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Positive Definite Kernels and Radial Distributions on the Euclidean Motion Group

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Abstract. Let $G = \mathbb{R}^2 \rtimes \mathbb{T}$ be the Euclidean motion group and let $K(\lambda, t) = I_0(\lambda)\delta(t)$ be a distribution on G, where $I_0(\lambda)$ is the Bessel function of order zero and $\delta(t)$ is the Dirac measure on $SO(2) \cong \mathbb{T}$, the circle group. In this work, it is proved, among other things, that the distribution $K(\lambda, t)$ is tempered, positive definite, bounded and radial. Furthermore, a description of Levy-Schoenberg Kernels on the homogenous space of $SE(2)$ is presented.

1. Introduction

Radial distributions on a group G that is locally compact is a probability distribution that depends on the radial distance of $g \in G$ from the identity of G. It is widely applied in modeling uncertain motion in robotics and computer vision, estimating motion distributions for visual tracking and also analysing motion related signals.

In this research, a kind of radial distribution on $SE(2)$ obtained as the product of Bessel function of order zero I_0 and the dirac function on $SE(2)$ is studied. It is demonstrated that this distribution is tempered, positive definite and bounded. I_0, used in defining the radial distribution on $SE(2)$, is obtained by solving the Laplace-Beltrami operator $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ radially using the method of separation of variable (see [3]). An explicit form of Levy-Schoenberg kernel on the the homogenous space $R^2 \cong SE(2)/SO(2)$ is given.

Preliminaries concerning the Euclidean motion group, its representation and invariant differential operators are presented in section two. Spaces of distributions on $SE(2)$ are presented in section three. It is also proved in this section that the Schwartz space of $SE(2)$ is a Frechect space

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and the convolution of functions is continuous in the Schwartz space of $SE(2)$. In section four, a radial distribution on $SE(2)$ is presented and is also shown to be tempered, bounded and positive definite. Lastly, a description of positive definite Levy-Schoenberg kernel on $SE(2)/SO(2)$ is presented in section five. These kernels are obtained by finding functions on G that are normalized, real-valued, K-spherical and continuous imbeddable positive definite functions on G.

2. Preliminaries

2.1 Euclidean Motion Group. The Euclidean Motion group is a non-compact and non-commutative solvable Lie group realised as a semi direct product of the additive group $\mathbb{R}^n$ with the Orthogonal group $O(n)$. This means that if $S(n)$ denotes the group, then

$$S(n) = \mathbb{R}^n \rtimes O(n).$$

The special Euclidean motion group is the semi direct product of $\mathbb{R}^n$ with the special orthogonal group, $SO(n)$. That is,

$$SE(n) = \mathbb{R}^n \rtimes SO(n),$$

where $SO(n) = SL(n) \cap O(n)$.

SE(n) is also called group of transformation of the Euclidean plane. Henceforth, SE(n) is considered for this research when $n = 2$. Elements of $SE(2)$ are given by $g = (x, \alpha) \in SE(2)$, where $\alpha \in SO(2)$ and $x \in \mathbb{R}^2$. For any $g = (x_1, \alpha_1)$ and $h = (x_2, \alpha_2)$, the group law of $SE(2)$ is given as

$$gh = (x_1, \alpha_1)(x_2, \alpha_2) = (x_1 + \alpha_1 x_2, \alpha_1 \alpha_2)$$

and the inverse g^{-1} is given as ([5])

$$g^{-1} = (-\alpha_1^T, x_1 \alpha_1^T).$$

Elements of $SE(2)$ may be identified as a 3×3 homogeneous transformation matrix of the form

$$H(g) = \begin{pmatrix} \alpha & x \\ 0^T & 1 \end{pmatrix},$$

where $0^T = (0, 0)$. $SE(2) = (\mathbb{R}^2 \rtimes SO(2)) \cong M \subseteq GL(3, \mathbb{R})$, where M is a subgroup of $GL(3, \mathbb{R})$. An element of $SE(2)$ may also be presented in rectangular coordinate as follows.

$$g(x_1, x_2, \phi) = \begin{pmatrix} cos\phi & -sin\phi & x_1 \\ sin\phi & cos\phi & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \phi \in [0, 2\pi], (x_1, x_2) \in \mathbb{R}^2$$

