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Hyers-Ulam Stability of Quartic Functional Equation in IFN-Spaces and 2-Banach Spaces by Classical Methods

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Abstract. There are many practical applications of functional equations that depend on the Ulam stability. Important for real-world applications, this stability idea makes sure that slight modifications to the functional equation don't cause significant modifications to the solutions. The purpose of this work is to examine the Hyers-Ulam stability of a finite-dimensional quartic functional equation in 2-Banach spaces and IFN-spaces (Intuitionistic Fuzzy Normed spaces) by utilizing fixed point and direct approaches. Within the context of this quartic functional equation, as an illustration of the stability of the equation can be regulated by sums and products of powers of norms, we present several instances.

1. Introduction

Since Ulam [20] initially posed the issue of the approximate stability of group homomorphisms in 1940, the notion of stability in functional equations has grown into a significant field of research. As a result of Hyers [8] in 1941 proof that additive mappings may be stable, the question posed

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by Ulam was promptly answered, and the concept of Hyers-Ulam stability was started. Fuzzy and intuitionistic fuzzy spaces are the latest extensions of this generalized theory, which has been applied to metric spaces, Banach spaces, and other functional equations and mathematical frameworks in recent years. Revelations into the estimated characteristics of solutions in complex spaces are gained through studying Ulam stability for various functional equations, which is important for theoretical and applied mathematics.

The result of Hyers is made longer by Aoki [2] by assuming that the Cauchy differences are not limited. As shown by Rassias [15], the additive mapping. The result of Rassias was summed up by Găvruţa [7].

Many various investigators have looked into the stability problems with various functional equations, and they have found a lot of novel findings (see [1, 12, 18, 19]).

Numerous researchers have investigated IFN-spaces and IF2N-spaces (Ref. [3], [5], [11]). Numerous scholars have explored the Ulam stability of functional equations in IFN-spaces (see [10], [16], [17], [21]

Since an example of a functional equation, the quartic functional equation has received a lot of interest. Since quartic functional equations have applications in dynamical systems and approximation theory, among others, studying their stability is an important mathematical issue. Ulam stability of the quartic functional equation in intuitionistic fuzzy normed spaces and 2-Banach spaces, two frameworks that provide different views on stability, are the main topics of this research. We can study stability in a broader context by using a 2-Banach space, which is an extension of the traditional Banach space. In contrast, intuitionistic fuzzy normed spaces build on the traditional idea of fuzzy spaces with acceptance and rejection levels, offering a more robust mathematical framework to represent the approximation character of findings. Subsequent to this groundwork, substantial strides were achieved in proving stability for quartic functional equations in broader frameworks such as 2-Banach and intuitionistic fuzzy normed spaces.

Introduced by Gähler [6] in 1964, the notion of 2-Banach spaces has offered a framework for investigating stability in spaces when certain generalised requirements are satisfied by norms. Since then, stability researchers have turned to 2-Banach spaces for their capacity to capture multi-dimensional norm structures in a flexible and broad way. 2-Banach spaces were recently shown to be useful for studying the stability of complicated functional equations, like the quartic functional equation [9].

Another way that stability theory has grown is through the use of intuitionistic fuzzy normed spaces. These spaces provide a more complex of stability, especially in domains where approximation and fuzzy logic are important, by adding a new parameter to reflect membership and non-membership degrees. Jung and Rassias [9] showed how intuitionistic fuzzy normed spaces may be used to analyse stability for different types of functional equations. These spaces offer a more robust mathematical framework that can handle the inherent uncertainty and variability in solution sets.

In order to demonstrate that functional equations are stable, the fixed point approach has been utilised extensively. This is due to the fact that Banach's fixed point theorem offers a solid framework for determining the convergence of approximate solutions [9]. Fixed point hypotheses have been shown to be useful in demonstrating Ulam consistency in a variety of mathematical situations, as proved by a number of research, such as those carried out by Radu [14]. Fixed point approaches may not be applicable to functional equations, but the direct approach is still an acceptable alternative. By using functional form-specific inequality and approximation, authors can derive stability constraints using the direct method.

In conclusion, although the fixed point and direct approaches have both demonstrated their efficacy in analysing the Ulam stability of functional equations, the application of both approaches to 2-Banach and intuitionistic fuzzy normed spaces provides a novel viewpoint on quartic functional equations for the first time. Through the application of both fixed-point and direct approaches, the purpose of this study is to contribute to the existing body of research on functional stability in generalised mathematical spaces. This will be accomplished by conducting a systematic analysis of the Ulam stability of quartic functional equations in these spaces.

Here, we examine the Hyers-Ulam stability of a finite-dimensional quartic functional equation

$$\begin{aligned}\sum_{1\le i\le m} f\left(-v_i+\sum_{j=1;i\neq j}^{m} v_j\right) &= (m-8)\sum_{1\le i<j<k<l\le m} f\left(v_i+v_j+v_k+v_l\right)\\ &-\left(m^2-12m+28\right)\sum_{1\le i<j<k\le m} f\left(v_i+v_j+v_k\right)\\ &+\left(\frac{m^3-15m^2+60m-68}{2}\right)\sum_{1\le i<j\le m} f\left(v_i+v_j\right)\\ &+2\sum_{1\le i<j\le m} f\left(v_i-v_j\right)+\sum_{i=1}^{m} f\left(3v_i\right)\\ &-\left(\frac{m^4-17m^3+86m^2-148m+558}{6}\right)\sum_{i=1}^{m} f\left(v_i\right) \qquad (1.1)\end{aligned}$$

where $m \geq 5$, in 2-Banach spaces and IFN-spaces by using direct and fixed point approaches. Within the context of this quartic functional equation, we provide examples that illustrate how the stability of the equation may be controlled by sums and products of powers of norms.

Theorem 1.1. *If a mapping $f : G \to T$ fulfills the equation (1.1), then the function $f : G \to T$ is quartic.*

Our notational convenience is ensured by the definition of a mapping $f : G \to T$ by use of

$$\begin{aligned}Df(v_1, v_2, \cdots, v_m) &= -\sum_{1\le i\le m} f\left(-v_i+\sum_{j=1;i\neq j}^{m} v_j\right)\\ &(m-8)\sum_{1\le i<j<k<l\le m} f\left(v_i+v_j+v_k+v_l\right)\end{aligned}$$

$$
\begin{aligned}
&-\left(m^2-12m+28\right)\sum_{1\le i<j<k\le m} f\left(v_i+v_j+v_k\right)\\
&+\left(\frac{m^3-15m^2+60m-68}{2}\right)\sum_{1\le i<j\le m} f\left(v_i+v_j\right)\\
&+2\sum_{1\le i<j\le m} f\left(v_i-v_j\right)+\sum_{i=1}^{m} f\left(3v_i\right)\\
&-\left(\frac{m^4-17m^3+86m^2-148m+558}{6}\right)\sum_{i=1}^{m} f\left(v_i\right)
\end{aligned}
$$

for every $v_1, v_2, \cdots, v_m \in G$.

