International Journal of Analysis and Applications
ISSN 2291-8639
Volume 6, Number 1 (2014), 63-81
http://www.etamaths.com

EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR p-LAPLACIAN FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

K. R. PRASAD[1] AND B. M. B. KRUSHNA[2,*]

Abstract. This paper deals with the existence of at least one and multiple positive solutions for p-Laplacian fractional order two-point boundary value problems,

$$D_{0^+}^{q_2}\left(\phi_p\left(D_{0^+}^{q_1}y(t)\right)\right)=f(t,y(t)),\ t\in(0,1),$$
$$y^{(j)}(0)=0,\ j=0,1,2,\cdots,n-2,\ y^{(q_4)}(1)=0,$$
$$\phi_p\left(D_{0^+}^{q_1}y(0)\right)=0=D_{0^+}^{q_3}\left(\phi_p\left(D_{0^+}^{q_1}y(1)\right)\right),$$

where $q_2\in(1,2],q_1\in(n-1,n],n\geq 2,q_3\in(0,1]$, $q_4\in[1,q_1-1]$ is a fixed integer, $\phi_p(y)=|y|^{p-2}y,\ p>1,\ \phi_p^{-1}=\phi_q,\ 1/p+1/q=1$, by applying Krasnosel'skii and five functionals fixed point theorems.

1. Introduction

The goal of differential equations is to understand the phenomena of nature by developing mathematical models. Fractional calculus is the field of mathematical analysis, which deals with investigation and applications of derivatives and integrals of an arbitrary order. Among all, a class of differential equations governed by nonlinear differential operators appears frequently and generated great deal of interest in studying such problems. In this theory, the most applicable operator is the classical p-Laplacian, given by $\phi_p(y)=|y|^{p-2}y,\ p>1$. These types of problems arise in applied fields such as turbulent flow of a gas in a porous medium, modeling of viscoelastic flows, biophysics, plasma physics and material science.

The existence of positive solutions of boundary value problems (BVPs) associated with integer order differential equations were studied by many authors [11, 1, 9, 2, 19] and extended to p-Laplacian boundary value problems [17, 14, 4, 23]. Later, these results are further extended to fractional order boundary value problems [6, 10, 5, 7, 8, 20, 21, 22] by utilizing various fixed point theorems on cones. In recent years, researchers are concentrating on the theory of fractional order boundary value problems related with p-Laplacian operator.

2010 *Mathematics Subject Classification.* 26A33, 34B18, 35J05.

Key words and phrases. Fractional derivative, p-Laplacian, boundary value problem, two-point, Green's function, positive solution.

In 2012, Chai [7] investigated the existence and multiplicity of positive solutions for a class of boundary value problem of fractional differential equation with p-Laplacian operator,

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t)))+f(t,u(t))=0,\ 0<t<1,$$
$$u(0)=0,u(1)+\sigma D_{0+}^{\gamma}u(1)=0,D_{0+}^{\alpha}u(0)=0,$$

where $1<\alpha\le 2, 0<\beta,\gamma\le 1, 0\le\alpha-\gamma-1, \sigma$ is a positive number, $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\gamma}$ are the standard Riemann-Liouville fractional order derivatives.

The purpose of this paper is to establish the existence of at least one and multiple positive solutions for p-Laplacian fractional order boundary value problems,

$$D_{0+}^{q_2}\left(\phi_p\left(D_{0+}^{q_1}y(t)\right)\right)=f(t,y(t)),\ t\in(0,1), \tag{1.1}$$

$$\left.\begin{aligned} y^{(j)}(0)=0,\ j=0,1,2,\cdots,n-2,\ y^{(q_4)}(1)=0,\\ \phi_p\left(D_{0+}^{q_1}y(0)\right)=0=D_{0+}^{q_3}\left(\phi_p\left(D_{0+}^{q_1}y(1)\right)\right),\end{aligned}\right\} \tag{1.2}$$

where $q_2\in(1,2], q_1\in(n-1,n], n\ge 2, q_3\in(0,1],\ q_4\in[1,q_1-1]$ is a fixed integer, $\phi_p(y)=|y|^{p-2}y,\ p>1,\ \phi_p^{-1}=\phi_q,\ 1/p+1/q=1,\ f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous and $D_{0+}^{q_i}$, for $i=1,2,3$ are the standard Riemann-Liouville fractional order derivatives.

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0=\lim_{y\to 0^+}\min_{t\in[0,1]}\frac{f(t,y)}{\phi_p(y)},\ f^0=\lim_{y\to 0^+}\max_{t\in[0,1]}\frac{f(t,y)}{\phi_p(y)},$$
$$f_\infty=\lim_{y\to\infty}\min_{t\in[0,1]}\frac{f(t,y)}{\phi_p(y)}\text{ and } f^\infty=\lim_{y\to\infty}\max_{t\in[0,1]}\frac{f(t,y)}{\phi_p(y)},$$

and also assume that they will exist.

The rest of the paper is organized as follows. In Section 2, we construct the Green functions for the homogeneous BVPs corresponding to (1.1)-(1.2) and estimate the bounds for the Green functions. In Section 3, we establish criteria for the existence of at least one positive solution for p-Laplacian fractional order BVP (1.1)-(1.2), by using Krasnosel'skii fixed point theorem. In Section 4, we derive sufficient conditions for the existence of at least three positive solutions for the p-Laplacian fractional order BVP (1.1)-(1.2), by applying five functionals fixed point theorem. We also establish the existence of at least $2k-1$ positive solutions for an arbitrary positive integer k. In Section 5, as an application, the results are demonstrated with examples.

2. Green Functions and Bounds

In this section, we construct the Green functions for the homogeneous BVPs and estimate the bounds for the Green functions, which are essential to establish the main results.

Let $G(t,s)$ be the Green's function for the homogeneous BVP,

$$-D_{0+}^{q_1}y(t)=0,\ t\in(0,1), \tag{2.1}$$

$$y^{(j)}(0)=0,\ j=0,1,\cdots,n-2,\ y^{(q_4)}(1)=0. \tag{2.2}$$

Lemma 2.1. *Let $h(t) \in C[0,1]$. Then the fractional order differential equation,*

$$D_{0^+}^{q_1} y(t) + h(t) = 0, \ t \in (0,1), \tag{2.3}$$

satisfying (2.2) *has a unique solution,*

$$y(t) = \int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = \begin{cases} G_1(t,s), \ 0 \le t \le s \le 1, \\ G_2(t,s), \ 0 \le s \le t \le 1, \end{cases} \tag{2.4}$$

$$\begin{aligned} G_1(t,s) =& \frac{t^{q_1-1}(1-s)^{q_1-q_4-1}}{\Gamma(q_1)}, \\ G_2(t,s) =& \frac{t^{q_1-1}(1-s)^{q_1-q_4-1} - (t-s)^{q_1-1}}{\Gamma(q_1)}. \end{aligned}$$

