Int. J. Anal. Appl. (2025), 23:43

International Journal of Analysis and Applications

Intuitionistic Fuzzy Soft Boolean Rings

Gadde Sambasiva Rao[1], D. Ramesh[1], Aiyared Iampan[2,*], B. Satyanarayana[3], P. Rajani[4]

[1] *Department of Engineering Mathematics, College of Engineering, Koneru Lakshmaiah Education Foundation, Vaddeswaram, Andhra Pradesh-522302, India*

[2] *Department of Mathematics, School of Science, University of Phayao, 19 Moo 2, Tambon Mae Ka, Amphur Mueang, Phayao 56000, Thailand*

[3] *Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, Guntur-522510, Andhra Pradesh, India*

[4] *Department of Basic Sciences and Humanities, Seshadri Rao Gudlavalleru Engineering College, Seshadri Rao Knowledge Village, Gudlavalleru, Andhra Pradesh-521356, India*

[*]*Corresponding author: aiyared.ia@up.ac.th*

Abstract. Maji et al. [10] presented the idea of intuitionistic fuzzy soft sets (IFSSs), which build on intuitionistic fuzzy sets (IFSs) and soft sets. In this paper, we extend the notion of IFSs to Boolean rings (BRs). We briefly overview intuitionistic fuzzy soft Boolean rings (IFSBRs) and list some of their fundamental characteristics. We define the intersection, union, AND, and OR operations of IFSBRs. We then present the definitions of IFSIs and consider a few associated findings.

1. Introduction

Imprecise data is used in numerous applications in business management, social science, engineering, economics, and environmental science. Uncertain data may result from incomplete or random data, measuring device limitations, delayed data updates, and other factors in those applications. The significance of those apps and the rapidly increasing volume of uncertain data being collected and accumulated have drawn a lot of attention to the need for research on efficient and effective methods for modeling uncertain data and addressing uncertainties. However, overall, this research has remained challenging. The literature is replete with numerous studies and

Received: Dec. 1, 2024.

2020 *Mathematics Subject Classification.* 06E05, 08A72, 06D72.

Key words and phrases. Boolean ring (BR); fuzzy soft set (FSS); intuitionistic fuzzy soft set (IFSS); intuitionistic fuzzy soft Boolean ring (IFSBR); intuitionistic fuzzy soft ideal (IFSI).

https://doi.org/10.28924/2291-8639-23-2025-43
© 2025 the author(s).
ISSN: 2291-8639

applications relating to specific specialized instruments, including interval mathematics, probability theory, fuzzy set theory, IFS theory, vague set theory, and rough set theory. In 1999, Russian researcher Molodtsov first proposed soft set theory [12] as a way to circumvent these problems.

The field of soft set theory is rapidly evolving. The concept of fuzzy soft sets (FSSs) was initially introduced by Maji et al. [11], who also offered an application for it. Additionally, Roy and Maji [20] presented a technique for object recognition using imprecise data collected from multiple observers.

The above discussion clarifies that Zadeh's fuzzy sets theory [23], which Atanassov extended to IFSs [1], serves as the basis for all of these works. As an extension of the concept of fuzzy sets, Atanassov presented and explored the idea of IFSs [2, 5]. In [3], a few uses of IFSs are covered. Some properties related to applying IFSs to algebraic hyperstructures by Davvaz et al. [4,6,9,10,23] are discussed. Rao et al. [14–19] studied fuzzy soft Boolean rings (FSBRs) and fuzzy soft Boolean near-rings (FSBNRs), examining their algebraic properties, generalizations, and structural implications. Key findings highlight that operations such as AND, restricted, and extended intersections preserve FSBNR structures, while FS-homomorphisms maintain FSBNR images and pre-images. Their work integrates soft set theory with Boolean near rings, broadening the theoretical understanding and applications of Boolean-like structures. İnan and Öztürk [7] investigated fuzzy soft rings and their generalization, $(\in, \in \vee q)$-fuzzy soft subrings, focusing on their fundamental properties. Ramesh et al. [8, 13, 21, 22] explored the interplay between intuitionistic fuzzy translations (IFTs) and IFSs in BF-algebras, introducing concepts such as IF-extensions, IF-multiplications, and their related properties. The study also examines their fundamental properties by examining intuitionistic fuzzy translations applied to implicative and commutative ideals in BCK-algebras. Building on the interval-valued intuitionistic fuzzy sets proposed by Atanassov and Gargov, the research introduces interval-valued intuitionistic fuzzy ideals and homomorphisms in BF-algebras, uncovering intriguing properties and extending the theoretical framework of fuzzy set theory in algebraic structures.

This paper deals with utilizing the algebraic framework of BRs using intuitionistic fuzzy soft theory. We characterize some algebraic properties and operation characterizations of IFSBRs.

2. Preliminaries

To be thorough, we list a few pertinent definitions and findings in this section.

Definition 2.1. *For a set $\mathbb{R}$, when two binary operations are available, namely addition $+$ and multiplication $\cdot$, if any of the following characteristics apply, it is considered to be a ring:*
(i) $\mathbb{R}$ is a group under $+$,
(ii) $\mathbb{R}$ is a semigroup under $\cdot$,
(iii) $(g+s)r = gr + sr$ and $g(s+r) = gs + gr$, $\forall g, s, r \in \mathbb{R}$.

Definition 2.2. *If $s^2 = s, \forall s \in \mathbb{R}$, then a ring $\mathbb{R}$ is a BR.*

Definition 2.3. *Let A represent the universe's beginning, π represent parameters, $\pi \subseteq A$, and $FS(A)$ indicate A's fuzzy power set. An FSS's A is a pair (G, π), in this case, $G : \pi \to FS(A)$ specifying the mapping G. An FSS is a family of fuzzy subsets of A with parameters.*

Definition 2.4. *Let A represent the universe's beginning, π represent parameters, $IFS(A)$ stand for A's IFS, and $\pi \subseteq A$. A pair (G, π) is called an IFSS over A; in this case, $G : \pi \to IFS(A)$ specifies the mapping G.*

An FSS is a particular case of an IFSS, a parameterized family of IFSSs of A. An IFSS degenerates into an FSS when all of A's IFSSs degenerate into fuzzy subsets.