Theorem 1.2. *[17] Let (G, d) be a generalized complete metric space and a strictly contractive function $\Omega : G \to G$ with $L < 1$. Then, for every $v_1 \in G$, either*

$$d\left(\Omega^m v_1, \Omega^{m+1} v_1\right) = \infty, \;\; m \ge m_0;$$

or there is an integer $m_0 > 0$ fulfills

(i) $d\left(\Omega^m v_1, \Omega^{m+1} v_1\right) < \infty, \;\; m \ge m_0;$
(ii) *the sequence $\{\Omega^m v_1\}_{m\in\mathbb{N}}$ converges to a fixed point v_1^* of Ω;*
(iii) *v_1^* is the unique fixed point of Ω in $G^* = \{v_2 \in G | d(\Omega^{m_0} v_1, v_2) < \infty\}$;*
(iv) *$d(v_2, v_1^*) \le \frac{1}{1-L} d(\Omega v_2, v_2)$, for every $v_2 \in G^*$,*

where L is a Lipschitz constant.

2. Hyers-Ulam Stability Results in IFN - spaces

Here, we take into consideration that G is a linear space, $(Z, N'_{\alpha_1,\alpha_2}, \Lambda)$ is a IFN-space, and $(T, N_{\alpha_1,\alpha_2}, \Lambda)$ complete IFN-space.

Definition 2.1. *[4] Let a membership degree α_1 and non-membership degree α_2 of an intuitionistic fuzzy set from $W \times (0, +\infty)$ to $[0, 1]$ such that $(\alpha_1)_v(t) + (\alpha_2)_v(t) \le 1$ for all $v \in W$ and $t > 0$. The triple $(W, N_{\alpha_1,\alpha_2}, \Upsilon)$ is called as an Intuitionistic Fuzzy Normed - space (briefly, IFN-space) if a vector space W, a continuous t-representable Υ and $N_{\alpha_1,\alpha_2} : W \times (0, +\infty) \to L^*$ satisfying as: for all $v_1, v_2 \in W$ and $t, s > 0$,*

(IFN1) $N_{\alpha_1,\alpha_2}(v_1, 0) = 0_{L^*}$;
(IFN2) $N_{\alpha_1,\alpha_2}(v_1, t) = 1_{L^*}$ *if and only if* $v_1 = 0$;
(IFN3) $N_{\alpha_1,\alpha_2}(\alpha v_1, t) = N_{\alpha_1,\alpha_2}\left(v_1, \frac{t}{|\alpha|}\right)$, *for all* $\alpha \neq 0$;
(IFN4) $N_{\alpha_1,\alpha_2}(v_1 + v_2, t + s) \ge_{L^*} \Upsilon(N_{\alpha_1,\alpha_2}(v_1, t), N_{\alpha_1,\alpha_2}(v_2, s))$.

In this case, N_{α_1,α_2} is called an intuitionistic fuzzy norm, where, $N_{\alpha_1,\alpha_2}(v_1, t) = ((\alpha_1)_{v_1}(t), (\alpha_2)_{v_1}(t))$.

2.1. **Direct Technique.**

Theorem 2.1. *If a function $g : G^m \to Z$ with $0 < \left(\frac{\psi}{3}\right) < 1$,*

$$N'_{\alpha_1,\alpha_2}\left(g(3v, 0, \cdots, 0), \epsilon\right) \ge_{L^*} N'_{\alpha_1,\alpha_2}\left(\psi(v, 0, \cdots, 0), \epsilon\right) \tag{2.1}$$

and

$$\lim_{l\to\infty} N'_{\alpha_1,\alpha_2}\left(g(3^l v_1, 3^l v_2, \cdots, 3^l v_m), 3^{4l}\epsilon\right) = 1_{L^*}$$

for every $v, v_1, v_2, \cdots, v_m \in G$ and every $\epsilon > 0$. If a function $f : G \to T$ fulfills

$$N_{\alpha_1,\alpha_2}\left(Df\left(v_1, v_2, \cdots, v_m\right), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v_1, v_2, \cdots, v_m), \epsilon\right) \tag{2.2}$$

for every $v_1, v_2, \cdots, v_m \in G$ and every $\epsilon > 0$, then the limit

$$N_{\alpha_1,\alpha_2}\left(Q_4(v) - \frac{f(3^l v)}{3^{4l}}, \epsilon\right) \to 1_{L^*} \text{ as } l \to \infty$$

exists and there is only one quartic solution $Q_4 : G \to T$ fulfilling the equation (1.1) and

$$N_{\alpha_1,\alpha_2}\left(g(v) - Q_4(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \epsilon(3^4 - \psi)\right) \tag{2.3}$$

for every $v \in G$ and every $\epsilon > 0$.

Proof. Fix $v \in G$ and every $\epsilon > 0$. Switching $(v_1, v_2, v_3, \cdots, v_m)$ by $(v, 0, 0, \cdots, 0)$ in (2.2), we arrive

$$N_{\alpha_1,\alpha_2}\left(f(3v) - 3^4 f(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \epsilon\right). \tag{2.4}$$

Switching v by $3^l v$ in (2.4) with using (IFN3), we obtain

$$N_{\alpha_1,\alpha_2}\left(\frac{f(3^{l+1}v)}{3^4} - f(3^l v), \left(\frac{\epsilon}{3^4}\right)\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(3^l v, 0, 0, \cdots, 0), \epsilon\right). \tag{2.5}$$

By the inequality (2.1) and (IFN3) in (2.5), we have

$$N_{\alpha_1,\alpha_2}\left(\frac{f(4^{l+1}v)}{3^4} - f(3^l v), \left(\frac{\epsilon}{3^4}\right)\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \frac{\epsilon}{\psi^l}\right). \tag{2.6}$$

Clearly, we can show from the inequality (2.6), that

$$N_{\alpha_1,\alpha_2}\left(\frac{f(3^{l+1}v)}{3^{4(l+1)}} - \frac{f(3^l v)}{3^{4l}}, \left(\frac{\epsilon}{3^{4(l+1)}}\right)\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \frac{\epsilon}{\psi^l}\right) \tag{2.7}$$

Replacing ϵ by $\psi^l \epsilon$ in (2.7), we get

$$N_{\alpha_1,\alpha_2}\left(\frac{f(3^{l+1}v)}{3^{4(l+1)}} - \frac{f(3^l v)}{3^{4l}}, \left(\frac{\psi^l \epsilon}{3^{4(l+1)}}\right)\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \epsilon\right). \tag{2.8}$$

Clearly,

$$\frac{f(3^l v)}{3^{4l}} - f(v) = \sum_{i=0}^{l-1} \frac{f(3^{i+1}v)}{3^{4(i+1)}} - \frac{f(3^i v)}{3^{4i}}. \tag{2.9}$$