Proof. Let $y(t) \in C^{[q_1]+1}[0,1]$ be the solution of fractional order differential equation (2.3) satisfying (2.2). Then

$$y(t) = \frac{-1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} h(s)ds + c_1 t^{q_1-1} + c_2 t^{q_1-2} + \cdots + c_n t^{q_1-n}.$$

From (2.2), $c_i = 0, i = 2, 3, \cdots, n$ and $c_1 = \int_0^1 \frac{(1-s)^{q_1-q_4-1}}{\Gamma(q_1)} h(s)ds$. Thus, the unique solution of (2.3) with (2.2) is

$$\begin{aligned} y(t) =& \frac{-1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} h(s)ds + \frac{t^{q_1-1}}{\Gamma(q_1)} \int_0^1 (1-s)^{q_1-q_4-1} h(s)ds \\ =& \int_0^1 G(t,s)h(s)ds, \end{aligned}$$

where $G(t,s)$ is the Green's function and given in (2.4). □

Lemma 2.2. *Let $z(t) \in C[0,1]$. Then the fractional order differential equation,*

$$D_{0^+}^{q_2} \Big(\phi_p \Big(D_{0^+}^{q_1} y(t) \Big) \Big) = z(t), \ t \in (0,1), \tag{2.5}$$

satisfying

$$\phi_p \Big(D_{0^+}^{q_1} y(0) \Big) = 0, D_{0^+}^{q_3} \Big(\phi_p \Big(D_{0^+}^{q_1} y(1) \Big) \Big) = 0, \tag{2.6}$$

has a unique solution,

$$y(t) = \int_0^1 G(t,s) \phi_q \Big(\int_0^1 H(s,\tau) z(\tau) d\tau \Big) ds,$$

where

$$H(t,s) = \begin{cases} \frac{t^{q_2-1}(1-s)^{q_2-q_3-1}}{\Gamma(q_2)}, \ 0 \le t \le s \le 1, \\ \frac{t^{q_2-1}(1-s)^{q_2-q_3-1} - (t-s)^{q_2-1}}{\Gamma(q_2)}, \ 0 \le s \le t \le 1. \end{cases} \tag{2.7}$$

Here $H(t,s)$ is the Green's function for

$$\begin{aligned} &- D_{0^+}^{q_2} \Big(\phi_p(x(t)) \Big) = 0, \ t \in (0,1), \\ &\phi_p(x(0)) = 0, \ D_{0^+}^{q_3} \Big(\phi_p(x(1)) \Big) = 0. \end{aligned}$$

Proof. An equivalent integral equation for (2.5) is given by

$$\phi_p\Big(D_{0^+}^{q_1}y(t)\Big) = \frac{1}{\Gamma(q_2)}\int_0^t (t-\tau)^{q_2-1}z(\tau)d\tau + c_1t^{q_2-1} + c_2t^{q_2-2}.$$

By (2.6), one can determine $c_2 = 0$ and $c_1 = \dfrac{-1}{\Gamma(q_2)}\displaystyle\int_0^1 (1-\tau)^{q_2-q_3-1}z(\tau)d\tau$. Then,

$$\begin{aligned}\phi_p\Big(D_{0^+}^{q_1}y(t)\Big) &= \frac{1}{\Gamma(q_2)}\int_0^t (t-\tau)^{q_2-1}z(\tau)d\tau - \frac{t^{q_2-1}}{\Gamma(q_2)}\int_0^1 (1-\tau)^{q_2-q_3-1}z(\tau)d\tau \\ &= -\int_0^1 H(t,\tau)z(\tau)d\tau.\end{aligned}$$

Therefore, $\phi_p^{-1}\Big(\phi_p\Big(D_{0^+}^{q_1}y(t)\Big)\Big) = -\phi_p^{-1}\Big(\int_0^1 H(t,\tau)z(\tau)d\tau\Big)$. Consequently, $D_{0^+}^{q_1}y(t) + \phi_q\Big(\int_0^1 H(t,\tau)z(\tau)d\tau\Big) = 0.$
Hence,

$$y(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)z(\tau)d\tau\Big)ds$$

is the solution of fractional order BVP (2.5), (1.2). □

Lemma 2.3. *For* $t \in I = \left[\frac{1}{4}, \frac{3}{4}\right]$ *, the Green's function* $G(t,s)$ *given in* (2.4) *satisfies the following inequalities*

$$\begin{aligned}&(i)\ G(t,s) \geq 0,\ \ for\ all\ (t,s) \in [0,1]\times[0,1],\\ &(ii)\ G(t,s) \leq G(1,s),\ \ for\ all\ (t,s) \in [0,1]\times[0,1],\\ &(iii)\ G(t,s) \geq \xi G(1,s),\ \ for\ all\ (t,s) \in I\times[0,1],\end{aligned}$$

where $\xi = \left(\frac{1}{4}\right)^{q_1-1}$.

Proof. The Green's function $G(t,s)$ of (2.1), (2.2) is given in (2.4).
For $0 \leq t \leq s \leq 1$,

$$G_1(t,s) = \frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-q_4-1}\Big] \geq 0.$$

For $0 \leq s \leq t \leq 1$,

$$\begin{aligned}G_2(t,s) =& \frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-q_4-1} - (t-s)^{q_1-1}\Big]\\ \geq& \frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-q_4-1} - (t-ts)^{q_1-1}\Big]\\ =& \frac{t^{q_1-1}}{\Gamma(q_1)}\Big[\Big(1+\delta s + \frac{q_4(q_4+1)}{2}s^2 + \cdots\Big) - 1\Big](1-s)^{q_1-1}\\ =& \frac{t^{q_1-1}}{\Gamma(q_1)}\Big[q_4 s + O(s^2)\Big](1-s)^{q_1-1} \geq 0.\end{aligned}$$

Hence the Green's function $G(t,s)$ is nonnegative.
For $0 \leq t \leq s \leq 1$,

$$\frac{\partial G_1(t,s)}{\partial t} = \frac{1}{\Gamma(q_1)}\Big[(q_1-1)t^{q_1-2}(1-s)^{q_1-q_4-1}\Big] \geq 0.$$

Therefore, $G_1(t,s)$ is increasing in t, which implies $G_1(t,s) \leq G_1(1,s)$. Now, for $0 \leq s \leq t \leq 1$,

$$\begin{aligned}
&\frac{\partial G_2(t,s)}{\partial t}\\
&= \frac{1}{\Gamma(q_1)}\Big[(q_1-1)t^{q_1-2}(1-s)^{q_1-q_4-1} - (q_1-1)(t-s)^{q_1-2}\Big]\\
&\geq \frac{1}{\Gamma(q_1)}\Big[(q_1-1)t^{q_1-2}(1-s)^{q_1-q_4-1} - (q_1-1)(t-ts)^{q_1-2}\Big]\\
&= \frac{(q_1-1)t^{q_1-2}}{\Gamma(q_1)}\Big[1-\Big(1-(q_4-1)s+\frac{(q_4-1)(q_4-2)}{2}s^2+\cdots\Big)\Big](1-s)^{q_1-q_4-1}\\
&= \frac{(q_1-1)t^{q_1-2}}{\Gamma(q_1)}\Big[(q_4-1)s+O(s^2)\Big](1-s)^{q_1-q_4-1} \geq 0.
\end{aligned}$$