For all $e \in \pi \subseteq A$, we have an IFSS on A, $G(e)$ referred to as the IFS of the parameter e. The intuitionistic value

$$\langle G_e(a), G_e'(a) \rangle$$

represents the extent to which an object $a \in A$ possesses parameter e. $G(e)$ has the following possible writing:

$$G(e) = \{\langle a, G_e(a), G_e'(a) \rangle | a \in A\}.$$

If for all $a \in A, G_e(a) + G_e'(a) \leq 1$, then $G(e)$ becomes an IFS; if for all $a \in A, e \in \pi \subseteq A, G_e(a) + G_e'(a) \leq 1$, then the IFSS (G, π) degenerates into an FSS.

Definition 2.5. *Let (G, π) and (S, ρ) be two IFSS's A. Then, (G, π) is said to be an IFS subset of (S, ρ) if*
(i) $\pi \subseteq \rho$,
(ii) $\forall e \in \pi$, $G(e)$ is an of $S(e)$ is an IF set .

The notation $(G, \pi) \subseteq (S, \rho)$ indicates the preceding relationship of inclusion.

Likewise, if (S, ρ) is an IFS subset of (G, π), then (G, π) is referred to as an IFS superset of (S, ρ), respectively. We indicated the relationship above by $(G, \pi) \supseteq (S, \rho)$. (G, π) and (S, ρ) are regarded as IFS equivalents if $(G, \pi) \subseteq (S, \rho)$ and $(S, \rho) \subseteq (G, \pi)$.

Definition 2.6. *Let (G, π) and (S, ρ) be IFSS's A, following that, the set*
(i) (G, π) AND (S, ρ), then $(G, \pi) \wedge (S, \rho)$ is characterized as $(G, \pi) \wedge (S, \rho) = (S, \sigma)$, where $\sigma = \pi \times \rho$ and $R(u, v) = G(u) \cap S(v), \forall (u, v) \in \pi \times \rho$.
(ii) (G, π) OR (S, ρ), then $(G, \pi) \vee (S, \rho)$ is characterized as $(G, \pi) \vee (S, \rho) = (S, \sigma))$, where $\sigma = \pi \times \rho$ and $R(u, v) = G(u) \cup S(v), \forall (u, v) \in \pi \times \rho$.

Definition 2.7. *An IFSS is defined as the intersection of IFSS's (G, π) and (S, ρ), as indicated by (R, σ), where $\sigma = \pi \cup \rho$ and $\forall e \in \sigma$,*

$$R_e = \begin{cases} G_e, & \text{if} \quad e \in \pi - \rho \\ S_e, & \text{if} \quad e \in \rho - \pi \\ G_e \cap S_e, & \text{if} \quad e \in \pi \cap \rho \end{cases}$$

Then it's represented as $(R, \sigma) = (G, \pi) \cap (S, \rho)$.

Definition 2.8. *An IFSS is defined as the union of IFSS's (G, π) and (S, ρ), as indicated by (R, σ), where $\sigma = \pi \cup \rho$ and $\forall e \in \sigma$,*

$$R_e = \begin{cases} G_e, & \text{if} \quad e \in \pi - \rho \\ S_e, & \text{if} \quad e \in \rho - \pi \\ G_e \cup S_e, & \text{if} \quad e \in \pi \cap \rho \end{cases}$$

Then it's represented as $(R, \sigma) = (G, \pi) \cup (S, \rho)$.

We may occasionally use alternative definitions of intersection and union, as follows, instead of the IFSS definitions previously provided.

Definition 2.9. *Let* (G, π) *and* (S, ρ) *be IFSS's A such that* $\pi \cap \rho \neq \emptyset$.
(i) The bi-union of (G, π) *and* (S, ρ) *is identified as the IFSS* (R, σ), *where* $\sigma = \pi \cap \rho$ *and* $R(a) = G(a) \cup S(a), \forall a \in \sigma$. *This is indicated by* $(R, \sigma) = (G, \pi) \sqcup (S, \rho)$.
(ii) The bi-intersection of (G, π) *and* (S, ρ) *is identified as the IFSS* (R, σ), *where* $\sigma = \pi \cap \rho$ *and* $R(a) = G(a) \cap S(a), \forall a \in \sigma$. *This is indicated by* $(R, \sigma) = (G, \pi) \sqcap (S, \rho)$.

Definition 2.10. *Let* (G, π) *and* (S, ρ) *be two IFSS's A. The product of* (G, π) *and* (S, ρ) *is described as the IFSS* $(G \circ S, \sigma)$, *where* $\sigma = \pi \cup \rho$ *and* $\forall e \in \sigma, a \in A$,

$$(G \circ S)_e = \begin{cases} G_e, & \text{if} \quad e \in \pi - \rho \\ S_e, & \text{if} \quad e \in \rho - \pi \\ \sup_{a=xy} G_e \cup S_e, & \text{if} \quad e \in \pi \cap \rho \end{cases}$$

and

$$(G \circ S)'_e = \begin{cases} G'_e, & \text{if} \quad e \in \pi - \rho \\ S'_e, & \text{if} \quad e \in \rho - \pi \\ \inf_{a=xy} G'_e \cup S'_e, & \text{if} \quad e \in \pi \cap \rho \end{cases}$$

The representation is $(G \circ S, \sigma) = (G, \pi) \circ (S, \rho)$.

3. Intuitionistic Fuzzy Soft Boolean Rings

This section provides an overview of IFSBR definitions and outlines some of their key characteristics. All IFSS are now taken into account over a BR $\mathbb{R}$.