Using (2.8) and (2.9), it is evident that

$$\begin{aligned} N_{\alpha_1,\alpha_2}\left(\frac{f(3^l v)}{3^{4l}} - f(v), \sum_{i=0}^{l-1} \frac{\psi^i \epsilon}{3^{4(i+1)}}\right) &\geq_{L^*} \Lambda_{i=0}^{l-1}\left\{N'_{\alpha_1,\alpha_2}\left(\frac{f(3^{i+1}v}{3^{4(i+1)}} - \frac{f(3^i v)}{3^{4i}}, \frac{\psi^i \epsilon}{3^{4(i+1)}}\right)\right\} \\ &\geq_{L^*} \Lambda_{i=0}^{l-1}\left\{N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \epsilon\right)\right\} \\ &\geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \epsilon\right) \end{aligned} \tag{2.10}$$

for every $v \in G$ and $\epsilon > 0$. Switching v by $3^j v$ in (2.10) and with the help of (2.1), we arrive

$$N_{\alpha_1,\alpha_2}\left(\frac{f(3^{l+j}v)}{3^{4(l+j)}} - \frac{f(3^j v)}{3^{4j}}, \sum_{i=0}^{l-1} \frac{\psi^i \epsilon}{3^{4(i+j)}}\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v,0,0,\cdots,0), \frac{\epsilon}{\psi^j}\right) \tag{2.11}$$

for every $j, l \geq 0$. Switching ϵ by $\psi^j \epsilon$ in (2.11), we reach

$$N_{\alpha_1,\alpha_2}\left(\frac{f(3^{l+j}v)}{3^{4(l+j)}} - \frac{f(3^j v)}{3^{4j}}, \sum_{i=j}^{l+j-1} \frac{\psi^i \epsilon}{3^{4(i+1)}}\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v,0,0,\cdots,0), \epsilon\right). \tag{2.12}$$

Utilizing (IFN3) in (2.12), we get

$$N_{\alpha_1,\alpha_2}\left(\frac{f(3^{l+j}v)}{3^{4(l+j)}} - \frac{f(3^j v)}{3^{4j}}, \epsilon\right) \geq_{L^*} \quad N'_{\alpha_1,\alpha_2}\left(g(v,0,0,\cdots,0), \frac{\epsilon}{\sum_{i=j}^{l+j-1} \frac{\psi^i}{3^{4i}3^4}}\right) \tag{2.13}$$

for every $j, l \geq 0$. As $0 < \psi < 3$ and $\sum_{i=0}^{l} \left(\frac{\psi}{3}\right)^i < \infty$. Thus, the sequence $\left\{\frac{f(3^l v)}{3^{4l}}\right\}$ is Cauchy sequence in $(T, N_{\alpha_1,\alpha_2}, \Lambda)$ is a complete IFN-space, this sequence $\left\{\frac{f(3^l v)}{3^{4l}}\right\}$ converges in $Q_4(v) \in T$. Then, we can define the mapping $Q_4 : G \to T$ by

$$N_{\alpha_1,\alpha_2}\left(Q_4(v) - \frac{f(3^l v)}{3^{4l}}\right) \to 1_{L^*} \text{ as } \ l \to \infty.$$

Setting $j = 0$ in inequality (2.13), we obtain

$$N_{\alpha_1,\alpha_2}\left(\frac{f(3^l v)}{3^{4l}} - f(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v,0,0,\cdots,0), \frac{\epsilon}{\sum_{i=0}^{l-1} \frac{\psi^i}{3^{4i}3^4}}\right). \tag{2.14}$$

Applying the limit as $l \to \infty$ in (2.14), we reach

$$N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v,0,0,\cdots,0), \epsilon(3^4 - \psi)\right).$$

After that, we aim to demonstrate that the functional equation (1.1) is satisfied by the function Q_4, replacing $(v_1, v_2, v_3, \cdots, v_m)$ by $(3^l v_1, 3^l v_2, 3^l v_3, \cdots, 3^l v_m)$ in (2.2) respectively, we have

$$N_{\alpha_1,\alpha_2}\left(\frac{1}{2^{4l}} Df\left(3^l v_1, 3^l v_2, 3^l v_3, \cdots, 3^l v_m\right), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g\left(3^l v_1, 3^l v_2, 3^l v_3, \cdots, 3^l v_m\right), 3^{4l}\epsilon\right)$$

for every $v_1, v_2, \cdots, v_m \in G$ and every $\epsilon > 0$. Since

$$\lim_{l \to \infty} N'_{\alpha_1,\alpha_2}\left(g(3^l v_1, 3^l v_2, 3^l v_3, \cdots, 3^l v_m), 3^{4l}\epsilon\right) = 1_{L^*}.$$

Therefore, according to the functional equation (1.1), the function Q_4 is valid. Therefore, Q_4 is a quartic function. Lastly, let's look at another quartic mapping Q'_4 to demonstrate that the function

Q_4 is unique.According to the functional equations (1.1) and (2.3), $G \to T$ is satisfied. Hence,

$$\begin{aligned} N_{\alpha_1,\alpha_2}\left(Q_4(v) - Q_4'(v), \epsilon\right) &= N_{\alpha_1,\alpha_2}\left(\frac{Q_4(3^l v)}{3^{4l}} - \frac{Q_4'(3^l v)}{3^{4l}}, \epsilon\right) \\ &\geq_{L^*} \Lambda\left\{N_{\alpha_1,\alpha_2}\left(\frac{Q_4(3^l v)}{3^{4l}} - \frac{f(3^l v)}{3^{4l}}, \frac{\epsilon}{2}\right), N_{\alpha_1,\alpha_2}\left(\frac{f(3^l v)}{3^{4l}} - \frac{Q_4'(3^l v)}{3^{4l}}, \frac{\epsilon}{2}\right)\right\} \\ &\geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g\left(3^l v, 0, 0, \cdots, 0\right), \frac{3^{4l}\epsilon(3^4 - \psi)}{2}\right) \\ &\geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g\left(v, 0, 0, \cdots, 0\right), \frac{3^{4l}\epsilon(3^4 - \psi)}{2\psi^l}\right) \end{aligned}$$

for every $v \in G$ and every $\epsilon > 0$. As

$$\lim_{l\to\infty} \frac{3^{4l}\epsilon(3^4 - \psi)}{2\psi^l} = \infty,$$

we obtain

$$\lim_{l\to\infty} N'_{\alpha_1,\alpha_2}\left(g\left(v, 0, 0, \cdots, 0\right), \frac{3^4 3^{4l}\epsilon(3^4 - \psi)}{2\psi^l}\right) = 1_{L^*}.$$