Therefore, $G_2(t,s)$ is increasing in t, which implies $G_2(t,s) \leq G_2(1,s)$.
Let $0 \leq t \leq s \leq 1$ and $t \in I$. Then

$$\begin{aligned}
G_1(t,s) =& \frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-q_4-1}\Big]\\
=& t^{q_1-1}\frac{1}{\Gamma(q_1)}\Big[(1-s)^{q_1-q_4-1}\Big]\\
=& t^{q_1-1}G_1(1,s)\\
\geq& \Big(\frac{1}{4}\Big)^{q_1-1}G_1(1,s).
\end{aligned}$$

Let $0 \leq s \leq t \leq 1$ and $t \in I$. Then

$$\begin{aligned}
G_2(t,s) =& \frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-q_4-1} - (t-s)^{q_1-1}\Big]\\
\geq& \frac{1}{\Gamma(q_1)}\Big[t^{q_1-1}(1-s)^{q_1-q_4-1} - (t-ts)^{q_1-1}\Big]\\
=& t^{q_1-1}G_2(1,s)\\
\geq& \Big(\frac{1}{4}\Big)^{q_1-1}G_2(1,s).
\end{aligned}$$

Hence the result. □

Lemma 2.4. *For $t,s \in [0,1]$, the Green's function $H(t,s)$ given in* (2.7) *satisfies the following inequalities*

$$\begin{aligned}
&(i)\ H(t,s) \geq 0,\\
&(ii)\ H(t,s) \leq H(s,s).
\end{aligned}$$

Proof. The Green's function $H(t,s)$ is given in (2.7). For $0 \leq t \leq s \leq 1$,

$$H(t,s) = \frac{1}{\Gamma(q_2)}\left[t^{q_2-1}(1-s)^{q_2-q_3-1}\right] \geq 0.$$

For $0 \le s \le t \le 1$,

$$\begin{aligned}
H(t,s) =& \frac{1}{\Gamma(q_2)}\Big[t^{q_2-1}(1-s)^{q_2-q_3-1} - (t-s)^{q_2-1}\Big]\\
\geq& \frac{1}{\Gamma(q_2)}\Big[t^{q_2-1}(1-s)^{q_2-q_3-1} - (t-ts)^{q_2-1}\Big]\\
=& \frac{t^{q_2-1}}{\Gamma(q_2)}\Big[(1-s)^{-q_3} - 1\Big](1-s)^{q_2-1}\\
=& \frac{t^{q_2-1}}{\Gamma(q_2)}\Big[\Big(1 + q_3 s + \frac{q_3(q_3+1)}{2}s^2 + \cdots\Big) - 1\Big](1-s)^{q_2-1}\\
=& \frac{t^{q_2-1}}{\Gamma(q_2)}\Big[q_3 s + O(s^2)\Big](1-s)^{q_2-1} \geq 0.
\end{aligned}$$

For $0 \le t \le s \le 1$,

$$\frac{\partial H(t,s)}{\partial t} = \frac{1}{\Gamma(q_2)}[(q_2-1)t^{q_2-2}(1-s)^{q_2-q_3-1}] \geq 0.$$

Therefore, $H(t,s)$ is increasing in t, for $s \in [0,1]$, which implies $H(t,s) \le H(s,s)$. Now, for $0 \le s \le t \le 1$,

$$\begin{aligned}
\frac{\partial H(t,s)}{\partial t} &= \frac{1}{\Gamma(q_2)}\Big[(q_2-1)t^{q_2-2}(1-s)^{q_2-q_3-1} - (q_2-1)(t-s)^{q_2-2}\Big]\\
&\le \frac{(q_2-1)}{\Gamma(q_2)}\Big[(1-s)^{q_2-q_3-1} - (1-s)^{q_2-2}\Big]\\
&= \frac{(q_2-1)}{\Gamma(q_2)}\Big[(1-s)^{-q_3+1} - 1\Big](1-s)^{q_2-2}\\
&= \frac{(q_2-1)}{\Gamma(q_2)}\Big[\Big(1 - (1-q_3)s + \frac{(1-q_3)(-q_3)}{2}s^2 + \cdots\Big) - 1\Big](1-s)^{q_2-2}\\
&= \frac{(q_2-1)}{\Gamma(q_2)}\Big[(q_3-1)s + O(s^2)\Big](1-s)^{q_2-2} \le 0.
\end{aligned}$$

Therefore, $H(t,s)$ is decreasing in t, for $s \in [0,1]$ which implies $H(t,s) \le H(s,s)$. Hence the result. $\square$

Lemma 2.5. *Let $\mu \in (\frac{1}{4}, \frac{3}{4})$. Then the Green's function $H(t,s)$ holds the inequality,*

$$\min_{t \in I} H(t,s) \geq \delta^*(s) H(s,s), \; for \; 0 < s < 1, \tag{2.8}$$

where

$$\delta^*(s) = \begin{cases} \frac{(\frac{3}{4})^{q_2-1}(1-s)^{q_2-q_3-1} - (\frac{3}{4}-s)^{q_2-1}}{s^{q_2-1}(1-s)^{q_2-q_3-1}}, & s \in (0,\mu],\\ \frac{1}{(4s)^{q_2-1}}, & s \in [\mu, 1). \end{cases} \tag{2.9}$$

Proof. For $s \in (0,1)$, $H(t,s)$ is increasing in t for $t \le s$ and decreasing in t for $s \le t$.

We define

$$\begin{aligned} h_1(t,s) =& \frac{[t^{q_2-1}(1-s)^{q_2-q_3-1} - (t-s)^{q_2-1}]}{\Gamma(q_2)} \\ h_2(t,s) =& \frac{[t^{q_2-1}(1-s)^{q_2-q_3-1}]}{\Gamma(q_2)}, \text{ and} \\ H(s,s) =& \frac{1}{\Gamma(q_2)}\left[s^{q_2-1}(1-s)^{q_2-q_3-1}\right]. \end{aligned}$$