Definition 3.1. *Let* (G, π) *be a non-null IFSS* $\mathbb{R}$. *Afterward,* (G, π) *is referred to as an IFSBR's* $\mathbb{R}$ *if for each* $e \in \pi$ *and* $\zeta, \theta \in \mathbb{R}$,
(i) $G_e(\zeta + \theta) \geq \min\{G_e(\zeta), G_e(\theta)\}$, $G'_e(\zeta + \theta) \leq \max\{G'_e(\zeta), G'_e(\theta)\}$
(ii) $G_e(\zeta\theta) \geq \min\{G_e(\zeta), G_e(\theta)\}$, $G'_e(\zeta\theta) \leq \max\{G'_e(\zeta), G'_e(\theta)\}$.

Example 3.1. *Let the binary operations* $+$ *and* $\cdot$ *be present in the nonempty set* $\mathbb{R} = \{0, g, s, r\}$ *in the following terms:*

$+$	0	g	s	r
0	0	g	s	r
g	g	0	r	s
s	s	r	0	g
r	r	s	g	0

$\cdot$	0	g	s	r
0	0	0	0	0
g	0	g	0	g
s	0	0	s	s
r	0	g	s	r

Then $(\mathbb{R},+,\cdot)$ *is a BR. Set the parameters to* $E=\{e_1,e_2,e_3\}$*, and define an IFSS* (G,π) *of* $\mathbb{R}$*,*

$$G(e_1)=\{(0,0.2),(g,0.2),(s,0.1),(r,0.1)\}$$
$$G(e_2)=\{(0,0.4),(g,0.4),(s,0.3),(r,0.3)\}$$
$$G(e_3)=\{(0,0.3),(g,0.3),(s,0.2),(r,0.2)\}$$
$$G^{'}(e_1)=\{(0,0.3),(g,0.4),(s,0.7),(r,0.7)\}$$
$$G^{'}(e_2)=\{(0,0.4),(g,0.5),(s,0.7),(r,0.7)\}$$
$$G^{'}(e_3)=\{(0,0.3),(g,0.5),(s,0.8),(r,0.8)\}$$

It follows that (G,π) *is an IFSSR's* $\mathbb{R}$*.*

Definition 3.2. *We state that* (G,π) *is an IFS-sub-BR of* (S,ρ) *for two IFSBRs* (G,π) *and* (S,ρ) *of* $\mathbb{R}$*. We also write* $(G,\pi)\subseteq(S,\rho)$ *if*
(i) $\pi\subseteq\rho$*,*
(ii) for each $h\in\mathbb{R}$ *and* $e\in\pi, G_e(\zeta)\leq S_e(\zeta)$ *and* $G_e^{'}(\zeta)\geq S_e^{'}(\zeta)$*.*

Definition 3.3. *If* $(G,\pi)\subseteq(S,\rho)$ *and* $(S,\rho)\subseteq(G,\pi)$*, then two IFSBRs* (G,π) *and* (S,ρ) *of* $\mathbb{R}$ *are equal.*

Definition 3.4. *Let* $(G,\pi)\cup(S,\rho)$ *be the union of IFSBRs* (G,π) *and* (S,ρ) *of* $\mathbb{R}$*. It is described using an IFSBR* $R:\pi\cup\rho\longrightarrow[0,1]^{\mathbb{R}}$ *which ensures that for every* $e\in\pi\cup\rho$*,*

$$R_e=\begin{cases}\langle\zeta,G_e(\zeta),G_e^{'}(\zeta)\rangle, & \text{if} \quad e\in\pi-\rho\\ \langle\zeta,S_e(\zeta),S_e^{'}(\zeta)\rangle, & \text{if} \quad e\in\rho-\pi\\ \langle\zeta,G_e(\zeta)\vee S_e(\zeta),G_e^{'}(\zeta)\wedge S_e^{'}(\zeta)\rangle, & \text{if} \quad e\in\pi\cap\rho\end{cases}$$

That is indicated by $(R,\sigma)=(G,\pi)\cup(S,\rho)$*, where* $\sigma=\pi\cup\rho$*.*

Theorem 3.1. *If* (G,π) *and* (S,ρ) *are IFSBR's* $\mathbb{R}$*, then* $(G,\pi)\cup(S,\rho)$ *is an IFSBR's* $\mathbb{R}$*.*

Proof. For any $e\in\pi\cup\rho$ and $h,i\in\mathbb{R}$, we take into account the following situations:

Case 1. Let $e\in\pi-\rho$. Then

$$\begin{aligned}R_e(\zeta+\theta) &= G_e(\zeta+\theta)\\ &\geq G_e(\zeta)\wedge G_e(\theta)\\ &= R_e(\zeta)\wedge R_e(\theta)\\ R_e(\zeta\theta) &= G_e(\zeta\theta)\\ &\geq G_e(\zeta)\wedge G_e(\theta)\\ &= R_e(\zeta)\wedge R_e(\theta)\\ R_e^{'}(\zeta+\theta) &= G_e^{'}(\zeta+\theta)\\ &\leq G_e^{'}(\zeta)\vee G_e^{'}(\theta)\\ &= R_e^{'}(\zeta)\vee R_e^{'}(\theta)\\ R_e^{'}(\zeta\theta) &= G_e^{'}(\zeta\theta)\\ &\leq G_e^{'}(\zeta)\vee G_e^{'}(\theta)\end{aligned}$$

$$= R'_e(\zeta) \vee R'_e(\theta).$$

Case 2. Let $e \in \rho - \pi$. Next, in line with Case 1's proof, we have

$$\begin{aligned}
R_e(\zeta+\theta) &= S_e(\zeta+\theta) \\
&\geq S_e(\zeta) \wedge S_e(i) \\
&= R_e(\zeta) \wedge R_e(\theta) \\
R_e(\zeta\theta) &= S_e(\zeta\theta) \\
&\geq S_e(\zeta) \wedge S_e(i) \\
&= R_e(\zeta) \wedge R_e(\theta) \\
R'_e(\zeta+\theta) &= S'_e(\zeta+\theta) \\
&\leq S'_e(\zeta) \vee S'_e(i) \\
&= R'_e(\zeta) \vee R'_e(\theta) \\
R'_e(\zeta\theta) &= S'_e(\zeta\theta) \\
&\leq S'_e(\zeta) \vee S'_e(i) \\
&= R'_e(\zeta) \vee R'_e(\theta).
\end{aligned}$$