Thus,

$$N_{\alpha_1,\alpha_2}\left(Q_4(v) - Q_4'(v), \epsilon\right) = 1_{L^*}.$$

Therefore, $Q_4(v) = Q_4'(v)$. The uniqueness of the quartic function $Q_4(v)$ is thus proven. □

Theorem 2.2. *If a function $g : G^m \to Z$ with $0 < \left(\frac{3}{\psi}\right) < 1$,*

$$N'_{\alpha_1,\alpha_2}\left(g\left(3^{-1}v, 0, 0, \cdots, 0\right), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\frac{1}{\psi} g(v, 0, 0, \cdots, 0), \epsilon\right) \tag{2.15}$$

and

$$\lim_{l\to\infty} N'_{\alpha_1,\alpha_2}\left(g\left(3^{-l}v_1, 3^{-l}v_2, \cdots, 3^{-l}v_m\right), 3^{-4l}\epsilon\right) = 1_{L^*}.$$

for every $v, v_1, v_2, \cdots, v_m \in G$ and every $\epsilon > 0$. If a function $f : G \to T$ fulfills (2.2), then the limit

$$N_{\alpha_1,\alpha_2}\left(Q_4(v) - 3^{4l} f\left(\frac{v}{3^l}\right), \epsilon\right) \to 1_{L^*} \ \ as \ \ l \to \infty$$

exists and there is only one quartic function $Q_4 : G \to T$ fulfilling the equation (1.1) and

$$N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\psi(v, 0, 0, \cdots, 0), 3^4\epsilon(\psi - 3^4)\right),$$

for every $v \in g$ and every $\epsilon > 0$.

Proof. Fix $v \in G$ and all $\epsilon > 0$. Setting $(v_1, v_2, v_3, \cdots, v_s)$ by $(v, 0, 0, \cdots, 0)$ in (2.2), we arrive

$$N_{\alpha_1,\alpha_2}\left(f(3v) - 3^4 f(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \epsilon\right). \tag{2.16}$$

Switching v by $\frac{v}{3}$ in (2.16), we get

$$N_{\alpha_1,\alpha_2}\left(f(v) - 3^4 f\left(\frac{v}{3}\right), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\varphi\left(\frac{v}{3}, 0, 0, \cdots, 0\right), \epsilon\right). \tag{2.17}$$

Switching v by $\frac{v}{3^l}$ in (2.17) and utilizing (IFN3), we have

$$N_{\alpha_1,\alpha_2}\left(f\left(\frac{v}{3^l}\right)-3^4 f\left(\frac{v}{3^{l+1}}\right),\epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g\left(\frac{v}{3^{l+1}},0,0,\cdots,0\right)\right). \tag{2.18}$$

By utilizing the inequality (2.15) and condition (IFN3) in (2.18), we arrive

$$N_{\alpha_1,\alpha_2}\left(f\left(\frac{v}{3^l}\right)-3^4 f\left(\frac{v}{3^{l+1}}\right),\epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v,0,0,\cdots,0),\epsilon\psi^{l+1}\right).$$

Utilizing the same methodology as Theorem 2.1, the remaining portion of the proof can be established. □

Corollary 2.1. *Let θ be in $\mathbb{R}^+$. If a mapping $f: G \to T$ such that*

$$N_{\alpha_1,\alpha_2}\left(Df(v_1,v_2,\cdots,v_m),\epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\theta,\epsilon\right)$$

for every $v_1,v_2,\cdots,v_m \in G$ and every $\epsilon > 0$, then there is only one quartic solution $Q_4: G \to T$ satisfies

$$N_{\alpha_1,\alpha_2}\left(f(v)-Q_4(v),\epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\theta,|3^4-1|\epsilon\right)$$

for every $v \in G$ and every $\epsilon > 0$.

Proof. Assuming $\psi = 3^0$ and $g(v_1,v_2,\cdots,v_m) = \theta$, the demonstration follows from Theorem 2.1 and Theorem 2.2. Considering $\psi = 3^0$ and $g(v_1,v_2,\cdots,v_m) = \theta$, the proof follows from Theorem 2.1 and Theorem 2.2. □

Corollary 2.2. *If a function $f: G \to T$ fulfills*

$$N_{\alpha_1,\alpha_2}\left(Df(v_1,v_2,\cdots,v_m),\epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\theta\sum_{i=1}^{m}||v_i||^{\xi},\epsilon\right)$$

for every $v_1,v_2,\cdots,v_m \in G$ and every $\epsilon > 0$, then there is only one quartic solution $Q_4: G \to T$ satisfies

$$N_{\alpha_1,\alpha_2}\left(f(v)-Q_4(v),\epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\theta||v||^{\xi},|3^4-3^{\xi}|\epsilon\right)$$

for every $v \in G$ and every $\epsilon > 0$, where θ and ξ are in $\mathbb{R}^+$ with $\xi \in (0,4)\cup(4,+\infty)$.

Proof. Assuming $g = 3^{\xi}$ and $g = \theta\sum_{i=1}^{m}||v_i||^{\xi}$, the proof follows from Theorem 2.1 and Theorem 2.2. □

Corollary 2.3. *Let $\theta,\xi,\gamma,\tau \in \mathbb{R}^+$ with $m\xi, m\tau \in (0,4)\cup(4,+\infty)$. If a function $f: G \to T$ fulfills*

$$N_{\alpha_1,\alpha_2}\left(Df(v_1,v_2,\cdots,v_m),\epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\theta\sum_{i=1}^{m}||v_i||^{m\xi}+\gamma\prod_{i=1}^{m}||v_i||^{\tau},\epsilon\right)$$

for every $v_1,v_2,\cdots,v_m \in G$ and every $\epsilon > 0$, then there is only one quartic solution $Q_4: G \to T$ satisfies

$$N_{\alpha_1,\alpha_2}\left(f(v)-Q_4(v),\epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\theta||v||^{m\xi},|3^4-3^{m\xi}|\epsilon\right)$$

for every $v \in G$ and every $\epsilon > 0$.

Proof. Assuming $\psi = 3^{m\xi}$ and $\theta \sum_{i=1}^{m} \|v_i\|^{m\xi} + \gamma \prod_{i=1}^{m} \|v_i\|^{\tau}$, the proof follows from Theorem 2.1 and Theorem 2.2. □

Corollary 2.4. *Let $\gamma, \tau \in \mathbb{R}^+$ with $0 < m\tau \neq 4$. If a function $f : G \to T$ fulfills*

$$N_{\alpha_1,\alpha_2}\left(Df(v_1, v_2, \cdots, v_m), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\gamma \prod_{i=1}^{m} \|v_i\|^{\tau}, \epsilon\right)$$

for every $v_1, v_2, \cdots, v_m \in G$ and every $\epsilon > 0$, then the solution f is quartic.