Then,

$$\begin{aligned} \min_{t\in I} H(t,s) &= \begin{cases} h_1(\frac{3}{4},s), & s\in(0,\frac{1}{4}], \\ \min\{h_1(\frac{3}{4},s), h_2(\frac{1}{4},s)\}, & s\in[\frac{1}{4},\frac{3}{4}], \\ h_2(\frac{1}{4},s), & s\in[\frac{3}{4},1), \end{cases} \\ &= \begin{cases} h_1(\frac{3}{4},s), & s\in(0,\mu], \\ h_2(\frac{1}{4},s), & s\in[\mu,1), \end{cases} \\ &= \begin{cases} \frac{(\frac{3}{4})^{q_2-1}(1-s)^{q_2-q_3-1}-(\frac{3}{4}-s)^{q_2-1}}{\Gamma(q_2)}, & s\in(0,\mu], \\ \frac{1}{\Gamma(q_2)}\frac{(1-s)^{q_2-q_3-1}}{4^{q_2-1}}, & s\in[\mu,1), \end{cases} \\ &\geq \begin{cases} \frac{(\frac{3}{4})^{q_2-1}(1-s)^{q_2-q_3-1}-(\frac{3}{4}-s)^{q_2-1}}{s^{q_2-1}(1-s)^{q_2-q_3-1}}H(s,s), & s\in(0,\mu], \\ \frac{1}{(4s)^{q_2-1}}H(s,s), & s\in[\mu,1), \end{cases} \\ &= \delta^*(s)H(s,s). \end{aligned}$$

□

Let $B = \{y : y \in C[0,1]\}$ be the real Banach space equipped with the norm

$$\|y\| = \max_{t\in[0,1]} |y(t)|.$$

Define a cone $P \subset B$ by

$$P = \Big\{y \in B : y(t) \geq 0,\ t \in [0,1] \text{ and } \min_{t\in I} y(t) \geq \xi\|y\|\Big\}.$$

Let

$$\mathcal{K} = \frac{1}{\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds}$$

and

$$\mathcal{L} = \frac{1}{\int_{s\in I} \xi G(1,s)\phi_q\Big(\int_{\tau\in I} \delta^*(\tau)H(\tau,\tau)d\tau\Big)ds}.$$

Let $T : P \to B$ be the operator defined by

$$Ty(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds. \tag{2.10}$$

If $y \in P$ is a fixed point of T, then y satisfies (2.10) and hence y is a positive solution of the p-Laplacian fractional order BVP (1.1)-(1.2).

Lemma 2.6. *The operator T defined by* (2.10) *is a self map on P.*

Proof. Let $y \in P$. Clearly, $Ty(t) \geq 0$, for all $t \in [0,1]$, and

$$\begin{aligned} Ty(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \\ &\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \end{aligned}$$

so that

$$\|Ty\| \leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds.$$

Next, if $y \in P$, then by the above inequality we have

$$\begin{aligned} \min_{t\in I} Ty(t) &= \min_{t\in I}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \\ &\geq \xi\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \\ &\geq \xi\|Ty\|. \end{aligned}$$

Hence, $Ty \in P$ and so $T : P \to P$. Standard arguments involving the Arzela-Ascoli theorem shows that T is completely continuous. □

3. **Existence of at Least One Positive Solution**

In this section, we establish criteria for the existence of at least one positive solution of the p-Laplacian fractional order BVP (1.1)-(1.2) by using Krasnosel'skii fixed point theorem.

To establish the existence of at least one positive solution for p-Laplacian fractional order BVP (1.1)-(1.2) by employing the following Krasnosel'skii fixed point theorem.

Theorem 3.1. [15] *Let X be a Banach Space, $P \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous operator such that either*

(i) $\| Tu \| \leq \| u \|,\ u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \geq \| u \|,\ u \in P \cap \partial\Omega_2$, *or*
(ii) $\| Tu \| \geq \| u \|,\ u \in P \cap \partial\Omega_1$ *and* $\| Tu \| \leq \| u \|,\ u \in P \cap \partial\Omega_2$ *holds.*

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.2. *If $f^0 = 0$ and $f_\infty = \infty$, then the p-Laplacian fractional order BVP* (1.1)-(1.2) *has at least one positive solution that lies in P.*

Proof. Let T be the cone preserving, completely continuous operator defined by (2.10). Since $f^0 = 0$, we may choose $H^1 > 0$ so that

$$\max_{t\in[0,1]} \frac{f(t,y)}{\phi_p(y)} \leq \eta_1, \text{ for } 0 < y \leq H^1,$$

where $\eta_1 > 0$ satisfies

$$\eta_1^{q-1}\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds = \eta_1^{q-1}\frac{1}{\mathcal{K}} \leq 1.$$

Thus, if $y \in P$ and $\|y\| = H^1$, then we have

$$\begin{aligned} Ty(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y)d\tau\Big)ds \\ &\le \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)\eta_1\phi_p(y)d\tau\Big)ds \\ &= \int_0^1 G(1,s)\eta_1^{q-1}\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)yds \\ &\le \eta_1^{q-1}\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds\|y\| \\ &= \eta_1^{q-1}\frac{1}{\mathcal{K}}\|y\| \le \|y\|. \end{aligned}$$

Therefore,

$$\|Ty\| \le \|y\|.$$

Now, if we let

$$\Omega_1 = \{y \in B : \|y\| < H^1\},$$

then

$$\|Ty\| \le \|y\|, \text{ for } y \in P \cap \partial\Omega_1. \tag{3.1}$$

Further, since $f_\infty = \infty$, there exists $\overline{H}^2 > 0$ such that

$$\min_{t\in[0,1]} \frac{f(t,y)}{\phi_p(y)} \ge \eta_2, \text{ for } y \ge \overline{H}^2,$$

where $\eta_2 > 0$ is chosen such that

$$\eta_2^{q-1}\xi^2 \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I}\delta^*(\tau)H(\tau,\tau)d\tau\Big)ds = \eta_2^{q-1}\xi^2\frac{1}{\mathcal{L}} \ge 1.$$

Let

$$H^2 = \max\Big\{2H^1, \frac{1}{\xi}\overline{H}^2\Big\},$$

and define

$$\Omega_2 = \{y \in B : \|y\| < H^2\}.$$

If $y \in P \cap \partial\Omega_2$ and $\|y\| = H^2$, then

$$\min_{t\in I} y(t) \ge \xi\|y\| \ge \overline{H}^2,$$

and so

$$\begin{aligned}
Ty(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y)d\tau\Big)ds \\
&\geq \int_{s\in I} \xi G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y)d\tau\Big)ds \\
&\geq \xi \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \delta^*(\tau)H(\tau,\tau)\eta_2\phi_p(y)d\tau\Big)ds \\
&= \xi\eta_2^{q-1} \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \delta^*(\tau)H(\tau,\tau)d\tau\Big)yds \\
&\geq \xi\eta_2^{q-1} \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \delta^*(\tau)H(\tau,\tau)d\tau\Big)\xi\|y\|ds \\
&= \xi^2\eta_2^{q-1} \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \delta^*(\tau)H(\tau,\tau)d\tau\Big)ds\|y\| \\
&= \eta_2^{q-1}\xi^2\frac{1}{\mathcal{L}}\|y\| \\
&\geq \|y\|.
\end{aligned}$$

Hence,

$$\|Ty\| \geq \|y\|, \text{ for } y \in P \cap \partial\Omega_2. \tag{3.2}$$

An application of Theorem 3.1 to (3.1) and (3.2) yields a fixed point of T that lies in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is the solution of the p-Laplacian fractional order BVP (1.1)-(1.2). □