$SE(2)$ is a non - compact and non- commutative Solvable Lie group [5]. Solvability of $SE(2)$ implies that there exist a sequence of closed subgroup $G_0 = G, G_1, ..., G_n, G_{n+1} = \{e\}$ such that G_{n+1} is normal in G_k and G_k/G_{k+1} is abelian [12]. Since all abelian and solvable Lie groups are amenable, it means that the Euclidean motion group is also an amenable group. $SE(n)$ is a group of affine maps induced by orthogonal transformation. It is also called a group of rigid motions and plays a fundamental role in robotic, dynamics and motion planning. The universal covering group of $SE(2)$ is the semi direct product group $\mathbb{R}^2 \rtimes \mathbb{R}$ whose multiplication is defined as

$$(x_1, \alpha_1)(x_2, \alpha_2) = (x_1 + e^{it} x_2, \alpha_1 + \alpha_2)$$

and its covering map is defined as

$$(x, \alpha) \mapsto (x, e^{it}).$$

For $x_1, x_2 \in \mathbb{R}^2$ and $\alpha \in SO(2)$, the invariant measure on $SE(n)$ is obtained as the product of Lebesque measure on $\mathbb{R}^2$ and the Haar measure on $SO(2)$ given as (see [2] and [4])

$$d\mu[(x, \alpha)] = dx_1 dx_2 d\alpha.$$

Let $H = L^2(SE(2), \mu)$ be the Hilbert space of square integrable functions on $SE(n)$. For $u \in H$ and $x, x' \in \mathbb{R}^2$, the right and left regular representations T^R and T^L of SE(2) are defined respectively as

$$\begin{aligned}(T^R_{(x_2,\alpha_2)} u)[(x_1, \alpha_2)] &= u[(x_1, \alpha_1)(x_2, \alpha_2)] \\ &= u[(x_1 + x_2\alpha_1, \alpha_1 + \alpha_2)]\end{aligned}$$

and

$$\begin{aligned}(T^L_{(x_2,\alpha_2)} u)[(x_1, \alpha_1)] &= u[(x_2, \alpha_1)^{-1}(x_1, \alpha_1)] \\ &= u[(-x_{2-\alpha'} + x_{12\pi-\alpha'}, 2\pi - \alpha_2 + \alpha_1)].\end{aligned}$$

Let $g(t_i)$ be the one-parameter subgroups of $SE(2)$ generated by X_i, $i = 1, 2, 3.$. Then

$$X_i u = \lim_{t \to 0} \left(\frac{T^L_{g(t_i)} - I}{t_i} u \right)$$

where u is an element of the Garding domain [2]. Explicitly,

$$X_1 = -\frac{\partial}{\partial x_1},$$

$$X_2 = -\frac{\partial}{\partial x_2},$$

$$X_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial \alpha}.$$

The generators of the left invariant Lie algebra of G are given as

$$Y_1 = cos\alpha \frac{\partial}{\partial x_1} + sin\alpha \frac{\partial}{\partial x_2},$$

$$Y_2 = -sin\alpha \frac{\partial}{\partial x_1} + cos\alpha \frac{\partial}{\partial x_2},$$

$$Y_3 = \frac{\partial}{\partial \alpha},$$

and they obey the following commutation relations $[Y_1, Y_2] = 0$, $[Y_2, Y_3] = Y_1$ and $[Y_3, Y_1] = Y_2$, where $[A, B]$ is the standard Lie bracket defined as $[A, B] = AB - BA$.

The spherical function for $SE(2)$ is the Bessel function of order zero I_0 and its integral representation is given as

$$J_0(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\lambda cos\phi} d\phi$$

. This is obtained by solving the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2.1}$$

on $SE(2)$ radially by separation of variable method.

3. Spaces of Distributions on SE(2)

In this section, descriptions of spaces of distributions and their respective topologies are presented. Further more, Fourier transform of functions on $SE(2)$ is discussed.