Case 3. Let $e \in \pi \cap \rho$. In this instance, it is an easy proof to follow. Therefore, in any event, as we have

$$\begin{aligned}
R_e(\zeta+\theta) &\geq R_e(\zeta) \wedge R_e(\theta) \\
R_e(\zeta\theta) &\geq R_e(\zeta) \wedge R_e(\theta) \\
R'_e(\zeta+\theta) &\leq R'_e(\zeta) \vee R'_e(\theta) \\
R'_e(\zeta\theta) &\leq R'_e(\zeta) \vee R'_e(\theta).
\end{aligned}$$

Therefore, $(G,\pi) \cup (S,\rho)$ is an IFSBR's $\mathbb{R}$. □

Definition 3.5. *Let $(G,\pi) \cap (S,\rho)$ represents the intersection of two IFSBRs (G,π) and (S,ρ) of $\mathbb{R}$, and its represented by*

$$R_e(\zeta) = \begin{cases} \langle \zeta, G_e(\zeta), G'_e(\zeta) \rangle, & \text{if} \quad e \in \pi - \rho \\ \langle \zeta, S_e(\zeta), S'_e(\zeta) \rangle, & \text{if} \quad e \in \rho - \pi \\ \langle \zeta, G_e(\zeta) \wedge S_e(\zeta), G'_e(\zeta) \vee S'_e(\zeta) \rangle, & \text{if} \quad e \in \pi \cap \rho \end{cases}$$

This is indicated by $(R,\sigma) = (G,\pi) \cap (S,\rho)$, where $\sigma = \pi \cup \rho$.

Theorem 3.2. *If (G,π) and (S,ρ) are IFSBR's $\mathbb{R}$, then $(G,\pi) \cap (S,\rho)$ is an IFSBR's $\mathbb{R}$.*

Proof. The proof is simple to understand. □

Definition 3.6. *Let (G,π) and (S,ρ) be IFSBR's $\mathbb{R}$. Then (G,π) AND (S,ρ) is indicated by $(G,\pi) \wedge (S,\rho)$, and its determined by $(G,\pi) \wedge (S,\rho) = (R,\sigma)$, where $\sigma = \pi \times \rho$ and $R : \sigma \to ([0,1] \times [0,1])^{\mathbb{R}}$ is determined as*

$$R(\zeta,\theta) = G(\zeta) \cap S(\theta), \forall(\zeta,\theta) \in \sigma.$$

Theorem 3.3. *If (G,π) and (S,ρ) are two IFSBR's $\mathbb{R}$, then $(G,\pi) \wedge (S,\rho)$ and $(G,\pi) \sqcap (S,\rho)$ are IFSBR's $\mathbb{R}$.*

Proof. For all $h, i \in \mathbb{R}$ and $(x,y) \in \pi \times \rho$, we have

$$\begin{aligned}
R_{(x,y)}(\zeta+\theta) &= G_x(\zeta+\theta) \cap S_y(\zeta+\theta) \\
&\geq (G_x(\zeta) \wedge G_x(\theta)) \cap (S_y(\zeta) \wedge S_y(\theta)) \\
&= (G_x(\zeta) \cap S_y(\zeta)) \wedge (G_x(\theta) \cap S_y(\theta)) \\
&= R_{(x,y)}(\zeta) \wedge R_{(x,y)}(\theta) \\
R_{(x,y)}(\zeta\theta) &= G_x(\zeta\theta) \cap S_y(\zeta\theta) \\
&\geq (G_x(\zeta) \wedge G_x(p)) \cap (S_y(\zeta) \wedge S_y(p)) \\
&= (G_x(\zeta) \cap S_y(\zeta)) \wedge \cap S_y(\theta)) \\
&= R_{(x,y)}(\zeta) \wedge R_{(x,y)}(\theta).
\end{aligned}$$

Parallel to this, we have

$$\begin{gathered}
R'_{(x,y)}(\zeta+\theta) \leq R'_{(x,y)}(\zeta) \vee R'_{(x,y)}(\theta) \\
R'_{(x,y)}(\zeta\theta) \leq R'_{(x,y)}(\zeta) \vee R'_{(x,y)}(\theta).
\end{gathered}$$

Hence, $(G,\pi) \wedge (S,\rho)$ is an IFSBR's $\mathbb{R}$. Similar proofs can be used for $(G,\pi) \sqcap (S,\rho)$. □

4. Intuitionistic Fuzzy Soft Ideals of Intuitionistic Fuzzy Soft Boolean Rings

Definition 4.1. *An IFSS (G,π) of $\mathbb{R}$ is referred to as an IFSI's $\mathbb{R}$ if for each $e \in \pi$ and $\zeta,\theta \in \mathbb{R}$,*
(i) $G_e(\zeta+\theta) \geq \min\{G_e(\zeta), G_e(\theta)\}$, $G'_e(\zeta+\theta) \leq \max\{G'_e(\zeta), G'_e(\theta)\}$
(ii) $G_e(\zeta\theta) \geq \max\{G_e(\zeta), G_e(\theta)\}$, $G'_e(\zeta\theta) \leq \min\{G'_e(\zeta), G'_e(\theta)\}$.