Theorem 3.3. *If $f_0 = \infty$ and $f^\infty = 0$, then the p-Laplacian fractional order BVP* (1.1)-(1.2) *has at least one positive solution that lies in P.*

Proof. Let T be the cone preserving, completely continuous operator defined by (2.10). Since $f_0 = \infty$, we choose $J^1 > 0$ such that

$$\min_{t\in[0,1]} \frac{f(t,y)}{\phi_p(y)} \geq \overline{\eta}_1, \text{ for } 0 < y \leq J^1,$$

where $\overline{\eta}_1 > 0$ satisfies

$$(\overline{\eta}_1)^{q-1}\xi^2 \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \delta^*(\tau)H(\tau,\tau)d\tau\Big)ds = (\overline{\eta}_1)^{q-1}\xi^2\frac{1}{\mathcal{L}} \geq 1.$$

In this case, we define

$$\Omega_1 = \{y \in B : \|y\| < J^1\},$$

we have $f(\tau,y)\geq \overline{\eta}_1\phi_p(y), \tau\in I$, and moreover $y(t)\geq \xi\|y\|,\ t\in I$
and so

$$\begin{aligned}
Ty(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y)d\tau\Big)ds\\
&\geq \int_{s\in I}\xi G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y)d\tau\Big)ds\\
&\geq \xi\int_{s\in I}G(1,s)\phi_q\Big(\int_{\tau\in I}\delta^*(\tau)H(\tau,\tau)\overline{\eta}_1\phi_p(y)d\tau\Big)ds\\
&= \xi(\overline{\eta}_1)^{q-1}\int_{s\in I}G(1,s)\phi_q\Big(\int_{\tau\in I}\delta^*(\tau)H(\tau,\tau)d\tau\Big)yds\\
&\geq \xi(\overline{\eta}_1)^{q-1}\int_{s\in I}G(1,s)\phi_q\Big(\int_{\tau\in I}\delta^*(\tau)H(\tau,\tau)d\tau\Big)\xi\|y\|ds\\
&\geq \xi^2(\overline{\eta}_1)^{q-1}\int_{s\in I}G(1,s)\phi_q\Big(\int_{\tau\in I}\delta^*(\tau)H(\tau,\tau)d\tau\Big)ds\|y\|\\
&= (\overline{\eta}_1)^{q-1}\xi^2\frac{1}{\mathcal{L}}\|y\|\\
&\geq \|y\|.
\end{aligned}$$

Thus,

$$\|Ty\|\geq\|y\|,\ \text{for } y\in P\cap\partial\Omega_1. \tag{3.3}$$

Now, since $f^\infty=0$, there exists $\overline{J}^2>0$ such that

$$\max_{t\in[0,1]}\frac{f(t,y)}{\phi_p(y)}\leq\overline{\eta}_2,\ \text{for } y\geq\overline{J}^2,$$

where $\overline{\eta}_2>0$ satisfies

$$(\overline{\eta}_2)^{q-1}\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds=(\overline{\eta}_2)^{q-1}\frac{1}{\mathcal{K}}\leq 1.$$

It follows that

$$f(t,y)\leq\overline{\eta}_2\phi_p(y),\ \text{for } y\geq\overline{J}^2.$$

We establish the result in two subcases.
Case(i) **:** f is bounded. Suppose $N>0$ is such that $\max_{t\in[0,1]}f(t,y)\leq N,$ for all $0<y<\infty$. In this case, we may choose

$$J^2=\max\Big\{2J^1,N^{q-1}\frac{1}{\mathcal{K}}\Big\},$$

and let

$$\Omega_2=\{y\in B:\|y\|<J^2\}.$$

Then for $y \in P \cap \partial\Omega_2$ and $\|y\| = J^2$, we have

$$\begin{aligned}
Ty(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y)d\tau\Big)ds \\
&\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)Nd\tau\Big)ds \\
&\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)N^{q-1}ds \\
&\leq N^{q-1}\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds \\
&= N^{q-1}\frac{1}{\mathcal{K}} \\
&= J^2 = \|y\|,
\end{aligned}$$

and therefore

$$\|Ty\| \leq \|y\|, \text{ for } y \in P \cap \partial\Omega_2. \tag{3.4}$$

Case(ii) **:** f is unbounded. Let $J^2 > \max\{2J^1, \overline{J}^2\}$ be such that $f(t,y) \leq f(t,J^2)$, for $0 < y \leq J^2$, and let $\Omega_2 = \{y \in B : \|y\| < J^2\}$. Choosing $y \in P \cap \partial\Omega_2$ with $\|y\| = J^2$, we have

$$\begin{aligned}
Ty(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y)d\tau\Big)ds \\
&\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y)d\tau\Big)ds \\
&\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)f(\tau,J^2)d\tau\Big)ds \\
&\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)\overline{\eta}_2\phi_p(J^2)d\tau\Big)ds \\
&= (\overline{\eta}_2)^{q-1}\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)dsJ^2 \\
&= (\overline{\eta}_2)^{q-1}\frac{1}{\mathcal{K}}J^2 \\
&\leq J^2 = \|y\|.
\end{aligned}$$

And so

$$\|Ty\| \leq \|y\|, \text{ for } y \in P \cap \partial\Omega_2. \tag{3.5}$$

An application of Theorem 3.1 to (3.3), (3.4) and (3.5) yields a fixed point of T that lies in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is the solution of the p-Laplacian fractional order BVP (1.1)-(1.2). □

4. Existence of Multiple Positive Solutions

In this section, we derive sufficient conditions for the existence of at least three positive solutions for the p-Laplacian fractional order BVP (1.1)-(1.2), by applying five functionals fixed point theorem. We also establish the existence of at least

$2k-1$ positive solutions for an arbitrary positive integer k.

Let γ, β, θ be nonnegative continuous convex functionals on P and α, ψ be nonnegative continuous concave functionals on P, then for nonnegative numbers h', a', b', d' and c', convex sets are defined.

$$
\begin{aligned}
P(\gamma, c') &= \{y \in P : \gamma(y) < c'\},\\
P(\gamma, \alpha, a', c') &= \{y \in P : a' \leq \alpha(y); \gamma(y) \leq c'\},\\
Q(\gamma, \beta, d', c') &= \{y \in P : \beta(y) \leq d'; \gamma(y) \leq c'\},\\
P(\gamma, \theta, \alpha, a', b', c') &= \{y \in P : a' \leq \alpha(y); \theta(y) \leq b'; \gamma(y) \leq c'\},\\
Q(\gamma, \beta, \psi, h', d', c') &= \{y \in P : h' \leq \psi(y); \beta(y) \leq d'; \gamma(y) \leq c'\}.
\end{aligned}
$$

In obtaining multiple positive solutions for the p-Laplacian fractional order B-VP (1.1)-(1.2), the following so called Five Functionals Fixed Point Theorem is fundamental.