3.1 The space $C^{\infty}(G)$. Given a solvable Lie group G endowed with invariant measure $d\mu(g)$, and $\mathfrak{g}$ its Lie algebra. Lets denote by m the dimension of $\mathfrak{g}$. Fix $\{X_1, ..., X_m\}$ a basis of $\mathfrak{g}$. To each $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$, we put $|\alpha| = \alpha_1 + ... + \alpha_n$ and associate a differential operator X^{α}, which is left invariant, on G acting on $f \in C^{\infty}(G)$, the space $C^{\infty}(G)$ of infinitely differentiable functions on G, by

$$X^{\alpha} f(g) = \frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} ... \frac{\partial^{\alpha_m}}{\partial t_m^{\alpha_m}} f(g\, exp(t_1 X_1) ... exp(t_m X_m))|_{t_1 = ... = t_m = 0.}.$$

The space $C^{\infty}(G)$ may be given a topology defined by a system of seminorms specified as

$$|f|_{\alpha,m} = Sup_{|\alpha| \leq m} |X^{\alpha} f(g)|.$$

With this topology, C^{∞}(G) is metrizable, locally convex and complete, hence, it is a Frechet space. This Frechet space may be denoted as $\xi(G)$

3.2 The space $C_c^{\infty}(G)$**.** This space $C_c^{\infty}(G)$ is the space of complex-valued C^{∞} function on G with compact support. For any $\epsilon > 0$, put

$$B_{\epsilon} = \left\{(\xi, \theta) \in G : \|\xi\| \leq \epsilon\right\}$$

and

$$\mathfrak{D}_{\epsilon} = \mathfrak{D}(B_{\epsilon}) = \left\{f \in C_c^{\infty}(G) : f(\xi, \theta) = 0, if\ \|\xi\| > \epsilon\right\}.$$

Then $\mathfrak{D}(B_{\epsilon})$ is a Frechet space with respect to the family semi norms defined as

$$\left\{P_{\alpha}(f) = \|D^{\alpha} f\|_{\infty} : \alpha \in \mathbb{N}^3\right\}.$$

$\mathfrak{D}(G) = \bigcup_{n=1}^{\infty} \mathfrak{D}(B_n)$ is topologised as the strict inductive limit of $\mathfrak{D}(B_n)$. A linear functional on the topological vector space $\mathfrak{D}(G)$ that is continuous is known as a distribution on G. Then $\mathfrak{D}'(G)$ is the space of distribution on G.

Given a manifold M and a distribution T, T is said to vanish on a subset $V \subset M$, which is open, if $T = 0$. Let $\{U_{\alpha}\}_{\alpha \in \omega}$ represents the collection of all open sets on which T vanishes and let U stand for the union of $\{U_{\alpha}\}_{\alpha \in \omega}$. $M - U$, regarded as the complement of M, is the support of T. We denote $\xi'(G)$ a distributions space with compact support.

3.3 The Schwartz space $\mathcal{S}(G)$. Consider the Euclidean motion group $SE(2)$ realised as $\mathbb{R} \rtimes \mathbb{T}$ where $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$. If we choose a system of coordinates ([10]) (x, y, θ) on G with $x, y \in \mathbb{R}$ and $\theta \in \mathbb{T}$, then a complex - valued C^∞ function f on $G = SE(2)$ is called rapidly decreasing if for any $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ we have

$$p_{N,\alpha}(f) = Sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2} \mid (1 + \|\xi\|^2)^N (D^\alpha f)(\xi, \theta) \mid < +\infty, \tag{3.1}$$

where

$$D^\alpha = \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \left(\frac{\partial}{\partial y}\right)^{\alpha_2} \left(\frac{\partial}{\partial \theta}\right)^{\alpha_3},$$

$(\alpha = (\alpha_1, \alpha_2, \alpha_3); \xi = (x, y))$. The space of all rapidly decreasing functions on G is denoted by $\mathcal{S} = \mathcal{S}(G)$. Then $\mathcal{S}$ is a Frechet space in the topology given by the family of semi-norms $\{P_{N,\alpha} : N \in \mathbb{N}, \alpha \in \mathbb{N}^3\}$. This result is stated formally with prove in the next proposition,