Theorem 4.1. *If (G,π) and (S,ρ) are IFSI's $\mathbb{R}$, then $(G,\pi) \wedge (S,\rho)$ and $(G,\pi) \sqcap (S,\rho)$ are IFSI's $\mathbb{R}$.*

Proof. Let us take $(G,\pi) \wedge (S,\rho) = (R,\sigma)$ respectively, where $\sigma = \pi \times \rho$ and $R(\zeta,\theta) = G(\zeta) \cap S(\theta), \forall(\zeta,\theta) \in \sigma$. Since (G,π) and (S,ρ) are two IFSI's $\mathbb{R}$, for all $\zeta,\theta \in \mathbb{R}$ and $(x,y) \in \sigma$,

$$\begin{aligned}
R_{(x,y)}(\zeta+\theta) &= \min\{G_x(\zeta+\theta), S_y(\zeta+\theta)\} \\
&\geq \min\{\min\{G_x(\zeta), G_x(\theta)\}, \min\{S_y(\zeta), S_y(\theta)\}\} \\
&= \min\{R_{(x,y)}(\zeta), R_{(x,y)}(\theta)\} \\
R'_{(x,y)}(\zeta+\theta) &= \max\{G'_x(\zeta+\theta), S'_y(\zeta+\theta)\} \\
&\leq \max\{\max\{(G'_x(\zeta), G'_x(\theta)\}, \max\{S'_y(\zeta), S'_y(\theta)\}\} \\
&\leq \max\{R'_{(x,y)}(\zeta), R'_{(x,y)}(\theta)\} \\
R_{(x,y)}(\zeta\theta) &= \min\{G_x(\zeta\theta), S_y(\zeta\theta)\}
\end{aligned}$$

$$\begin{aligned}
&\geq \min\{\max\{G_x(\zeta), G_x(\theta)\}, \max\{S_y(\zeta), S_y(\theta)\}\} \\
&= \max\{\min\{G_x(\zeta), S_y(\zeta)\}, S_y(\theta)\}\} \\
&= \max\{R_{x,y}(\zeta), R_{x,y}(\theta)\} \\
R'_{(x,y)}(\zeta\theta) &= \max\{G'_x(\zeta+\theta), S'_y)(\zeta+\theta)\} \\
&\leq \max\{\min\{G'_x(\zeta), G'_x(\theta)\}, \min\{S'_y)(\zeta), S'_y(\theta)\} \\
&= \min\{\max\{G'_x)(\zeta), S'_y(\zeta)\}, \max\{G'_x(\theta), S'_y(\theta)\}\} \\
&= \min\{R'_{(x,y)}(\zeta), R'_{(x,y)}(\theta)\}.
\end{aligned}$$

Consequently, $(G,\pi) \wedge (S,\rho)$ is an IFSI's $\mathbb{R}$. Hence, $(G,\pi) \sqcap (S,\rho)$ is similarly proved. □

Theorem 4.2. *If (G,π) and (S,ρ) are IFSI's $\mathbb{R}$, then $(G,\pi) \cap (S,\rho)$ is an IFSI's $\mathbb{R}$.*

Proof. For any $\zeta, \theta \in \mathbb{R}$ and $e \in \sigma$. Examine the following scenarios:

Case 1. Let $e \in \pi - \rho$. Then

$$\begin{aligned}
R_e(\zeta+\theta) &= G_e(\zeta+\theta) \\
&\geq \min\{G_e(\zeta), G_e(\theta)\} \\
&= \min\{R_e(\zeta), R_e(\theta)\} \\
R'_e(\zeta+\theta) &= G'_e(\zeta+\theta) \\
&\leq \max\{G'_e(\zeta), G'_e(\theta)\} \\
&= \max\{R'_e(\zeta), R'_e(\theta)\} \\
R_e(\zeta\theta) &= G_e(\zeta\theta) \\
&\geq \max\{G_e(\zeta), G_e(\theta)\} \\
&= \max\{R_e(\zeta), R_e(\theta)\} \\
R'_e(\zeta\theta) &= G'_e(\zeta\theta) \\
&\leq \min\{G'_e(\zeta), G'_e(\theta)\} \\
&= \min\{R'_e(\zeta), R'_e(\theta)\}.
\end{aligned}$$

Case 2. Let $e \in \rho - \pi$. Then

$$\begin{aligned}
R_e(\zeta+\theta) &= S_e(\zeta+\theta) \\
&\geq \min\{S_e(\zeta), S_e(i)\} \\
&= \min\{R_e(\zeta), R_e(\theta)\} \\
R'_e(\zeta+\theta) &= S'_e(\zeta+\theta) \\
&\leq \max\{S'_e(\zeta), S'_e(i)\} \\
&= \max\{R'_e(\zeta), R'_e(\theta)\}
\end{aligned}$$

$$\begin{aligned} R_e(\zeta\theta) &= S_e(\zeta\theta) \\ &\geq \max\{S_e(\zeta), S_e(i)\} \\ &= \max\{R_e(\zeta), R_e(\theta)\} \\ R_e^{'}(\zeta\theta) &= S_e^{'}(\zeta\theta) \\ &\leq \min\{S_e^{'}(\zeta), S_e^{'}(i)\} \\ &= \min\{R_e^{'}(\zeta), R_e^{'}(\theta)\}. \end{aligned}$$

Case 3. Let $e \in \pi \cap \rho$. Then

$$\begin{aligned} R_e(\zeta+\theta) &= \min\{G_e(\zeta+\theta), S_e(\zeta+\theta)\} \\ &\geq \min\{\min\{G_e(\zeta), G_e(\theta)\}, \min\{S_e(\zeta), S_e(\theta)\}\} \\ &\geq \min\{\min\{G_e(\zeta), S_e(\zeta)\}, \min\{G_e(\theta), S_e(\theta)\}\} \\ &= \min\{R_e(\zeta), R_e(\theta)\} \\ R_e^{'}(\zeta+\theta) &= \max\{G_e^{'}(\zeta+\theta), S_e^{'}(\zeta+\theta)\} \\ &\leq \max\{\max\{G_e^{'}(\zeta), G_e^{'}(\theta)\}, \max\{S_e^{'}(\zeta), S_e^{'}(\theta)\}\} \\ &\leq \max\{\max\{G_e^{'}(\zeta), S_e^{'}(\zeta)\}, \max\{G_e^{'}(\theta), S_e^{'}(\theta)\}\} \\ &= \max\{R_e^{'}(\zeta), R_e^{'}(\theta)\}. \end{aligned}$$

Now, $R_e(\zeta\theta) \geq \max\{R_e(\zeta), R_e(\theta)\}$ and $R_e^{'}(\zeta\theta) \leq \min\{R_e^{'}(\zeta), R_e^{'}(\theta)\}$ are similarly proved. Consequently, $(G, \pi) \cap (S, \rho)$ is an IFSI's $\mathbb{R}$. □

