.** We mention a result on convergence of functions in $H^{l}_{\delta,p}\left(R^d;R^d\right)$ which is needed for constructing a solution.

Lemma 8.1. *Let $p>1$, $l\geq 0$, $\delta\geq 0$. If $\zeta_n\to\zeta$ in $H^{l}_{\delta,p}\left(R^d;R^d\right)$ and $|\zeta_n|_{H^{l+1}_{\delta+1,p}}\leq M<\infty$ for all $n\geq 1$ then $\zeta\in H^{l+1}_{\delta+1,p}\left(R^d;R^d\right)$.*

Proof. Without loss of generality, we assume that $\zeta_n,\zeta\in H^{l}_{\delta,p}\left(R^d;R\right)$ instead of $H^{l}_{\delta,p}\left(R^d;R^d\right)$. The idea is to show that $w^{\sigma}\partial^{l+1}\zeta\in L_p\left(R^d\right)$ where $\sigma=\delta+1-\frac{d}{p}$. Let $\gamma\in N^{d}_{0}$ with $\left|\gamma\right|=l$ and $1\leq i\leq d$.
By a standard argument, we take $\varphi\in C^{\infty}_{0}\left(R^d\right)$ then on some subsequence of $\{\zeta_n\}$ there exist $f,g\in L_p\left(R^d\right)$ such that

$$\int\left(w^{\sigma}\partial_i\partial^{\gamma}\zeta_n\right)(x)\,\varphi(x)\,dx\to\int f(x)\,\varphi(x)\,dx$$
$$\int\left(w^{\sigma-1}\partial_i w\partial^{\gamma}\zeta_n\right)(x)\,\varphi(x)\,dx\to\int g(x)\,\varphi(x)\,dx.$$

On the other hand, using integration by parts,

$$-\int\left(w^{\sigma}\partial_{i}\partial^{\gamma}\zeta_{n}\right)(x)\,\varphi(x)+\left(\sigma w^{\sigma-1}\partial_{i}w\partial^{\gamma}\zeta_{n}\right)(x)\,\varphi(x)\,dx$$
$$=\int\left(w^{\sigma}\partial^{\gamma}\zeta_{n}\right)(x)\,\partial_{i}\varphi(x)\,dx.$$

Taking limit $n\to\infty$,

$$-\int f(x)\,\varphi(x)\,dx-\sigma\int g(x)\,\varphi(x)\,dx=\int\left(w^{\sigma}\partial^{\gamma}\zeta\right)(x)\,\partial_{i}\varphi(x)\,dx.$$

Therefore, $w^{\sigma}\partial^{\gamma}\zeta$ has a weak derivative, namely $f+\sigma g\in L_{p}\left(R^{d}\right)$. Now because $w^{\sigma-1}\partial^{\gamma}\zeta\in L_{p}\left(R^{d}\right)$, we conclude that $w^{\sigma}\partial_{i}\partial^{\gamma}\zeta\in L_{p}\left(R^{d}\right)$. Since γ,i are arbitrary, the conclusion follows. □

8.2. **Weight Function.** Let $B_{r}(x)=\left\{y\in R^{d}:\left|x-y\right|<r\right\},x\in R^{d},B_{r}=B_{r}(0),r>0$. Let $p>1,\frac{1}{p}+\frac{1}{q}=1$. We say that a non-negative function w on R^{d} is of class A_{p} if

$$\left(\frac{1}{|B_{R}|}\int_{B_{R}}w(x+y)\,dy\right)\left(\frac{1}{|B_{R}|}\int_{B_{R}}w(x+y)^{-q/p}\,dy\right)^{p/q}\leq C,\ x\in R^{d},\ R>0.$$

We show an important property of the weight function $w(x)=\left(1+|x|^{2}\right)^{1/2},x\in R^{d}$.

Lemma 8.2. *Let $p>1$, $d\geq 1$, $\alpha\in(-d,d(p-1))$, then $w^{\alpha}\in A_{p}$.*

Remark 8.1. *$d-\frac{\alpha q}{p}>0$ is equivalent to $\alpha<d(p-1)$. Also, for $\theta\in(1,d)$, $\alpha=\theta-\frac{d}{p}\in(-d,d(p-1))$.*

Proof. For each $\alpha\in R$,

$$\frac{1}{|B_{R}|}\int_{B_{R}}w(y)^{\alpha}\,dy\leq CR^{-d}\int_{0}^{R}(1+r)^{\alpha}\,r^{d}\frac{dr}{r}\leq C\text{ if }R\in(0,1].$$