3.4 Proposition. The Schwartz space $\mathcal{S}(SE(2))$ is a Frechet space.

Proof. Let us denote the system of seminorm defined in (5) by $\|.\|^\infty_{n,\alpha}$, $n \in N$, $\alpha \in N^m$. $\|.\|^\infty_{n,\alpha}$ is countable and separable on $\mathcal{S}(SE(2))$. This is because $\|\varsigma\|^\infty_{0,0} = \|\varsigma\|^\infty_{L^1(G)} = 0 \Rightarrow \varsigma = 0$. This separability condition defines a locally convex topology on $\mathcal{S}(SE(2))$. Next is to prove that $\mathcal{S}(SE(2))$ is a Frechet space. In order to do this, we need to show that it is complete. To this end, let $\{\varsigma\}_{n\in N} \subset \mathcal{S}(SE(2))$ be a sequence that is Cauchy in nature for the semi norms $\|.\|^\infty_{n,\alpha}$. Let X^α be as defined in 2.1, $X^\alpha\varsigma_n$ converges to a bounded function $\varsigma_{n,\alpha}$ uniformly for every $n \in N$ and $\alpha \in N^m$. Next is for us to prove that

$$\varsigma_{n,\alpha} = X^\alpha \varsigma_{0,0}\; n \in N,\; \alpha \in N^m. \tag{3.2}$$

The prove of (6) is to establish that $\varsigma_{0,0} \in \mathcal{S}(SE(2))$ and $\varsigma_n \to \varsigma_{0,0}$ in $\mathcal{S}(SE(2))$ and by implication, it will mean that $\mathcal{S}(SE(2))$ is complete. Therefore, let us prove that (6) is true. For $n = 0$ and α of length one, say $\alpha = \alpha_i$ with all coordinates equal to zero but the i^{th} equal to one, we have for all $t \in N$

$$\varsigma_n(gexp(tX_i) = \varsigma_n(g) + \int_0^t X_i\varsigma_n(\eta X_i))d\eta, \tag{3.3}$$

when $n \to \infty$ (7) becomes

$$\varsigma_{0,0}(gexp(tX_i)) = \varsigma_{0,0}(g) + \int_0^t \varsigma_{0,\alpha_i}(gexp(\eta X_i))d\eta. \tag{3.4}$$

If we differentiate (8) with respect to t at 0, it shows that $\varsigma_{0,0}$ is continuously differentiable in the direction X_i with

$$X_i\varsigma_{0,0}(g) = \varsigma_{0,\alpha_i}g.$$

If this argument is repeated, it shows that $\varsigma_{0,0} \in C^\infty(G)$ with $X^\alpha\varsigma_{0,0} = \varsigma_{0,\alpha}$, $\forall \alpha \in N^m$. This means that for all $n \in N$ and $\alpha \in N^m$, $X^\alpha\varsigma_n$ converges pointwise to $X^\alpha\varsigma_{0,0}$. By hypothesis, $X^\alpha\varsigma_n$ converges to $\varsigma_{n,\alpha}$, therefore $\varsigma_{n,\alpha} = X^\alpha\varsigma_{0,0}$. Since $\varsigma_{n,\alpha} \in L^\infty(G)$, this shows that $\varsigma_{0,0} \in \mathcal{S}(SE(2))$ and that ς_n converges to $\varsigma_{0,0}$ in $\mathcal{S}(SE(2))$. Hence, $\mathcal{S}(SE(2))$ is complete and therefore Frchet. □

The space $\mathcal{S}'(G)$ of (continuous) linear functionals on $S(G)$ is referred to as the space of tempered distributions on $G = SE(2)$. This space can be topologised by strong dual topology, which is defined as the topology of uniform convergence on the bounded subsets of $\mathcal{S}(G)$ generated by the seminorms $p_\varphi(u) = |u(\varphi)|$, where $u : \mathcal{S}(G) \to \mathbb{R}$ and $\varphi \in \mathcal{S}(G)$.

Let $f_1, f_2 \in \mathcal{S}(G)$ or $L^2(G)$. The convolution of f_1 and f_2 is defined as

$$\begin{aligned}(f_1 * f_2)(g) &= \int_G f_1(h) f_2(h^{-1}g) d\mu_G(h) \\ &= \int_G f_1(gh) f_2(h^{-1}) d\mu_G(h).\end{aligned}$$

The convolution operation obeys the associativity property

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3),$$

whenever all the integrals are absolutely convergent (cf: [5,10,11]).

The next result shows continuity of convolution of functions in $\mathcal{S}(SE(2))$. It is presented below as proposition 3.5 with proof.