Theorem 4.3. *If (G, π) and (S, ρ) are two IFSI's $\mathbb{R}$, then $(G, \pi) \circ (S, \rho)$ is an IFSI's $\mathbb{R}$.*

Proof. For any $\zeta, \theta \in \mathbb{R}$ and $e \in \pi \cup \rho$. Examine the following scenarios:

Case 1. Let $e \in \pi - \rho$. Then

$$\begin{aligned} (G\circ S)_e(\zeta+\theta) &= G_e(\zeta+\theta) \\ &\geq \min\{G_e(\zeta), G_e(\theta)\} \\ &= \min\{(G\circ S)_e(\zeta), (G\circ S)_e(\theta)\} \\ (G\circ S)_e^{'}(\zeta+\theta) &= G_e^{'}(\zeta+\theta) \\ &\leq \max\{G_e^{'}(\zeta), G_e^{'}(\theta)\} \\ &= \max\{(G\circ S)_e^{'}(\zeta), (G\circ S)_e^{'}(\theta)\} \\ (G\circ S)_e(\zeta\theta) &= G_e(\zeta\theta) \\ &\geq \max\{G_e(\zeta), G_e(\theta)\} \\ &= \max\{(G\circ S)_e(\zeta), (G\circ S)_e(\theta)\} \\ (G\circ S)_e^{'}(\zeta\theta) &= G_e^{'}(\zeta\theta) \\ &\leq \min\{G_e^{'}(\zeta), G_e^{'}(\theta)\} \end{aligned}$$

$$= \min\{(G\circ S)'_e(\zeta),(G\circ S)'_e(\theta)\}.$$

Case 2. Assume that $e\in\rho-\pi$. This instance resembles Case 1.

Case 3. Let $e\in\pi\cap\rho$. Then

$$\begin{aligned}(G\circ S)_e(\zeta) &= \sup_{\zeta=\zeta_1\zeta_2}\min\{G_e(\zeta_1),S_e(\zeta_2)\}\\ &\le \sup_{\zeta p=\zeta_1\zeta_2 p}\min\{G_e(\zeta_1 p),S_e(\zeta_2 p)\}\\ &\le \sup_{\zeta p=zt}\min\{G_e(z),S_e(t)\}\\ &= (G\circ S)_e(\zeta\theta).\end{aligned}$$

Similarly, we can write $(G\circ S)_e(\theta)\le(G\circ S)_e(\zeta\theta)$. Therefore, $(G\circ S)_e(\zeta\theta)\ge\max\{G\circ S)_e(\zeta),G\circ S)_e(\theta)\}$. Also,

$$\begin{aligned}(G\circ S)'_e(\zeta) &= \inf_{\zeta=\zeta_1\zeta_2}\max\{G'_e(\zeta_1),S'_e(\zeta_2)\}\\ &\ge \inf_{\zeta p=\zeta_1\zeta_2 p}\max\{G'_e(\zeta_1 p),S'_e(\zeta_2 p)\}\\ &\ge \inf_{\zeta p=zt}\max\{G'_e(z),S'_e(t)\}\\ &= (G\circ S)'_e(\zeta\theta).\end{aligned}$$

Similarly, we can write $(G\circ S)'_e(p)\ge(G\circ S)'_e(\zeta\theta)$. Hence, $(G\circ S)'_e(\zeta\theta)\le\min\{(G\circ S)'_e(\zeta),(G\circ S)'_e(\theta)\}$. Therefore, $(G,\pi)\circ(S,\rho)$ is an IFSI's $\mathbb{R}$. □

5. Idealistic Intuitionistic Fuzzy Soft Boolean Rings

Definition 5.1. *Let (G,π) be an IFSBR's $\mathbb{R}$. Then (G,π) is referred to as an IIFSBR's $\mathbb{R}$ if $G(e)$ is an IFI of $\mathbb{R}$, for all $e\in supp(G,\pi)$, i.e., $\forall\zeta,\theta,\vartheta\in\mathbb{R}$,*
(i) $G_e(\zeta+\theta)\ge\min\{G_e(\zeta),G_e(\theta)\}$, $G'_e(\zeta+\theta)\le\max\{G'_e(\zeta),G'_e(\theta)\}$
(ii) $G_e(\zeta\theta)\ge\min\{G_e(\zeta),G_e(\theta)\}$, $G'_e(\zeta\theta)\le\max\{G'_e(\zeta),G'_e(\theta)\}$
(iii) $G_e(-\zeta)\ge G_e(\zeta)$, $G'_e(-\zeta)\le G'_e(\zeta)$
(iv) $G_e(\zeta)=G_e(\theta+\zeta-\theta)$, $G'_e(\zeta)\le G'_e(\theta+\zeta-\theta)$
(v) $G_e[(\zeta+\vartheta)\theta-\zeta\theta]\ge G_e(\vartheta)$, $G'_e[(\zeta+\vartheta)\theta-\zeta\theta]\le G'_e(\vartheta)$.

Example 5.1. *Let the binary operations $+$ and $\cdot$ be applied to the nonempty set $\mathbb{R}=\{0,g,s,r\}$ in the following terms:*

$+$	0	g	s	r
0	0	g	s	r
g	g	0	r	s
s	s	r	0	g
r	r	s	g	0

$\cdot$	0	g	s	r
0	0	0	0	0
g	0	g	0	g
s	0	0	s	s
r	0	g	s	r

Then $(\mathbb{R},+,\cdot)$ is a BR. Define an IFSS (G,π) over $\mathbb{R}$ by letting $E=\{e_1,e_2,e_3\}$ be the parameters,

$G(e_1) = \{(0,0.2),(g,0.2),(s,0.1),(r,0.1)\}, G^{'}(e_1) = \{(0,0.3),(g,0.4),(s,0.7),(r,0.7)\}$
$G(e_2) = \{(0,0.4),(g,0.4),(s,0.3),(r,0.3)\}, G^{'}(e_2) = \{(0,0.4),(g,0.5),(s,0.7),(r,0.7)\}$
$G(e_3) = \{(0,0.3),(g,0.3),(s,0.2),(r,0.2)\}, G^{'}(e_3) = \{(0,0.3),(g,0.5),(s,0.8),(r,0.7)\}$

Then (G,π) *is an IIFSBR's* $\mathbb{R}$.