If $R>1$ then

$$\begin{aligned}\int_{0}^{R}(1+r)^{\alpha}\,r^{d}\frac{dr}{r}&=\int_{0}^{1}(1+r)^{\alpha}\,r^{d}\frac{dr}{r}+\int_{1}^{R}(1+r)^{\alpha}\,r^{d}\frac{dr}{r}\\&\leq CR^{d+\alpha}\text{ if }d+\alpha>0.\end{aligned}$$

Therefore, for $R>1$, noting $d+\alpha>0$,

$$\frac{1}{|B_{R}|}\int_{B_{R}}w(y)^{\alpha}\,dy\leq CR^{\alpha}. \tag{8.1}$$

Hence, for $R>1$, noting $d+\alpha>0$ and $d-\frac{\alpha q}{p}>0$,

$$\left(\frac{1}{|B_{R}|}\int_{B_{R}}w(y)^{\alpha}\,dy\right)\left(\frac{1}{|B_{R}|}\int_{B_{R}}w(y)^{-\alpha q/p}\,dy\right)^{p/q}\leq C. \tag{8.2}$$

Consider now

$$\frac{1}{\left|B_{R}(x)\right|}\int_{B_{R}(x)}w(y)^{\alpha}\,dy=\frac{1}{|B_{R}|}\int_{B_{R}}w(x+y)^{\alpha}\,dy,\ x\in R^{d}.$$

Since

$$\frac{1}{2}w(x)\,w(y)^{-1}\leq w(x+y)\leq 2w(x)\,w(y)\,,x,y\in R^{d},$$

it follows that

$$\frac{1}{|B_R|}\int_{B_R} w(x+y)^\alpha\, dy \le 2^\alpha w(x)^\alpha \frac{1}{|B_R|}\int_{B_R} w(y)^\alpha\, dy \text{ if } \alpha \ge 0,$$

and

$$\frac{1}{|B_R|}\int_{B_R} w(x+y)^\alpha\, dy \le 2^{-\alpha} w(x)^\alpha \frac{1}{|B_R|}\int_{B_R} w(y)^{-\alpha}\, dy \text{ if } \alpha < 0.$$

Hence, for $R \in (0,1]$,

$$\begin{aligned}
&\left(\frac{1}{|B_R|}\int_{B_R} w(x+y)^\alpha\, dy\right)\left(\frac{1}{|B_R|}\int_{B_R} w(x+y)^{-\alpha q/p}\, dy\right)^{p/q} \\
\le\ & C\left(\frac{1}{|B_R|}\int_{B_R} w(y)^{|\alpha|}\, dy\right)\left(\frac{1}{|B_R|}\int_{B_R} w(y)^{|\alpha| q/p}\, dy\right)^{p/q} \le C. \qquad (8.3)
\end{aligned}$$

Let $R > 1, |x| > 2R$. Then with $\left|y\right| \le R$ we have $2|x| \ge \left|y+x\right| \ge \frac{1}{2}|x|$. Hence for each $\alpha \in R$, there is $C > 0$ so that

$$\frac{1}{|B_R|}\int_{B_R} w(x+y)^\alpha\, dy \le C w(x)^\alpha.$$

The conclusion follows from (8.2) and (8.3).
Let $R > 1, |x| \le 2R$. By (8.1),

$$\begin{aligned}
\frac{1}{|B_R|}\int_{B_R} w(x+y)^\alpha\, dy &\le C\frac{1}{|B_{3R}|}\int_{|x+y|\le 3R} w(x+y)^\alpha\, dy \\
&\le CR^\alpha.
\end{aligned}$$

Hence,

$$\left(\frac{1}{|B_R|}\int_{B_R} w(x+y)^\alpha\, dy\right)\left(\frac{1}{|B_R|}\int_{B_R} w(x+y)^{-\alpha q/p}\, dy\right)^{\frac{p}{q}} \le C,\ x \in R^d,\ R > 0.$$

□

8.3. **Newton Potential Estimates.** We investigate some fundamental properties of Newton potential in weighted Sobolev spaces.

Lemma 8.3. *Let $p > 1, \Delta u = 0$ and $u \in H^0_{\delta,p}\left(R^d\right)$ with $\delta \ge 0$. Then $u = 0$.*

Proof. Let $u_1, u_2 \in H^0_{\delta,p}\left(R^d\right)$ be the solutions of $\Delta u = 0$. Let $v = u_1 - u_2$ and

$$v_\varepsilon = v * \varphi_\varepsilon$$

with

$$\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon), \quad x \in R^d,$$

and $\varphi \in C_0^\infty\left(R^d\right), \varphi \ge 0, \int \varphi = 1$.
As a bounded harmonic function $v_\epsilon = c$. By definition, $cw^{\delta - d/p} \in L_p\left(R^d\right)$ and thus $c = 0$. □