3.5 Proposition. Convolution of functions is continuous from $\mathcal{S}(SE(2)) \times \mathcal{S}(SE(2))$ to $\mathcal{S}(SE(2))$

Proof. Let us recall that the convolution of two functions on $SE(2)$, provided the integral converges, is defined as

$$\begin{aligned}(f_1 * f_2)(g) &= \int_G f_1(h) f_2(h^{-1}g) d\mu_G(h) \\ &= \int_G f_1(gh) f_2(h^{-1}) d\mu_G(h).\end{aligned}$$

Since the differential operators X^α are left invariant, they act on the convolution as follows

$$X^\alpha (f_1 * f_2) = f_1 * X^\alpha f_2,$$

therefore,

$$|X^\alpha (f_1 * f_2)(g)| = |\int_G X^\alpha f_1(h) f_2(h^{-1}g) dh|$$

We note that $SE(2)$ is unimodular (see [6], p.326), this means

$$\int_G f(hg) dg = \int_G f(gh) dg = \int_G f(g^{-1}) dg = \int_G f(g) dg.$$

So, by putting $g = hg$, we get

$$\begin{aligned}|X^\alpha (f_1 * f_2)(g)| &= |\int_G X^\alpha f_1(h) f_2(h^{-1}(hg) dh| \\ &= |\int_G X^\alpha f_1(h) f_2(g) dh| \\ &\le \int_G |X^\alpha f_1(h) f_2(g) dh|.\end{aligned}$$

We know that

$$X^{\alpha}(f_1 * f_2)(g) = f_1 * X^{\alpha} f_2(g).$$

But

$$f_1 * X^{\alpha} f_2(h) = \int_G f_1(h) X^{\alpha}(h^{-1}g) dh.$$

Therefore

$$\begin{aligned} |X^{\alpha}(f_1 * f_2)(g)| &\leq \int_G |f_1(h)||X^{\alpha} f_2(g)| dh \\ &\leq \int_G |f_1(h)| dh \frac{C}{|(1+\|\xi\|^2)^m|}, \end{aligned}$$

because $|X^{\alpha} f_2(g)| \leq \frac{C_{\alpha,N}}{|(1+\|\xi\|^2)^m|}$. Since $\|\xi\|$ is a positive real constant, we may put $Q_N = |(1+\|\xi\|^2)^N|$, so that

$$\begin{aligned} |X^{\alpha}(f_1 * f_2)(g)| &\leq \int_G |f_1(h)||X^{\alpha} f_2(g)| dh \\ &\leq \int_G |f_1(h)| dh \frac{C}{|(1+\|\xi\|^2)^N|} \\ &\leq \frac{C}{Q_N} \int_G |f_1(h)| dh \\ &= \frac{C}{Q_N} \|f_1\|_{L^1(G)} \end{aligned}$$

$$\begin{aligned} |(1+\|\xi\|^2)^N X^{\alpha}(f_1 * f_2)(\xi,\theta)| &= |(1+\|\xi\|^2)^N||X^{\alpha}(f_1 * f_2)(\xi,\theta)| \\ &\leq C\|f\|_{L^1(G)} < +\infty. \end{aligned}$$

On taking supremum, we have

$$P_{\alpha,N}(f_1 * f_2) = sup_{\theta \in \mathbb{T}, \xi \in \mathbb{R}^2} |(1+\|\xi\|^2)^N X^{\alpha}(f_1 * f_2)(\xi,\theta)| < +\infty.$$

4. Radial Distribution on SE(2)

4.1 Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called radial if $\exists\, \phi$ defined on $[0,\infty)$ in such a way that $f(x) = \phi(|x|)$, $\forall\, x \in \mathbb{R}^n$. For a transformation ρ on $\mathbb{R}^n$, ρ is called orthogonal if $\exists$ linear operator on $\mathbb{R}^n$ such that $\langle \rho x, \rho y \rangle = \langle x, y \rangle\ \forall\, x, y \in \mathbb{R}^n$.

A Schwartz function φ is called radial if for all $A \in O(n)$ (that is to say, for all rotations on R^n) the following equation holds

$$\varphi = \varphi o A.$$

A collection of all radial Schwartz functions is denoted as $\mathcal{S}_{rad}(R)$ and $\mathcal{S}'(R)$ the space of tempered distributions on R^n. A distribution $u \in \mathcal{S}'(R)$ is called radial if for all $A \in O(n)$, we have

$$u = u o A.$$

This means that for all Schwartz functions φ on R^n, we have

$$\langle u, \varphi \rangle = \langle u, \varphi o A \rangle$$

and $\mathcal{S}'_{rad}(R)$ is the space of all radial tempered distributions on R^n.