Theorem 5.1. *If* (G,π) *and* (S,ρ) *are two IIFSBR's* $\mathbb{R}$, *then* $(G,\pi)\sqcap(S,\rho)$ *is an IIFSBR's* $\mathbb{R}$ *if it is non-null.*

Proof. Let $(R,\sigma) = (G,\pi)\cap(S,\rho)$, where $\forall e \in \sigma, R_e = G_e \cap S_e$. Suppose (R,σ) is non-null, so there exists $e \in Supp(R,\sigma)$ such that $R_e = G_e \cap S_e \neq \emptyset$. That is, $R_e(\zeta) = G_e(\zeta)\wedge S_e(\zeta)$ and $R_e^{'}(\zeta) = G_e^{'}(\zeta)\vee S_e^{'}(\zeta), \forall\zeta \in \mathbb{R}$.
Since (G,π) is an IIFSBR's $\mathbb{R}$, we have
(i) $G_e(\zeta+\theta) \geq \min\{G_e(\zeta), G_e(\theta)\}$, $G_e^{'}(\zeta+\theta) \leq \max\{G_e^{'}(\zeta), G_e^{'}(\theta)\}, \forall\zeta,\theta \in \mathbb{R}$
(ii) $G_e(\zeta\theta) \geq \min\{G_e(\zeta), G_e(\theta)\}$, $G_e^{'}(\zeta\theta) \leq \max\{G_e^{'}(\zeta), G_e^{'}(\theta)\}, \forall\zeta,\theta \in \mathbb{R}$
(iii) $G_e(-\zeta) \geq G_e(\zeta)$, $G_e^{'}(-\zeta) \leq G_e^{'}(\zeta), \forall\zeta \in \mathbb{R}$
(iv) $G_e(\zeta) = G_e(\theta+\zeta-\theta)$, $G_e^{'}(\zeta) \leq G_e^{'}(\theta+\zeta-\theta), \forall\zeta,\theta \in \mathbb{R}$
(v) $G_e[(\zeta+\vartheta)\theta-\zeta\theta] \geq G_e(\vartheta)$, $G_e^{'}[(\zeta+\vartheta)\theta-\zeta\theta] \leq G_e^{'}(\vartheta), \forall\zeta,\theta,\vartheta \in \mathbb{R}$,
and also we have the same properties for fuzzy sets S_e and $S_e^{'}$. For any $\zeta,\theta,\vartheta \in \mathbb{R}$, we obtain

$$\begin{aligned}
R_e(\zeta+\theta) &= (G_e \wedge S_e)(\zeta+\theta)\\
&= G_e(\zeta+\theta)\wedge S_e(\zeta+\theta)\\
&\geq (G_e(\zeta)\wedge G_e(\theta))\wedge(S_e(\zeta)\wedge S_e(\theta))\\
&= (G_e(\zeta)\wedge S_e(\zeta))\wedge(G_e(\theta)\wedge S_e(\theta))\\
&= (G_e\wedge S_e)(\zeta)\wedge(G_e\wedge S_e)(\theta)\\
&= R_e(\zeta)\wedge R_e(\theta).
\end{aligned}$$

Similarly, we get

$$R_e^{'}(\zeta+\theta) \leq R_e^{'}(\zeta)\vee R_e^{'}(\theta).$$

Now, we have that

$$\begin{aligned}
R_e[(\zeta+\vartheta)\theta-\zeta\theta] &= G_e[(\zeta+\vartheta)\theta-\zeta\theta]\wedge S_e[(\zeta+\vartheta)\theta-\zeta\theta]\\
&\geq G_e(\vartheta)\wedge S_e(\vartheta)\\
&= R_e(\vartheta)\\
R_e^{'}[(\zeta+\vartheta)\theta-\zeta\theta] &= G_e^{'}[(\zeta+\vartheta)\theta-\zeta\theta]\vee S_e^{'}[(\zeta+\vartheta)\theta-\zeta\theta]\\
&\leq G_e^{'}(\vartheta)\vee S_e^{'}(\vartheta)\\
&= R_e^{'}(\vartheta).
\end{aligned}$$

The other equalities are proved similarly for each $\zeta,\theta \in \mathbb{R}$. Hence, $(G,\pi)\sqcap(S,\rho)$ is an IIFSBR's $\mathbb{R}$, as desired. □

Theorem 5.2. *If (G,π) and (S,ρ) are IIFSBR's $\mathbb{R}$, then $(G,\pi)\wedge(S,\rho)$ is an IIFSBR's $\mathbb{R}$.*

Proof. Let $(G,\pi)\wedge(S,\rho)=(R,\sigma)$, where $R_e=G_e\cap S_e, \forall e\in\sigma$. Let $e\in Supp(R,\sigma)$. Then $R_e=G_e\cap S_e\neq\emptyset$. For simplicity, we only show that $R_e(\zeta\theta)\geq R_e(\theta)$ and $R'_e(\zeta\theta)\leq R'_e(\theta), \forall\zeta,\theta\in\mathbb{R}$. For any $\zeta,\theta\in\mathbb{R}$, we have

$$\begin{aligned}
R_e(\zeta\theta) &= G_e(\zeta\theta)\wedge S_e(\zeta\theta)\\
&\geq G_e(\theta)\wedge S_e(\theta)\\
&= G_e(\theta)\\
R'_e(\zeta\theta) &= G'_e(\zeta\theta)\vee S'_e(\zeta\theta)\\
&\leq G'_e(\theta)\vee S'_e(\theta)\\
&= R'_e(\theta).
\end{aligned}$$

The other equalities are easily satisfied. Hence, $(G,\pi)\wedge(S,\rho)$ is an IIFSBR's $\mathbb{R}$. □