For $f \in C_0^\infty\left(R^d\right)$ we denote $N(f)$ the Newton potential of f :

$$N(f) = \int \Gamma(\cdot - y) f(y)\, dy.$$

Lemma 8.4. *Let $p > 1, d \geq 2, \theta \in (1, d)$. Then there exists $C > 0$ such that for all $f \in H^0_{\theta,p}\left(R^d\right)$ and $u = \nabla N(f)$, i.e., $u(x) = \int \nabla\Gamma(x-y) f(y)\,dy$, $x \in R^d$,*

$$|u|_{H^1_{\theta,p}(R^d)} \leq C\left|f\right|_{H^0_{\theta,p}}.$$

Proof. Let $f \in C_0^\infty\left(R^d\right), \psi = N(f)$ and

$$u(x) = \nabla\psi(x) = \int \nabla\Gamma(x-y) f(y)\,dy, \quad x \in R^d.$$

By the estimate of [20, Theorem 9.9], for each $p > 1$ there exists $C > 0$ such that

$$|\nabla u|_p \leq C\left|f\right|_p.$$

In particular, $|\nabla u|_2 \leq C\left|f\right|_2$. Also, it is straightforward to verify that $\left|D^\alpha D^2\Gamma(x)\right| \leq |x|^{-d-\alpha}$ for all $|x| \neq 0$ and $|\alpha| \leq 1$. Hence, by Lemma 8.2, [21, Theorem 2, Chapter V.4] and [21, Theorem 1, Chapter V.3], for each $\theta \in (1, d)$ and $p > 1$ there is $C > 0$ so that

$$\left|w^{\theta-d/p}\nabla u\right|_p \leq C\left|w^{\theta-d/p} f\right|_p.$$

We will now apply generalized Hardy-Littlewood inequality (see e.g. [13, Theorem 1.3.5]) to show that

$$\left|w^{\theta-d/p-1} u\right|_p \leq C\left|w^{\theta-d/p} f\right|_p. \tag{8.4}$$

Consider,

$$\begin{aligned}\left|w^{\theta-d/p-1}(x)\, u(x)\right| &= C\left|\int \frac{w(x)^{\theta-d/p-1}}{w(y)^{\theta-d/p}} \frac{x_i - y_i}{\left|x-y\right|^d} w(y)^{\theta-d/p} f(y)\,dy\right| \\ &\leq C\int \frac{w(x)^{\theta-d/p-1}}{w(y)^{\theta-d/p}} \left|x-y\right|^{1-d} w(y)^{\theta-d/p}\left|f(y)\right| dy.\end{aligned}$$

Case i. $\theta - \frac{d}{p} - 1 \geq 0$,

$$\frac{w(x)^{\theta-d/p-1}}{w(y)^{\theta-d/p}} \leq C\left(|x|^{\theta-d/p-1}\left|y\right|^{-(\theta-d/p)} + \left|y\right|^{-1}\right).$$

Case ii. $\theta - \frac{d}{p} - 1 < 0$,

$$\frac{w(x)^{\theta-d/p-1}}{w(y)^{\theta-d/p}} \leq C\left(|x|^{\theta-d/p-1}\left|y\right|^{-(\theta-d/p)}\right).$$

Because $\theta \in (1, d)$, the condition of [13, Theorem 1.3.5] is now easily verified and (8.4) is proved. Therefore,

$$|u|_{H^1_{\theta,p}} \leq C\left|f\right|_{H^0_{\theta,p}}.$$

The estimate for general $f \in H^0_{\theta,p}\left(R^d\right)$ follows by passing to the limit. □

We will need higher order estimates of the Newton potential.

Lemma 8.5. *Let $p > 1$, $l \geq 0$, $d \geq 2$, $\theta \in (1, d)$. Then there exists $C > 0$ so that for all $f \in H^l_{\theta+l,p}\left(R^d\right)$,*

$$\left|\nabla N(f)\right|_{H^{l+1}_{\theta+l,p}} \leq C\left|f\right|_{H^l_{\theta+l,p}}. \tag{8.5}$$