4.2 Definition. A positive definite function

$$f : G \to \mathbb{C}$$

satisfies the following inequality

$$\sum_{i,j=1^m} \alpha_i \overline{\alpha_k} f(g_i^{-1} g_k) \geq 0 \tag{4.1}$$

for all subsets $\{g_1, ..., g_m\} \in G$ and all sequences $\{\alpha_1, ..., \alpha_m\} \in \mathbb{C}$. The integral analogue of the inequality (5) is given by

$$\int_G \int_G f(g_i^{-1} g_k) \varphi(g_i) \varphi(g_k) dg_i dg_k \geq 0 \tag{4.2}$$

where φ ranges over $L^1(G)$ or $C_c(G)$. If f is a continuous functions, (9) and (10) are equivalent.

A spherical function that also satisfies (5) is referred to as positive definite spherical function. Let $\widehat{(G,K)}$ stand for the set of spherical functions on G and let $\widehat{(G,K)}_+$ denotes the subset of $\widehat{(G,K)}$ that is positive definite. The set $\widehat{(G,K)}_+$ is isomorphic with $\mathbb{R}^+$. A measure π on $\widehat{(G,K)}$ such that for $f \in L^1(G,K)$ the plancherel theorem holds, that is

$$\int_{(K\backslash G/K)} |f(a)|^2 d(kak) = \int_{\widehat{(G,K)}} |\widehat{f(\varphi)}|^2 d\pi(\varphi).$$

π is referred to as plancherel measure and its support is the full set $\widehat{(G,K)}$.

4.3 Definition [7]. A positive definite distribution T on a Lie group G is a distribution that satisfies $T(\widetilde{\phi} * \phi) \geq 0 \ \forall \phi \in \mathcal{D}(G)$. If in addition to the above condition, $\phi \in C_c(K\backslash G/K)$, such a distribution is known as a K-bi-invariant distribution on G.

Let us look at the following regular distributions on $SE(n)$

(1) Let f be a continuous function on $\mathbb{R}^2$, μ a Radon measure on the compact subgroup of G. The linear functional $f \otimes \mu$ on $\mathfrak{D}(G)$ defined by

$$\varphi \mapsto \left\langle \varphi, f \otimes \mu \right\rangle = \int_{\mathbb{R}^2} \int_{SO(2)} f(a) \varphi(g) da dA$$

is a distribution on $G = SE(2), \varphi \in \mathfrak{D}(G)$.

(2) The character function χ_a is a linear functional on $\mathfrak{D}(G)$ or distribution on G defined by

$$\chi_a : f \mapsto \sum_{n=-\infty}^{\infty} \int_G f(g) \big(U_g^a \chi_n, \chi_n\big) dg = \sum_{n=-\infty}^{\infty} \left(U_g^a \chi_n, \chi_n\right) = Tr U_f^a,$$

where

$$U_f^a = \int_G f(g) U_g^a dg.$$

It is our interest in this section to show that the type of distribution mentioned in 2 above is a radial, positive definite, tempered and bounded. We proceed as follows.

There is a relationship between the spherical function I_0 of SE(2) and χ_a on $\mathcal{D}(G)$. This may be found in [10] as theorem 3.3. It is stated here as theorem 4.4, without proof.

4.4 Theorem For any fixed $\sigma > 0$, the linear functional

$$\chi_a f \mapsto TrU_f^\sigma$$

is a distribution on $G = SE(2)$. In fact, χ_a is equal to $I_0(t\|\xi\|) \otimes \delta(t)$ where J_0 is the Bessel function of order 0 and δ is the Dirac measure of order zero on $T \cong K$ and TrU_f^σ stands for the trace of the representation U_f^σ. That is to say, our distribution under consideration is $K(\lambda, t) = I_0(\lambda) \otimes \delta(t)$, where $\lambda = t\|\xi\|$. □

$\mathcal{S}'(SE(2))$, as earlier defined, is the space of tempered distributions on $SE(2)$. There are three kinds of topology that can be given to $\mathcal{S}'(SE(2))$, namely, strong dual topology, weak topology and the weak $*$ topology. A tempered distribution $u \in \mathcal{S}'(SE(2))$ is a continuous linear functional on $\mathcal{S}(SE(2))$.