6. Conclusion

This research explores the integration of intuitionistic fuzzy soft sets (IFSSs) into the algebraic framework of Boolean rings (BRs), presenting the concept of intuitionistic fuzzy soft Boolean rings (IFSBRs). It builds on previous advancements in fuzzy set theory and soft set theory, highlighting their applications in managing uncertainty across disciplines. The study systematically defines operations on IFSBRs, including union, intersection, AND, and OR, extending classical Boolean ring properties to an intuitionistic fuzzy soft context. Key findings demonstrate that these operations preserve the IFSBR structure, offering insights into their algebraic and operational behaviours. Additionally, the paper introduces the notion of intuitionistic fuzzy soft ideals (IFSIs) and investigates their compatibility with IFSBRs, providing theorems that affirm structural integrity under various operations. The work concludes with a theoretical foundation for advancing soft set theory and its potential application to other algebraic structures, emphasizing the role of intuitionistic fuzzy logic in enhancing the computational modeling of uncertain data. This comprehensive approach aligns intuitionistic fuzzy soft sets with practical applications, supporting decision-making processes in areas where data imprecision is prevalent.

Acknowledgment: This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] K.T. Atanassov, Intuitionistic Fuzzy Sets, in: Intuitionistic Fuzzy Sets, Physica-Verlag HD, Heidelberg, 1999: pp. 1–137. https://doi.org/10.1007/978-3-7908-1870-3_1.

[2] B. Davvaz, P. Corsini, V. Leoreanu-Fotea, Atanassov's Intuitionistic (S,T)-Fuzzy n-Ary Sub-Hypergroups and Their Properties, Inf. Sci. 179 (2009), 654–666. https://doi.org/10.1016/j.ins.2008.10.023.

[3] B. Davvaz, W.A. Dudek, Y.B. Jun, Intuitionistic Fuzzy H_ν-Submodules, Inf. Sci. 176 (2006), 285-300. https://doi.org/10.1016/j.ins.2004.10.009.

[4] B. Davvaz, S.K. Majumder, Atanassov's Intuitionistic Fuzzy Interior Ideals of Γ-Semigroups, UPB Sci. Bull. Ser. A, 73 (2011), 45-60.

[5] S.K. De, R. Biswas, A.R. Roy, Some Operations on Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 114 (2000), 477–484. https://doi.org/10.1016/S0165-0114(98)00191-2.

[6] W. Dudek, B. Davvaz, Y. Jun, On Intuitionistic Fuzzy Sub-Hyperquasigroups of Hyperquasigroups, Inf. Sci. 170 (2005), 251–262. https://doi.org/10.1016/j.ins.2004.02.025.

[7] E. İnan, M.A. Öztürk, Fuzzy Soft Rings and Fuzzy Soft Ideals, Neural Comput. Appl. 21 (2012), 1–8. https://doi.org/10.1007/s00521-011-0550-5.

[8] U.B. Madhavi, V.J. Sree, D. Ramesh, B. Satyanarayana, Intuitionistic Fuzzy Translations of Implicative Ideals of BCK-Algebra, J. Gujarat Res. Soc. 21 (2019), 528-533.

[9] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (2001), 589-602.

[10] P.K. Maji, R. Biswas, A.R. Roy, Intuitionistic Fuzzy Soft Sets, J. Fuzzy Math. 9 (2001), 677-692.

[11] P.K. Maji, R. Biswas, A.R. Roy, Soft Set Theory, Comput. Math. Appl. 45 (2003), 555–562. https://doi.org/10.1016/S0898-1221(03)00016-6.

[12] D. Molodtsov, Soft Set Theory–First Results, Comput. Math. Appl. 37 (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5.

[13] D. Ramesh, K.K. Rao, R.D. Prasad, N. Srimannarayana, B. Satyanarayana, Translations of Intuitionistic Fuzzy Subalgebras in BF-Algebras, Adv. Math.: Sci. J. 9 (2020), 8837–8844. https://doi.org/10.37418/amsj.9.10.107.

[14] S.R. Gadde, Pushpalatha. Kolluru, B.P. Munagala, A Note on Soft Boolean Near-Rings, AIP Conf. Proc. 2707 (2023), 020013. https://doi.org/10.1063/5.0148444.

[15] S.R. Gadde, P. Kolluru, N. Thandu, Soft Intersection Boolean Near-Rings with Its Applications, AIP Conf. Proc. 2707 (2023), 020012. https://doi.org/10.1063/5.0148443.

[16] G.S. Rao, K. Pushpalatha, Ramesh D., S.V.M. Sarma, G. Vijayalakshmi, A Study on Boolean like Algebras, AIP Conf. Proc. 2375 (2021), 020018. https://doi.org/10.1063/5.0066293.

[17] G.S. Rao, D. Ramesh, A. Iampan, B. Satyanarayana, Fuzzy Soft Boolean Rings, Int. J. Anal. Appl. 21 (2023), 60. https://doi.org/10.28924/2291-8639-21-2023-60.

[18] G.S. Rao, R. Dasari, A. Iampan, B. Satyanarayana, Fuzzy Soft Boolean Near-Rings and Idealistic Fuzzy Soft Boolean Near-Rings, ICIC Express Lett. 18 (2024), 677-684. https://doi.org/10.24507/icicel.18.07.677.

[19] G.S. Rao, D. Ramesh, and B. Satyanarayana, $(\in, \in \vee q_k)$-Fuzzy soft Boolean near rings, Asia Pac. J. Math. 10 (2023), 50 https://doi.org/10.28924/APJM/10-50.

[20] A.R. Roy, P.K. Maji, A Fuzzy Soft Set Theoretic Approach to Decision Making Problems, J. Comput. Appl. Math. 203 (2007), 412–418. https://doi.org/10.1016/j.cam.2006.04.008.

[21] B. Satyanarayana, D. Ramesh, Y.P. Kumar, Interval Valued Intuitionistic Fuzzy Homomorphism of BF-Algebras, Math. Theory Model. 3 (2013), 15-23.

[22] B. Satyanarayana, D. Ramesh, R.D. Prasad, On Interval-Valued Intuitionistic Fuzzy Ideals of BF-Algebras, J. Comput. Math. Sci. 3 (2012), 83-96.

[23] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.