Theorem 4.1. [3] *Let P be a cone in the real Banach space B. Suppose α and ψ are nonnegative continuous concave functionals on P and γ, β, θ are nonnegative continuous convex functionals on P, such that for some positive numbers c' and e', $\alpha(y) \leq \beta(y)$ and $\| y \| \leq e'\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose further that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$ is completely continuous and there exist constants h', d', a' and $b' \geq 0$ with $0 < d' < a'$ such that each of the following is satisfied.*

$(D1)$ $\{y \in P(\gamma, \theta, \alpha, a', b', c') : \alpha(y) > a'\} \neq \emptyset$ *and*
$\alpha(Ty) > a'$ *for* $y \in P(\gamma, \theta, \alpha, a', b', c')$,

$(D2)$ $\{y \in Q(\gamma, \beta, \psi, h', d', c') : \beta(y) > d'\} \neq \emptyset$ *and*
$\beta(Ty) > d'$ *for* $y \in Q(\gamma, \beta, \psi, h', d', c')$,

$(D3)$ $\alpha(Ty) > a'$ *provided* $y \in P(\gamma, \alpha, a', c')$ *with* $\theta(Ty) > b'$,

$(D4)$ $\beta(Ty) < d'$ *provided* $y \in Q(\gamma, \beta, \psi, h', d', c')$ *with* $\psi(Ty) < h'$.

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that $\beta(y_1) < d', a' < \alpha(y_2)$ and $d' < \beta(y_3)$ with $\alpha(y_3) < a'$.

Define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on P by

$$
\alpha(y) = \min_{t \in I} y(t), \psi(y) = \min_{t \in I_1} y(t),
$$
$$
\gamma(y) = \max_{t \in [0,1]} y(t), \beta(y) = \max_{t \in I_1} y(t), \theta(y) = \max_{t \in I} y(t),
$$

where $I_1 = \left[\frac{1}{3}, \frac{2}{3}\right]$. For any $y \in P$,

$$
\alpha(y) = \min_{t \in I} y(t) \leq \max_{t \in I_1} y(t) = \beta(y) \tag{4.1}
$$

and

$$
\|y\| \leq \frac{1}{\xi} \min_{t \in I} y(t) \leq \frac{1}{\xi} \max_{t \in [0,1]} y(t) = \frac{1}{\xi} \gamma(y). \tag{4.2}
$$

Theorem 4.2. *Suppose there exist* $0 < a' < b' < \frac{b'}{\xi} \leq c'$ *such that* f *satisfies the following conditions:*

$$(C1)\ f(t,y(t)) < \phi_p\Big(a'\mathcal{K}\Big),\ t \in [0,1] \text{ and } y \in [\xi a', a'],$$

$$(C2)\ f(t,y(t)) > \phi_p\Big(b'\mathcal{L}\Big),\ t \in I \text{ and } y \in \Big[b', \frac{b'}{\xi}\Big],$$

$$(C3)\ f(t,y(t)) < \phi_p\Big(c'\mathcal{K}\Big),\ t \in [0,1] \text{ and } y \in [0, c'].$$

Then the p-Laplacian fractional order BVP (1.1)-(1.2) *has at least three positive solutions* $y_1,\ y_2$ *and* y_3 *such that* $\beta(y_1) < a',\ b' < \alpha(y_2)$ *and* $a' < \beta(y_3)$ *with* $\alpha(y_3) < b'$.

Proof. We seek three fixed points $y_1, y_2, y_3 \in P$ of T defined by (2.10). From Lemma 2.6, (4.1) and (4.2), for each $y \in P$, $\alpha(y) \leq \beta(y)$ and $\|y\| \leq \frac{1}{\xi}\gamma(y)$. We show that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. Let $y \in \overline{P(\gamma, c')}$. Then $0 \leq y \leq c'$. We may use the condition $(C3)$ to obtain

$$\begin{aligned}
\gamma(Ty) &= \max_{t\in[0,1]} \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \\
&\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)f(\tau,y(\tau))d\tau\Big)ds \\
&\leq \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)\phi_p\Big(c'\mathcal{K}\Big)d\tau\Big)ds \\
&< c'\mathcal{K}\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds = c'.
\end{aligned}$$

Therefore $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. Now we verify the conditions $(D1)$ and $(D2)$ of Theorem 4.1 are satisfied. It is obvious that

$$\frac{b' + \frac{b'}{\xi}}{2} \in \Big\{y \in P\Big(\gamma, \theta, \alpha, b', \frac{b'}{\xi}, c'\Big) : \alpha(y) > b'\Big\} \neq \emptyset$$

and

$$\frac{\xi a' + a'}{2} \in \Big\{y \in Q\Big(\gamma, \beta, \psi, \xi a', a', c'\Big) : \beta(y) < a'\Big\} \neq \emptyset.$$

Next, let $y \in P(\gamma, \theta, \alpha, b', \frac{b'}{\xi}, c')$ or $y \in Q(\gamma, \beta, \psi, \xi a', a', c')$. Then, $b' \leq y \leq \frac{b'}{\xi}$ and $\eta a' \leq y \leq a'$. Now, we may apply the condition $(C2)$ to get

$$\begin{aligned}
\alpha(Ty) &= \min_{t\in I} \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds \\
&\geq \xi \int_{s\in I} G(1,s)\phi_q\Big(\int_{\tau\in I} \delta^*(\tau)H(\tau,\tau)\phi_p\Big(b'\mathcal{L}\Big)d\tau\Big)ds \\
&> b'\mathcal{L}\int_{s\in I} \xi G(1,s)\phi_q\Big(\int_{\tau\in I} \delta^*(\tau)H(\tau,\tau)d\tau\Big)ds = b'.
\end{aligned}$$

Clearly, by the condition $(C1)$, we have

$$\begin{aligned}
\beta(Ty) &= \max_{t\in I_1}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&\le \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)\phi_p\Big(a'\mathcal{K}\Big)d\tau\Big)ds\\
&< a'\mathcal{K}\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds\\
&= a'.
\end{aligned}$$

To see that $(D3)$ is satisfied, let $y\in P(\gamma,\alpha,b',c')$ with $\theta(Ty)>\frac{b'}{\xi}$. Then

$$\begin{aligned}
\alpha(Ty) &= \min_{t\in I}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&\ge \xi\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&\ge \xi\max_{t\in[0,1]}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&\ge \xi\max_{t\in I}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&= \xi\theta(Ty)\\
&> b'.
\end{aligned}$$

Finally, we show that $(D4)$ holds. Let $y\in Q(\gamma,\beta,a',c')$ with $\psi(Ty)<\xi a'$. Then, we have

$$\begin{aligned}
\beta(Ty) &= \max_{t\in I_1}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&\le \max_{t\in[0,1]}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&\le \int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&= \frac{1}{\xi}\Big[\xi\int_0^1 G(1,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)\Big]\\
&\le \frac{1}{\xi}\min_{t\in I}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&\le \frac{1}{\xi}\min_{t\in I_1}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau,y(\tau))d\tau\Big)ds\\
&= \frac{1}{\xi}\psi(Ty)<a'.
\end{aligned}$$