Let $T \in \mathcal{S}'(SE(2))$ and let f be an arbitrary C^∞ function on G. There is a condition for $fT \in \mathcal{S}'(SE(2))$. This leads us to the following definition of a Lie group with polynomial growth.

4.5 Definition [1], p.7 Let G be a Lie group and let μ be the left Haar measure of G. G is said to have polynomial growth if $\exists$ a compact symmetric neighborhood U of $e \in G$ that generates G and such that the sequence $(\mu(U^n)_{n\in\mathbb{N}}$ has polynomial growth as $n \to \infty$. A function $f \in C^\infty(G)$ is said to have a polynomial growth if G has a polynomial growth. A Gelfand pair (G, K) has polynomial growth if G has polynomial growth ([1], p.7). The Gelfand pair $(SE(2), SO(2) \cong \mathbb{T})$ is a pair with polynomial growth, then $SE(2)$ is a a Lie group with polynomial growth.

Given $T \in \mathcal{S}'(G)$ and $f \in C^\infty(G)$, the condition for $fT \in \mathcal{S}'(G)$ is that f must be a function with polynomial growth. I_0 is the spherical function on $SE(2)$. It is bounded, positive definite and has polynomial growth. (see [6]). Also, $\delta(t) \in C^\infty(G)$. Following this development, we have that $I_0(\lambda)\delta(t) \in \mathcal{S}'(G)$, where $\lambda = \sigma\|\xi\|$. Further more, I_0 being the spherical function of $SE(2)$ is also a radial function. This is because elementary spherical functions are also radial functions. Since $I_o(\lambda)$ is radial, bounded and positive definite (see [6], Prop. 2.4) and $K(\lambda, t) = I_0(\lambda)\delta(t)$ is compactly supported in $\mathcal{S}'(R)$ at the identity, it therefore means that it also belongs to the space of tempered radial distributions $\mathcal{S}'_{rad}(G)$ on $G = SE(2)$.

5. Levy-Schoenberg Kernel on SE(2)

Let G be a separable topological group and K its subgroup that is compact. G/K being the qotient group is given the quotient topology. If $g_1, g_2, ...$ are members of G, then members of G/K will be $g_1K, g_2K, ..., etc$. In this section, we shall consider a homogenous space where $G = SE(2)$ and $K = SO(2)$ and $R^2 \cong SE(2)/SO(2)$

5.1 Definition. Let S be a space that is Hausdorff. A kernel on S is a continuous complex valued function on $S \times S$.

5.2 Definition. A kernel on the homogenous space $M(2)/SO(2)$ is said to be positive definite if for any $a_1, ..., a_n \in M(2)/SO(2)$ and complex numbers $\alpha_1, \alpha_2, ..., \alpha_n$ one has

$$\Sigma_{i=1}^{n} \Sigma_{j=1}^{n} \alpha_i \overline{\alpha_j} f(a_i, a_j) \geq 0. \tag{5.1}$$

5.3 Definition. A kernel f on G/K is called $Levy - Schoenberg$ kernel if the following properties are satisfied
(i) $f(a, b) = f(b, a)\ \forall a, b \in G/K$
(ii) there exists a point $\xi \in G/K$ such that $f(a, \xi) = 0$ for all $\xi \in G/K$.
(iii) the kernel r on G/K given by $r(a, b) = f(a, a) + f(b, b) - 2f(a, b)$ is invariant under G. That is $r(ga, gb) = r(a, b)$ for all $g \in G, a, b \in G/K$.
(iv) f is positive definite.
Following conditions (i) and (iv), a Levy-Schoenberg kernel is real valued.

Let $\{\xi(a) : a \in R^n\}$ be a Gaussian process with parameter a running over the Euclidean space R^n with center zero. That is to say, if E stands fo expectation, then $E(\xi(a)) = 0, a \in R^n$. Its covariance is defined by the Kernel f on $R^n \times R^n$ by

$$f(a, b) = \frac{1}{2}(|a| + |b| - |a - b|),\ a, b \in R^n, \tag{5.2}$$

$|a|$ stands for the length of $a \in R^n$. (12) is an example of Levy-Schoenberg kernel. It is real, symmetric and positive definite. R^2 in this case is seen as the homogeneous space of all rigid motions of R^2, modulo the compact subgroup K. This means that the kernel defined by (12) lives in $R^2 \cong SE(2)/SO(2)$.