Proof. The proof is valid for Theorem 2.1 and Theorem 2.2 through the setting of $g(v_1, v_2, \cdots, v_m) = \gamma \prod_{i=1}^{m} \|v_i\|^{\tau}$. □

2.2. **Fixed Point Technique.** Before we start, let's look at a constant β_a that

$$\varrho_i = \begin{cases} 3, & \text{if } i = 0, \\ \frac{1}{3}, & \text{if } i = 1 \end{cases}$$

and Φ is the set such that $\Phi = \{t_1 | t_1 : G \to T, t_1(0) = 0\}$.

Theorem 2.3. *Let a mapping $f : G \to T$ for which there is a mapping $g : G^m \to Z$ with*

$$\lim_{l\to\infty} N'_{\alpha_1,\alpha_2}\left(g(3^l v_1, 3^l v_2, \cdots, 3^l v_m), 3^{4l}\epsilon\right) = 1_{L^*} \tag{2.19}$$

and fulfilling the inequality (2.2). *If there is $L = L(i)$ such that $v \to \mu(v) = \frac{1}{3^4} g\left(\frac{v}{3}, 0, 0, \cdots, 0\right)$ has the property*

$$N'_{\alpha_1,\alpha_2}\left(L \frac{1}{\varrho_i^4} \mu(\varrho_i v), \epsilon\right) = N'_{\alpha_1,\alpha_2}\left(\mu(v), \epsilon\right), \tag{2.20}$$

then there is only one quartic function $Q_4 : G \to T$ fulfills the equation (1.1) and

$$N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\frac{L^{1-i}}{1-L} \mu(v), \epsilon\right)$$

for every $v \in G$ and every $\epsilon > 0$.

Proof. Consider ψ be a general metric on Φ:

$$\psi(t_1, t_2) = \inf\left\{j \in (0, \infty) | N_{\alpha_1,\alpha_2}\left(t_1(v) - t_2(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(j\mu(v), \epsilon\right), v \in G, \epsilon > 0\right\}.$$

Clearly, (Φ, ψ) is complete. Define a mapping $\Upsilon : \Phi \to \Phi$ by $\Upsilon t_1(v) = \frac{1}{\varrho_i} t_1(\varrho_i v)$, for every $v \in G$. For $t_1, t_2 \in \Phi$, we have

$$\begin{aligned} \psi(t_1, t_2) &\leq j \\ \Rightarrow N_{\alpha_1,\alpha_2}(t_1(v) - t_2(v), \epsilon) &\geq_{L^*} N'_{\alpha_1,\alpha_2}(j\mu(v), \epsilon) \\ \Rightarrow N_{\alpha_1,\alpha_2}\left(\frac{t_1(\varrho_i v)}{\varrho_i^4} - \frac{t_2(\varrho_i v)}{\varrho_i^4}, \epsilon\right) &\geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\frac{j\mu(\varrho_i v)}{\varrho_i^4}, \epsilon\right) \\ \Rightarrow N_{\alpha_1,\alpha_2}\left(\Upsilon t_1(v) - \Upsilon t_2(v), \epsilon\right) &\geq_{L^*} N_{\alpha_1,\alpha_2}\left(j\, L\mu(v), \epsilon\right) \\ \Rightarrow \psi\left(\Upsilon t_1(v), \Upsilon t_2(v)\right) &\leq j\, L \\ \Rightarrow \psi\left(\Upsilon t_1, \Upsilon t_2\right) &\leq L\psi\left(t_1, t_2\right). \end{aligned}$$

Thus, the function Υ is strictly contractive on Φ with L (Lipschitz constant). Replacing $(v_1, v_2, v_3, \cdots, v_m)$ by $(v, 0, 0, \cdots, 0)$ in (2.2), we have

$$N_{\alpha_1,\alpha_2}\left(f(3v) - 3^4 f(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g(v, 0, 0, \cdots, 0), \epsilon\right). \tag{2.21}$$

Using (IFN3) in (2.21), we have

$$N_{\alpha_1,\alpha_2}\left(\frac{f(3v)}{3^4} - f(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\left(\frac{1}{3^4}\right) g(v, 0, 0, \cdots, 0), \epsilon\right).$$

Utilizing (2.20) for $i = 0$, that

$$\begin{aligned} N_{\alpha_1,\alpha_2}\left(\frac{f(3v)}{3^4} - f(v), \epsilon\right) &\geq_{L^*} N'_{\alpha_1,\alpha_2}\left(L\mu(v), \epsilon\right) \\ \Rightarrow \psi\left(\Upsilon f, f\right) &\leq L = L^1 = L^{1-i}. \end{aligned} \tag{2.22}$$

Setting v by $\frac{v}{3}$ in (2.21), we have

$$N_{\alpha_1,\alpha_2}\left(f(v) - 3^4 f\left(\frac{v}{3}\right), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(g\left(\frac{v}{3}, 0, 0, \cdots, 0\right), \epsilon\right) \tag{2.23}$$

for every $v \in G$ and every $\epsilon > 0$, using (2.20) for $i = 1$, that

$$\begin{aligned} N_{\alpha_1,\alpha_2}\left(f(v) - 3^4 f\left(\frac{v}{3}\right), \epsilon\right) &\geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\mu(v), \epsilon\right) \\ \Rightarrow \psi\left(f, \Upsilon f\right) &\leq 1 = L^0 = L^{1-i}. \end{aligned} \tag{2.24}$$

We can conclude from (2.22) and (2.24), that

$$\psi\left(f, \Upsilon f\right) \leq L^{1-i} < \infty.$$

From the fixed point view, it has a fixed point Q_4 of Υ in Φ that allows it possible for

$$\lim_{l\to\infty} N_{\alpha_1,\alpha_2}\left(\frac{f(\varrho_i^l v)}{\varrho_i^{4l}} - Q_4(v), \epsilon\right) \to 1_{L^*},\ v \in G, \epsilon > 0.$$

Replacing $(v_1, v_2, \cdots, v_m)$ by $(\varrho_i v_1, \varrho_i v_2, \cdots, \varrho_i v_m)$ in (2.2), we obtain

$$N_{\alpha_1,\alpha_2}\left(\frac{1}{\varrho_i^4} Df\left(\varrho_i v_1, \varrho_i v_2, \cdots, \varrho_i v_m\right), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\psi(\varrho_i v_1, \varrho_i v_2, \cdots, \varrho_i v_m), \varrho_i^4 \epsilon\right)$$

for every $v_1, v_2, \cdots, v_m \in G$ and every $\epsilon > 0$. By applying the same method as in Theorem 2.1, we are able to demonstrate that the function Q_4 is in accordance with the functional equation (1.1). Considering that Q_4 is a singular fixed point of Υ in the context of Theorem 1.2, $\Delta = \{f \in \Phi | \psi(f, Q_4) < \infty\}$. Therefore, the function Q_4 is unique such that

$$N_{\alpha_1,\alpha_2}\left(Q_4(v) - f(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(j\, \mu(v), \epsilon\right),\ t > 0.$$