Proof. We prove the claim by induction. The case of $l = 0$ follows from Lemma 8.4. Let $\psi = N(f), f \in C_0^\infty\left(R^d\right)$. Assume that we proved (8.5) for $0 \le l \le n$ and for each $p > 1$. Consider for a multi-index $\beta \in N_0^d$ with $\left|\beta\right| = n+1$,

$$\psi_\beta = D^\beta \psi = N\left(f_\beta\right),$$

where $f_\beta = D^\beta f$. Then

$$\Delta\psi_\beta = f_\beta, \quad \Delta\left(\nabla\psi_\beta\right) = \nabla f_\beta \text{ in } R^d,$$

and $v = w^{n+1}\psi_\beta$ solves

$$\Delta v = F, \quad \Delta\left(\nabla v\right) = \nabla F,$$

where $F = w^{n+1} f_\beta + \Delta\left(w^{n+1}\right)\psi_\beta + 2\nabla\left(w^{n+1}\right)\cdot\nabla\psi_\beta$.
By (8.5) for $l = n$,

$$\left|w^{\theta-d/p+n-1}\psi_\beta\right|_p + \left|w^{\theta-d/p+n}\nabla\psi_\beta\right|_p \le \left|f\right|_{H^n_{\theta+n,p}}. \tag{8.6}$$

Therefore,

$$\begin{aligned} \left|w^{\theta-d/p}F\right|_p &\le C\left[\left|w^{\theta-d/p+n+1}f_\beta\right|_p + \left|w^{\theta-d/p+n-1}\psi_\beta\right|_p + \left|w^{\theta-d/p+n}\nabla\psi_\beta\right|_p\right] \\ &\le C\left|f\right|_{H^{n+1}_{\theta+n+1,p}}. \end{aligned} \tag{8.7}$$

Now,

$$\begin{aligned} w^{\theta-d/p-1}\nabla v &= w^{\theta-d/p-1}\left[\nabla\left(w^{n+1}\right)\psi_\beta + w^{n+1}\nabla\psi_\beta\right] \\ &= (n+1)\,w^{\theta-d/p+n-1}\nabla w\psi_\beta + w^{\theta-d/p+n}\nabla\psi_\beta, \end{aligned}$$

and by (8.6),

$$\left|w^{\theta-d/p-1}\nabla v\right|_p \le C\left|f\right|_{H^n_{\theta+n,p}}.$$

Clearly, $\nabla N(F)$ solves $\Delta\left(\nabla N(F)\right) = \nabla F$. Due to Lemma 8.4 and (8.7),

$$\left|\nabla N(F)\right|_{H^1_{\theta,p}} \le C\left|F\right|_{H^0_{\theta,p}} \le C\left|f\right|_{H^{n+1}_{\theta+n+1,p}} < \infty. \tag{8.8}$$

Hence,

$$\left|w^{\theta-d/p-1}\left(\nabla v - \nabla N(F)\right)\right|_p \le C\left|f\right|_{H^{n+1}_{\theta+n+1,p}} < \infty.$$

We conclude by Lemma 8.3 that $\nabla v = \nabla N(F)$.
From (8.8) for each $p > 1$, there is $C > 0$ so that for all multi-index $\mu \in N_0^d$ with $\left|\mu\right| = 1$,

$$\left|w^{\theta-d/p}D^\mu\nabla v\right|_p \le C\left|f\right|_{H^{n+1}_{\theta+n+1,p}}.$$

We have, recalling that $v = w^{n+1}\psi_\beta$, $\nabla v = (n+1)\,w^n\nabla w\psi_\beta + w^{n+1}\nabla\psi_\beta$ and

$$\begin{aligned} D^\mu\nabla v &= (n+1)\left[nw^{n-1}D^\mu w\nabla w + w^n D^\mu\nabla w\right]\psi_\beta \\ &+ (n+1)\,w^n D^\mu w\nabla\psi_\beta + w^{n+1}D^\mu\nabla\psi_\beta \\ &= B + w^{n+1}D^\mu\nabla\psi_\beta, \end{aligned}$$

where by (8.6)

$$\begin{aligned} \left|w^{\theta-d/p}B\right|_p &\le C\left(\left|w^{\theta-d/p+n-1}\psi_\beta\right|_p + \left|w^{\theta-d/p+n}\nabla\psi_\beta\right|_p\right) \\ &\le C\left|f\right|_{H^n_{\theta+n,p}}. \end{aligned}$$

Hence,

$$\left|w^{\theta-d/p+n+1}D^\mu\nabla\psi_\beta\right|_p \le C\left|f\right|_{H^{n+1}_{\theta+n+1,p}}$$

and the statement is proved. The estimate for general $f \in H^l_{\theta+l,p}\left(R^d\right)$ follows by passing to the limit. □

Conclusion

We employed the Euler-Lagrangian approach to prove the local existence of the Navier-Stokes equations in weighted Sobolev spaces on the full domain. This paper is the first to cover general $p > d \ge 2$. In the future, we will expand our approach and prove similar results with stochastic integrals as the forcing terms.

Funding: This research was funded by College of Industrial Technology, King Mongkut's University of Technology North Bangkok (Grant No. Res-CIT320/2023).

Conflict of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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