5.4 Definition. A complex valued function ϕ on $SE(2)$ is said to be K-spherical if for $g \in SE(2)$, $k_1, k_2 \in SO(2)$, one has $\phi(k_1 g k_2) = \phi(g)$ and is called normalized if $\phi(e) = 1$. ϕ on $SE(2)$ is also said to be imbeddable if for each $t \geq 0$, ϕ^t is positive definite and $\phi^t(g) \to 1$ for each $g \in SE(2)$ as $t \downarrow 0$. Any imbeddable function is positive definite (see [13]).

5.5 Definition. The collection of all K-spherical functions that are continuous with complex values that are normalized and imbeddable on $SE(2)$ are referred to as class 1 for $(SE(2), SO(2))$.

5.6 Definition. A continuous positive functions ϕ on $SE(2)$ is infinitely divisible if for each $n > 0$, $\exists\, \phi_n$ on $SE(2)$ such that $\phi(g) = (\phi^n(g))^n$.

5.7 Definition. Class D for the pair $(SE(2), SO(2))$ simply means the set of all complex valued valued continuous K-spherical, normalized, infinitely divisible positive definite function on $SE(2)$.

Let ϕ be a radial function on $SE(2)$ then ϕ may be lifted to a function ϕ^* on $SE(2)/SO(2) \cong R^2$ by setting $\phi^* \circ \pi = \phi$ where $\pi : SE(2) \to SE(2)/SO(2)$ is a natural projection. This action turns ϕ^* to a radial function. ϕ is positive definite on $SE(2)$, it follows that ϕ^* is also positive definite on the quotient group $SE(2)/SO(2)$ and belongs to the class D (or 1) for the pair $(R^2, \{0\})$, implying that ϕ is also in the class D (or 1) for the pair $(SE(2), SO(2))$.

Levy-Khinchine formula (see [13]), that explains the Fourier transforms of probability measure on R^n that are infinitely divisible under convolution, is the description of the class D for the pair $(R^n, \{0\})$.

Example5.8 An explicit form of ϕ^* for the quotient group $SE(2)/SO(2)$ is presented as follows. A function ϕ^* on $SE(2)/SO(2) = R^n$ with $n \geq 2$ is a radial function in the class D for the pair $(R^n, \{0\})$ and has the representation below:

$$\phi^*(\alpha) = exp - \{g^*(a) + \int_{\lambda \geq 0} (1 - I_n(\lambda |a|))dL^*(\lambda)\} \tag{5.3}$$

where I_n is the Bessel function and

$$I_n(t) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}.\Gamma(\frac{n-1}{2})}.\int_0^{\pi} e^{itcos\phi Sin^{n-2}\phi d\phi} = \Gamma(\frac{n}{2})(2t^{-1})^{\frac{n-2}{2}}.J_{\frac{n-2}{2}}(t),\ t \geq 0$$

g^*, L^* are measurable functions that satisfy
(a) L^* is a non negative measure
(b) g^* is a function on R^n, such that $g^*(a) = c|a|^2$, where c is a constant ≥ 0 and $|a|$ is the length of a.
(13) is an explicit form of a radial function on the homogenous space $SE(2)/SO(2)$. It is positive definite, bounded and infinitely differentiable. It is compactly supported in $[-1, 1]$, hence it is a distribution.
Using (13) above, a kernel f on $G/K = R^n$ is a Levy-Schoenberg kernel if $f(a, b) = \frac{1}{2}(r(a, \xi)) + r(b, \xi) - r(a, b), a, b \in R^n$. Where $r(a, b) = \Psi^*(a - b)$ and Ψ^* is a function on R^n of the form

$$\Psi^*(a) = g^*(a) + \int_{\lambda \geq 0} (1 - I_n(\lambda |a|))dL^*(\lambda)$$

6. Conclusion

In this research, a kind of radial distribution on $SE(2)$ was obtained as the product of Bessel function of order zero I_0 and the dirac function on $SE(2)$. We have demonstrated that this distribution is tempered, positive definite and bounded. The I_0 used in defining the radial distribution on $SE(2)$ was obtained by solving the Laplace-Beltrami operator $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ radially using the method of separation of variable (see [3]). An explicit form of Levy-Schoenberg kernel on the the homogenous space $R^2 \cong SE(2)/SO(2)$ has been given as example 5.8.

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