If we choose the fixed point alternative, we will get at

$$\begin{aligned} \psi(f, Q_4) &\leq \frac{1}{1-L}\psi(f, \Upsilon f) \\ \Rightarrow \psi(f, Q_4) &\leq \frac{L^{1-i}}{1-L} \\ \Rightarrow N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) &\geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\frac{L^{1-i}}{1-L}\mu(v), \epsilon\right) \end{aligned}$$

for every $v \in G$ and every $\epsilon > 0$. Hence the proof. □

Corollary 2.5. *Let θ and ξ are in $\mathbb{R}^+$ with $\theta > 0$. If a function $f : G \to T$ fulfills*

$$N_{\alpha_1,\alpha_2}\left(Df\left(v_1, v_2, \cdots, v_m\right), \epsilon\right) \geq_{L^*} \begin{cases} N'_{\alpha_1,\alpha_2}\left(\theta, \epsilon\right), \\ N'_{\alpha_1,\alpha_2}\left(\theta \sum_{j=1}^m \|v_j\|^\xi, \epsilon\right), \\ N'_{\alpha_1,\alpha_2}\left(\theta\left(\prod_{j=1}^m \|v_j\|^\xi + \sum_{j=1}^m \|v_j\|^{m\xi}\right), \epsilon\right), \end{cases}$$

for all $v_1, v_2, \cdots, v_m \in G$ and $\epsilon > 0$, then there is only one quartic function $Q_4 : G \to T$ satisfies

$$N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) \geq_{L^*} \begin{cases} N'_{\alpha_1,\alpha_2}\left(\theta, |3^4 - 1|\, \epsilon\right) \\ N'_{\alpha_1,\alpha_2}\left(\theta\|v\|^m, |3^4 - 3^\xi|\epsilon\right), & \xi < 4 \text{ or } \xi > 4; \\ N'_{\alpha_1,\alpha_2}\left(\theta\|v\|^m, |3^4 - 3^{m\xi}|\epsilon\right), & \xi < \frac{4}{m} \text{ or } \xi > \frac{4}{m}; \end{cases}$$

for every $v \in G$ and every $\epsilon > 0$.

Proof. Setting

$$g(v_1, v_2, \cdots, v_m) = \begin{cases} \theta, \\ \theta \sum_{j=1}^m \|v_j\|^\xi, \\ \theta\left(\prod_{j=1}^m \|v_j\|^\xi + \sum_{j=1}^m \|v_j\|^{m\xi}\right). \end{cases}$$

Then,

$$N^{'}_{\alpha_1,\alpha_2}\left(g\left(\varrho_i^l v_1,\varrho_i^l v_2,\cdots,\varrho_i^l v_m\right),\varrho_i^{4l}\epsilon\right)=\begin{cases}N^{'}_{\alpha_1,\alpha_2}\left(\theta,(\varrho_i)^{4l}\epsilon\right),\\ N^{'}_{\alpha_1,\alpha_2}\left(\theta\sum_{j=1}^m\|v_j\|^{\xi},\left(\varrho_i^{1-\xi}\right)^{4l}\epsilon\right),\\ N^{'}_{\alpha_1,\alpha_2}\left(\theta\left(\prod_{j=1}^m\|v_j\|^{\xi}+\sum_{j=1}^m\|v_j\|^{m\xi}\right),\left(\varrho_i^{1-m\xi}\right)^{4l}\epsilon\right),\end{cases}$$

$$=\begin{cases}\to 1_{L^*}\ \text{as}\ l\to\infty,\\ \to 1_{L^*}\ \text{as}\ l\to\infty,\\ \to 1_{L^*}\ \text{as}\ l\to\infty.\end{cases}$$

Thus, (2.19) is holds. But we have $\eta(v)=g\left(\frac{v}{3},0,0,\cdots,0\right)$ has the property

$$N^{'}_{\alpha_1,\alpha_2}\left(L\frac{1}{\varrho_i^4}\mu(\varrho_i v),\epsilon\right)\geq_{L^*}N^{'}_{\alpha_1,\alpha_2}\left(\mu(v),\epsilon\right),\ v\in G,\epsilon>0.$$

Hence,

$$\begin{aligned}N^{'}_{\alpha_1,\alpha_2}\left(\mu(v),\epsilon\right) &= N^{'}_{\alpha_1,\alpha_2}\left(g\left(\frac{v}{2},0,0,\cdots,0\right),\epsilon\right)\\ &= \begin{cases}N^{'}_{\alpha_1,\alpha_2}\left(\theta,\epsilon\right),\\ N^{'}_{\alpha_1,\alpha_2}\left(\frac{\theta}{3^{\xi}}\|v\|^{\xi},\epsilon\right),\\ N^{'}_{\alpha_1,\alpha_2}\left(\frac{\theta}{3^{m\xi}}\|v\|^{m\xi},\epsilon\right).\end{cases}\end{aligned}$$

Now,

$$\begin{aligned}N^{'}_{\alpha_1,\alpha_2}\left(\frac{1}{\varrho_i^4}\mu(\varrho_i v),\epsilon\right) &= \begin{cases}N^{'}_{\alpha_1,\alpha_2}\left(\frac{\theta}{\varrho_i^4},\epsilon\right),\\ N^{'}_{\alpha_1,\alpha_2}\left(\frac{\theta}{3^{\xi}\varrho_i^4}\|\varrho_i v\|^{\xi},\epsilon\right),\\ N^{'}_{\alpha_1,\alpha_2}\left(\frac{\theta}{3^{m\xi}\varrho_i^4}\|\varrho_i v\|^{m\xi},\epsilon\right),\end{cases}\\ &= \begin{cases}N^{'}_{\alpha_1,\alpha_2}\left(\varrho_i^{-4}\mu(v),\epsilon\right),\\ N^{'}_{\alpha_1,\alpha_2}\left(\varrho_i^{\xi-4}\mu(v),\epsilon\right),\\ N^{'}_{\alpha_1,\alpha_2}\left(\varrho_i^{m\xi-4}\mu(v),\epsilon\right).\end{cases}\end{aligned}$$

We are able to verify the following situations for conditions of ϱ_i by basing our verification on inequality (2.20).