We have proved that all the conditions of Theorem 4.1 are satisfied. Therefore, the p-Laplacian fractional order BVP (1.1)-(1.2) has at least three positive solutions y_1, y_2 and y_3 such that $\beta(y_1)<a'$, $b'<\alpha(y_2)$ and $a'<\beta(y_3)$ with $\alpha(y_3)<b'$. This completes the proof. □

Theorem 4.3. *Let k be an arbitrary positive integer. Assume that there exist numbers $a_r(r = 1, 2, \cdots, k)$ and $b_s(s = 1, 2, \cdots, k-1)$ with $0 < a_1 < b_1 < \frac{b_1}{\xi} < a_2 < b_2 < \frac{b_2}{\xi} < \cdots < a_{k-1} < b_{k-1} < \frac{b_{k-1}}{\xi} < a_k$ such that f satisfies the following conditions:*

$$(C4)\ f(t, y(t)) < \phi_p\Big(a_r\mathcal{K}\Big),\ t \in [0, 1] \text{ and } y \in [\xi a_r, a_r], r = 1, 2, \cdots, k,$$

$$(C5)\ f(t, y(t)) > \phi_p\Big(b_s\mathcal{L}\Big),\ t \in I \text{ and } y \in \Big[b_s, \frac{b_s}{\xi}\Big], s = 1, 2, \cdots, k-1.$$

Then the p-Laplacian fractional order BVP (1.1)-(1.2) *has at least $2k-1$ positive solutions in $\overline{P}_{a_k}$.*

Proof. We use induction on k. First, for $k = 1$, we know from the condition $(C4)$ that $T : \overline{P}_{a_1} \to P_{a_1}$, then it follows from the Schauder fixed point theorem that the p-Laplacian fractional order BVP (1.1)-(1.2) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $k = l$. In order to prove that this conclusion holds for $k = l + 1$. We suppose that there exist numbers $a_r(r = 1, 2, \cdots, l+1)$ and $b_s(s = 1, 2, \cdots, l)$ with $0 < a_1 < b_1 < \frac{b_1}{\xi} < a_2 < b_2 < \frac{b_2}{\xi} < \cdots < a_l < b_l < \frac{b_l}{\xi} < a_{l+1}$ such that f satisfies the following conditions:

$$f(t, y(t)) < \phi_p\Big(a_r\mathcal{K}\Big),\ t \in [0, 1] \text{ and } y \in [\xi a_r, a_r], r = 1, 2, \cdots, l+1, \tag{4.3}$$

$$f(t, y(t)) > \phi_p\Big(b_s\mathcal{L}\Big),\ t \in I \text{ and } y \in \Big[b_s, \frac{b_s}{\xi}\Big], s = 1, 2, \cdots, l. \tag{4.4}$$

By assumption, the fractional order BVP (1.1)-(1.2) has at least $2l-1$ positive solutions $y_i^*(i = 1, 2, \cdots, 2l-1)$ in $\overline{P}_{a_l}$. At the same time, it follows from Theorem 4.2, (4.3) and (4.4) that the p-Laplacian fractional order BVP (1.1)-(1.2) has at least three positive solutions y_1, y_2 and y_3 in $\overline{P}_{a_{l+1}}$ such that $\beta(y_1) < a_l$, $b_l < \alpha(y_2)$ and $a_l < \beta(y_3)$ with $\alpha(y_3) < b_l$. Obviously y_2 and y_3 are distinct from $y_i^*(i = 1, 2, \cdots, 2l-1)$ in $\overline{P}_{a_l}$. Therefore, the p-Laplacian fractional order BVP (1.1)-(1.2) has at least $2l+1$ positive solutions in $\overline{P}_{a_{l+1}}$, which shows that this conclusion also holds for $k = l + 1$. This completes the proof. □

5. Examples

In this section, as an application, the results of the previous sections are demonstrated with examples.

Example 5.1 Consider the p-Laplacian fractional order BVP,

$$D_{0^+}^{1.7}\Big(\phi_p\Big(D_{0^+}^{2.6}y(t)\Big)\Big) = f(t, y(t)),\ t \in (0, 1), \tag{5.1}$$

$$y(0) = y'(0) = y''(1) = 0, \phi_p\Big(D_{0^+}^{2.6}y(0)\Big) = 0 = D_{0^+}^{0.6}\Big(\phi_p\Big(D_{0^+}^{2.6}y(1)\Big)\Big), \tag{5.2}$$

where

$$f(t, y) = \frac{y^2(650 - 649e^{-3y})}{5}.$$

Then the Green's function $G(t,s)$ for the homogeneous BVP,

$$-D_{0^+}^{2.6}y(t)=0,\ t\in(0,1),$$
$$y(0)=y'(0)=y''(1)=0,$$

and is given by

$$G(t,s)=\begin{cases}\frac{t^{1.6}(1-s)^{-0.6}}{\Gamma(2.6)},\ t\le s,\\ \frac{t^{1.6}(1-s)^{-0.6}-(t-s)^{1.6}}{\Gamma(2.6)},\ s\le t.\end{cases}$$

The Green's function $H(t,s)$ for the BVP,

$$-D_{0^+}^{1.7}\Big(\phi_p\Big(D_{0^+}^{2.6}y(t)\Big)\Big)=0,\ t\in(0,1),$$
$$\phi_p\Big(D_{0^+}^{2.6}y(0)\Big)=0=D_{0^+}^{0.6}\Big(\phi_p\Big(D_{0^+}^{2.6}y(1)\Big)\Big),$$

and is given by

$$H(t,s)=\begin{cases}\frac{t^{0.7}(1-s)^{0.1}}{\Gamma(1.7)},\ t\le s,\\ \frac{t^{0.7}(1-s)^{0.1}-(t-s)^{0.7}}{\Gamma(1.7)},\ s\le t.\end{cases}$$

Clearly, the Green functions $G(t,s)$ and $H(t,s)$ are positive. Let $p=2$. By direct calculations, $\xi=0.1088$, $\mathcal{K}=1.0077$, $\mathcal{L}=27.1209$, $f^0=0$ and $f_\infty=\infty$. Then, all the conditions of Theorem 3.2 are satisfied. Thus by Theorem 3.2, the p-Laplacian fractional order BVP (5.1)-(5.2) has at least one positive solution.