Case:1 $L=3^{-4}$ if $i=0$.

$$N_{\alpha_1,\alpha_2}\left(f(v)-Q_4(v),\epsilon\right)\geq_{L^*}N^{'}_{\alpha_1,\alpha_2}\left(\frac{3^{-4}}{1-3^{-4}}\mu(v),\epsilon\right)=N^{'}_{\alpha_1,\alpha_2}\left(\theta,80\,\epsilon\right).$$

Case:2 $L=3^{4}$ if $i=1$.

$$N_{\alpha_1,\alpha_2}\left(f(v)-Q_4(v),\epsilon\right)\geq_{L^*}N^{'}_{\alpha_1,\alpha_2}\left(\frac{1}{1-3^{4}}\mu(v),\epsilon\right)=N^{'}_{\alpha_1,\alpha_2}\left(\theta,-80\,\epsilon\right).$$

Case:3 $L = 3^{\xi-4}$ for $\xi < 4$ if $i = 0$.

$$N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\frac{3^{\xi-4}}{1-3^{\xi-4}}\mu(v), \epsilon\right) = N'_{\alpha_1,\alpha_2}\left(\theta\|v\|^{\xi}, (3^4 - 3^{\xi})\epsilon\right).$$

Case:4 $L = 3^{4-\xi}$ for $\xi > 4$ if $i = 1$.

$$N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\frac{1}{1-3^{4-\xi}}\mu(v), \epsilon\right) = N'_{\alpha_1,\alpha_2}\left(\theta\|v\|^{\xi}, (3^{\xi} - 3^4)\epsilon\right).$$

Case:5 $L = 3^{m\xi-4}$ for $\xi < \frac{4}{m}$ if $i = 0$.

$$N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\frac{3^{m\xi-4}}{1-3^{m\xi-4}}\mu(v), \epsilon\right) = N'_{\alpha_1,\alpha_2}\left(\theta\|v\|^{m\xi}, (3^4 - 3^{m\xi})\epsilon\right).$$

Case:6 $L = 3^{4-m\xi}$ for $\xi > \frac{4}{m}$ if $i = 1$.

$$N_{\alpha_1,\alpha_2}\left(f(v) - Q_4(v), \epsilon\right) \geq_{L^*} N'_{\alpha_1,\alpha_2}\left(\frac{1}{1-3^{4-m\xi}}\mu(v), \epsilon\right) = N'_{\alpha_1,\alpha_2}\left(\theta\|v\|^{m\xi}, (3^{m\xi} - 3^4)\epsilon\right).$$

□

3. Hyers-Ulam Stability Results in 2-Banach Spaces

Definition 3.1. *[13] Let G be a linear space over $\mathbb{R}$ with a dimension greater than 1, and consider a mapping $\|\cdot,\cdot\| : G^2 \to \mathbb{R}$ that fulfills the subsequent conditions:*

(a) $\|p_1, p_2\| = 0$ *iff p_1 and p_2 are linearly dependent.*
(b) $\|p_1, p_2\| = \|p_2, p_1\|$,
(c) $\|\omega p_1, p_2\| = |\omega|\, \|p_1, p_2\|$,
(d) $\|p_1, p_2 + p_3\| \leq \|p_1, p_2\| + \|p_1, p_3\|$

for every $p_1, p_2, p_3 \in G$ and $\omega \in \mathbb{R}$.

Therefore, $\|\cdot,\cdot\|$ is referred to as a 2-norm on G, and $(G, \|\cdot,\cdot\|)$ is termed a linear 2-normed space. The space $\mathbb{R}^2$, endowed with the 2-norm defined as $|p_1, p_2|$ = the area of the triangle formed by the vertices 0, p_1, and p_2, exemplifies a 2-normed space.

Lemma 3.1. *[13] Let $(G, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $p_1 \in G$ and $\|p_1, p_2\| = 0$ for every $p_2 \in G$, then $p_1 = 0$.*

Lemma 3.2. *[13] For a convergent sequence $\{(p_1)_j\}$ in a linear 2-normed space G,*

$$\lim_{j\to\infty} \|(p_1)_j, p_2\| = \|\lim_{j\to\infty}(p_1)_j, p_2\|$$

for every $p_2 \in G$.

In this part, we treat G and T are the normed linear space and the 2-Banach space, respectively.

Theorem 3.1. *Let a mapping $g : G \to [0, +\infty)$ fulfills*

$$\lim_{i\to\infty} \frac{1}{3^{4i}} g\left(3^i v_1, 3^i v_2, \cdots, 3^i v_m, s\right) = 0 \tag{3.1}$$

for every $v_1, v_2, \cdots, v_m, s \in G$. Suppose that $f : G \to T$ is a mapping with $f(0) = 0$ and satisfies

$$\left\|Df(v_1, v_2, v_3, \cdots, v_m), s\right\| \leq g(v_1, v_2, v_3, \cdots, v_m, s) \tag{3.2}$$

and

$$\hat{g}(v, s) =: \sum_{j=0}^{\infty} \frac{1}{3^{4j}} g(3^j v, 0, 0, \cdots, 0, s) < \infty$$

exists for every $v_1, v_2, \cdots, v_m, s \in G$. Then there is only one quartic function $Q_4 : G \to T$ satisfies

$$\left\|f(v) - Q_4(v), s\right\| \leq \frac{1}{3^4} \hat{g}(v, s) \tag{3.3}$$

for every $v, s \in G$.

Proof. Setting $(v_1, v_2, v_3, \cdots, v_m)$ by $(v, 0, 0, \cdots, 0)$ in (3.2), we arrive

$$\left\|f(3v) - 3^4 f(v), s\right\| \leq g(v, 0, 0, \cdots, 0, s) \tag{3.4}$$

for every $v, s \in G$. Switching v by $3^n v$ in (3.4) and dividing both sides by 3^{n-4}, we attain

$$\left\|\frac{1}{3^{4(n+1)}} f(3^{n+1} v) - \frac{1}{3^{4n}} f(3^n v), s\right\| \leq g(3^i v, 0, 0, \cdots, 0, s)$$

for every $v, s \in G$ and every positive integer i. Hence,

$$\begin{aligned} \left\|\frac{1}{3^{4(i+1)}} f(3^{i+1} v) - \frac{1}{3^{4m}} f(3^m v), s\right\| &\leq \sum_{j=m}^{i} \left\|\frac{1}{3^{4(j+1)}} f(3^{j-1} v) - \frac{1}{3^{4j}} f(3^j v), s\right\| \\ &\leq \sum_{j=m}^{i} \frac{1}{3^{4j}} g(3^j v, 0, \cdots, 0, s) \end{aligned} \tag{3.5}$$

for every $v, s \in G$ and every positive integers m and i with $i \geq m$. Consequently, given (2.2) and (2.5), the sequence $\{\frac{1}{3^{4i}} f(3^i v)\}$ is Cauchy in T for every $v \in G$. Given that T is complete, the sequence $\{\frac{1}{3^{4i}} f(3^i v)\}$ converges in T for every $v \in G$. Consequently, we may define a mapping $Q_4 : G \to T$ by

$$Q_4(v) := \lim_{i \to \infty} \frac{1}{3^{4i}} f(3^i v) \tag{3.6}$$

for every $v \in G$. Then,

$$\lim_{i \to \infty} \left\|\frac{1}{3^{4i}} f(3^i v) - Q_4(v), s\right\| = 0$$

for every $v, s \in G$. By setting $m = 0$ and evaluating a limit as $i \to \infty$ in equation (3.5), we obtain equation (3.3). Subsequently, we aim to demonstrate that the function Q_4 is of quartic degree. From inequalities (3.1), (3.2), (3.6), and Lemma 3.2, that

$$\begin{aligned} \left\|Df(v_1, v_2, \cdots, v_m), s\right\| &= \lim_{i \to \infty} \left\|Df(3^i v_1, 3^i v_2, \cdots, 3^i v_m), s\right\| \\ &\leq \lim_{i \to \infty} \frac{1}{3^{4i}} g(3^i v_1, 3^i v_2, \cdots, 3^i v_m, s) = 0 \end{aligned}$$

for every $v_1, v_2, \cdots, v_m, s \in G$. By Lemma 3.1,

$$DQ_4(v_1, v_2, v_3, \cdots, v_m) = 0$$

for every $v_1, v_2, v_3, \cdots, v_m \in G$. Therefore, based on Theorem 1.1, the mapping $Q_4 : G \to T$ is of quartic dimension.