Example 5.2 Consider the p-Laplacian fractional order BVP,

$$D_{0^+}^{1.8}\Big(\phi_p\Big(D_{0^+}^{2.7}y(t)\Big)\Big)=f(t,y(t)),\ t\in(0,1),\tag{5.3}$$

$$y(0)=y'(0)=y''(1)=0,\phi_p\Big(D_{0^+}^{2.7}y(0)\Big)=0=D_{0^+}^{0.7}\Big(\phi_p\Big(D_{0^+}^{2.7}y(1)\Big)\Big),\tag{5.4}$$

where

$$f(t,y)=\begin{cases}\frac{t}{100}+\frac{15}{32}y^7,\ 0\le y\le 2,\\ y+\frac{t}{100}+\frac{116}{2},\ y>2.\end{cases}$$

Then the Green's function $G(t,s)$ for the homogeneous BVP,

$$-D_{0^+}^{2.7}y(t)=0,\ t\in(0,1),$$
$$y(0)=y'(0)=y''(1)=0,$$

and is given by

$$G(t,s)=\begin{cases}\frac{t^{1.7}(1-s)^{-0.7}}{\Gamma(2.7)},\ t\le s,\\ \frac{t^{1.7}(1-s)^{-0.7}-(t-s)^{1.7}}{\Gamma(2.7)},\ s\le t,\end{cases}$$

The Green's function $H(t,s)$ for the BVP,

$$-D_{0^+}^{1.8}\Big(\phi_p\Big(D_{0^+}^{2.7}y(t)\Big)\Big)=0,\ t\in(0,1),$$
$$\phi_p\Big(D_{0^+}^{2.7}y(0)\Big)=0=D_{0^+}^{0.7}\Big(\phi_p\Big(D_{0^+}^{2.7}y(1)\Big)\Big),$$

and is given by

$$H(t,s)=\begin{cases}\frac{t^{0.8}(1-s)^{0.1}}{\Gamma(1.8)},\ t\le s,\\ \frac{t^{0.8}(1-s)^{0.1}-(t-s)^{0.8}}{\Gamma(1.8)},\ s\le t.\end{cases}$$

Clearly, the Green functions $G(t,s)$ and $H(t,s)$ are positive and f is continuous and increasing on $[0,\infty)$. Let $p=2$. By direct calculations, $\xi = 0.0947$, $\mathcal{K} = 1.8901$ and $\mathcal{L} = 29.6238$. Choosing $a' = 1, b' = 2$ and $c' = 100$, then $0 < a' < b' < \frac{b'}{\xi} \leq c'$ and f satisfies

$$(i)\ f(t,y) < 1.8901 = \phi_p\Big(a'\mathcal{K}\Big),\ t \in [0,1] \text{ and } y \in [0.0947, 1],$$

$$(ii)\ f(t,y) > 59.25 = \phi_p\Big(b'\mathcal{L}\Big),\ t \in \Big[\frac{1}{4}, \frac{3}{4}\Big] \text{ and } y \in [2, 21.12],$$

$$(iii)\ f(t,y) < 189.01 = \phi_p\Big(c'\mathcal{K}\Big),\ t \in [0,1] \text{ and } y \in [0, 100].$$

Then, all the conditions of Theorem 4.2 are satisfied. Thus by Theorem 4.2, the p-Laplacian fractional order BVP (5.3)-(5.4) has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\max_{t\in[\frac{1}{3},\frac{2}{3}]} y_1(t) < 1, 2 < \min_{t\in[\frac{1}{4},\frac{3}{4}]} y_2(t),$$

$$1 < \max_{t\in[\frac{1}{3},\frac{2}{3}]} y_3(t), \min_{t\in[\frac{1}{4},\frac{3}{4}]} y_3(t) < 2.$$

References

[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations,* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[2] D. R. Anderson and J. M. Davis, Multiple positive solutions and eigenvalues for third order right focal boundary value problems, *J. Math. Anal. Appl.,* **267**(2002), 135–157.

[3] R. I. Avery, A generalization of the Leggett-Williams fixed point theorem, *Math. Sci. Res. Hot-line,* **3**(1999), 9–14.

[4] R. I. Avery, J. Henderson, Existence of three positive pseudo-symmetric solutions for a one-dimensional p-Laplacian, *J. Math. Anal. Appl.,* **277**(2003), 395–404.

[5] C. Bai, Existence of positive solutions for boundary value problems of fractional functional differential equations, *Electronic Journal of Qualitative Theory of Differential Equations,* **30**(2010), 1–14.

[6] Z. Bai and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.,* **311**(2005), 495–505.

[7] G. Chai, Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator, *Boundary Value Problems,* **2012**(2012), 1–18.

[8] T. Chen and W. Liu, An anti-periodic boundary value problem for the fractional differential equation with a p-Laplacian operator, *Appl. Math. Lett.,* **25**(2012), 1671–1675.

[9] J. M. Davis, J. Henderson, K. R. Prasad and W. Yin, Eigenvalue intervals for non-linear right focal problems, *Appl. Anal.,* **74**(2000), 215–231.

[10] R. Dehghani and K. Ghanbari, Triple positive solutions for boundary value problem of a non-linear fractional differential equation, *Bulletin of the Iranian Mathematical Society,* **33**(2007), 1–14.

[11] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.,* **120**(1994), 743–748.

[12] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones,* Acadamic Press, San Diego, 1988.

[13] A. A. Kilbas, H. M. Srivasthava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematics Studies, Vol. **204**, Elsevier Science, Amsterdam, 2006.

[14] L. Kong and J. Wang, Multiple positive solutions for the one-dimensional p-Laplacian, *Nonlinear Analysis,* **42**(2000), 1327–1333.

[15] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations,* Noordhoff, Groningen, 1964.

[16] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operator on order Banach spaces, *Indiana Univer. Math. J.*, **28**(1979), 673–688.

[17] D. O'Regan, Some general existence principles and results for $(\phi(y'))' = qf(t, y, y'), 0 < t < 1$, *SIAM J. Math. Appl.,* **24**(1993), 648–668.

[18] I. Podulbny, *Fractional Diffrential Equations,* Academic Press, San Diego, 1999.

[19] K. R. Prasad and P. Murali, Eigenvalue intervals of two-point boundary value problems for general n^{th} order differential equations, *Nonlinear Studies,* Vol. **18**, No.**2**(2011), 167–176.

[20] K. R. Prasad and B. M. B. Krushna, Multiple positive solutions for a coupled system of Riemann-Liouville fractional order two-point boundary value problems, *Nonlinear Studies,* Vol.**20**, No.**4**(2013), 501–511.

[21] K. R. Prasad, B. M. B. Krushna and N. Sreedhar, Eigenvalues for iterative systems of (n, p)-type fractional order boundary value problems, *International Journal of Analysis and Applications*, Vol. **5,** No. **2**(2014), 136–146.

[22] K. R. Prasad and B. M. B. Krushna, Eigenvalues for iterative systems of Sturm-Liouville fractional order two-point boundary value problems, *Fract. Calc. Appl. Anal.,* Vol. **17**, No.**3**(2014), 638–653, DOI: 10.2478/s13540-014-0190-4.

[23] C. Yang and J. Yan, Positive solutions for third-order Sturm-Liouville boundary value problems with p-Laplacian, *Comput. Math. Appl.,* **59**(2010), 2059–2066.

[1]Department of Applied Mathematics, Andhra University, Visakhapatnam, 530 003, India,

[2]Department of Mathematics, MVGR College of Engineering, Vizianagaram, 535 005, India,

*Corresponding author