To demonstrate the uniqueness of the function Q_4, we assume the existence of another quartic solution $Q_4^{'} : G \to T$ that satisfies (3.3). Subsequently

$$\begin{aligned}\left\|Q_4(v) - Q_4^{'}(v), s\right\| &= \lim_{i\to\infty} \frac{1}{3^{4i}} \left\|Q_4(3^i v) - f(3^i v) + f(3^i v) - Q_4^{'}(3^i v), s\right\| \\ &\leq \lim_{i\to\infty} \frac{1}{3^{4i}} \hat{g}(3^i v, s) = 0\end{aligned}$$

for all $v, s \in G$. By Lemma 3.1, $Q_4(v) - Q_4^{'}(v) = 0$ for every $v \in G$. Hence, $Q_4 = Q_4^{'}$. □

Remark 3.1. *A theorem similar to 3.1 can be established, wherein the sequence is defined as follows:*

$$Q_4(v) := \lim_{i\to\infty} 3^{4i} f\left(\frac{v}{3^i}\right)$$

is defined using suitable assumptions for g.

Corollary 3.1. *Let $\omega : [0, \infty) \to [0, \infty)$ be a function fulfilling $\omega(0) = 0$ and*

(i) $\omega(pq) \leq \omega(p)\omega(q)$.
(ii) $\omega(p) < p$ *for every* $p > 4$.

If a function $f : G \to T$ fulfills

$$\left\|Df(v_1, v_2, \cdots, v_m), s\right\| \leq \sum_{i=1}^{m} \omega(\|v_i\|) + \omega(\|s\|)$$

for every $v_1, v_2, \cdots, v_m, s \in G$, then there is only one quartic solution $Q_4 : G \to T$ satisfies

$$\left\|f(v) - Q_4(v), s\right\| \leq \left[\frac{\omega(\|v\|)}{3^4 - \omega(3)} + \omega(\|s\|)\right] \tag{3.7}$$

for every $v, s \in G$.

Proof. Setting

$$g(v_1, v_2, v_3, \cdots, v_m, s) = \sum_{1\leq i\leq m} \omega(\|v_i\|) + \omega(\|s\|)$$

for every $v_1, v_2, v_3, \cdots, v_m, s \in G$. We can say from (i) that

$$\omega(3^{4i}) \leq (\omega(3))^{4i}$$

and

$$g(3^i v_1, 3^i v_2, \cdots, 3^i v_m, s) \leq (\omega(3))^{4i} \left(\sum_{1\leq i\leq m} \omega(\|v_i\|)\right) + \omega(\|s\|).$$

We get (3.7) by using Theorem 3.1. □

Corollary 3.2. *Let $q < 4$ and a homogeneous function $H : [0, \infty) \times [0, \infty) \to [0, \infty)$. If a function $f : G \to T$ fulfills*

$$\left\|Df(v_1, v_2, v_3, \cdots, v_m), s\right\| \leq H\left(\|v_1\|, \|v_2\|, \|v_3\|, \cdots, \|v_m\|\right) + \|s\|$$

for every $v_1, v_2, v_3, \cdots, v_m, s \in G$, then there is only one quartic function $Q_4 : G \to T$ fulfills

$$\left\| f(v) - Q_4(v), s \right\| \leq \frac{H(\|v\|, 0, 0, \cdots, 0) + \|s\|}{3 - q} \tag{3.8}$$

for every $v, s \in G$ and every $q \in\in \mathbb{R}^+$.

Proof. Setting

$$g(v_1, v_2, v_3, \cdots, v_m, s) = H\left(\|v_1\|, \|v_2\|, \|v_3\|, \cdots, \|v_m\|\right) + \|s\|$$

for every $v_1, v_2, v_3, \cdots, v_m, s \in G$. By utilizing Theorem 3.1, we reach (3.8). □

Corollary 3.3. *Let $q < 4$ and a homogeneous function $H : [0, \infty) \times [0, \infty) \to [0, \infty)$ with degree q. If a function $f : G \to T$ fulfills*

$$\left\| Df(v_1, v_2, v_3, \cdots, v_m), s \right\| \leq H\left(\|v_1\|, \|v_2\|, \|v_3\|, \cdots, \|v_m\|\right) \|s\|$$

for every $v_1, v_2, v_3, \cdots, v_m, s \in G$, then there is only one quartic solution $Q_4 : G \to T$ satisfies

$$\left\| f(v) - Q_4(v), s \right\| \leq \frac{H(\|v\|, 0, 0, \cdots, 0)\|s\|}{3 - 3^q} \tag{3.9}$$

for every $v, s \in G$ and $q \in \mathbb{R}^+$.

Proof. Setting

$$g(v_1, v_2, v_3, \cdots, v_m, s) = H\left(\|v_1\|, \|v_2\|, \|v_3\|, \cdots, \|v_m\|\right) \|s\|$$

for every $v_1, v_2, v_3, \cdots, v_m, s \in G$. By utilizing Theorem 3.1, we arrive (3.9). □

Corollary 3.4. *Let $p < 4$ and a function $f : G \to T$ fulfills*

$$\left\| Df(v_1, v_2, \cdots, v_m), s \right\| \leq \sum_{i=1}^{m} \|v_i\|^p + \|s\|$$

for every $v_1, v_2, \cdots, v_m, s \in G$, then there is only one quartic solution $Q_4 : G \to T$ satisfies

$$\left\| f(v) - Q_4(v), s \right\| \leq \frac{\|v\|^p + \|s\|}{3 - p}$$

for every $v, s \in G$ and $p \in \mathbb{R}^+$.

4. Conclusion

In this current work, we examined the Hyers-Ulam stability of a finite-dimensional quartic functional equation in IFN-spaces and 2-Banach spaces by utilizing fixed point and direct approaches. Within the context of this quartic functional equation, as an illustration of the stability of the equation can be regulated by sums and products of powers of norms, we present several instances.

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