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Exploring Solutions to the Stochastic Fractional Zakharov-Kuznetsov Equation Influenced by Space-Time White Noise Using the Tanh-Coth Method

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ABSTRACT. This study investigates the stochastic fractional Zakharov-Kuznetsov equation (SFZKE) influenced by space-time white noise, utilizing the conformable fractional derivative (CFD). The primary objective is to employ the Tanh-Coth method to derive soliton, wave, and periodic solutions for SFZKE under varying conditions of space-time white noise and fractional order. A broader spectrum of exact analytical solutions for the SFZKE has been achieved. Graphical representations are provided to highlight the physical properties of the obtained solutions. The Tanh-Coth method is demonstrated to be a reliable and effective approach for solving stochastic fractional partial differential equations.

1. Introduction

Stochastic partial differential equations (SPDEs) are extensively applicable across diverse scientific disciplines, including pure and applied mathematics, physics, biology, and engineering (see, for example, [1], [2], [3], [4], [5]). The Zakharov-Kuznetsov equation (ZKE), originally proposed by Zakharov and Kuznetsov [6], characterizes the behavior of nonlinear ion-acoustic waves in a highly magnetized, lossless plasma within a two-dimensional context. Numerous researchers have employed various numerical methods to achieve exact solutions for nonlinear

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SPDEs (NLSPDEs) and nonlinear stochastic FPDEs (NLSFPDEs). Notably, these methodologies include the Homotopy Perturbation Method [7], the Improved Fractional Sub-Equation Method [8], iterative approaches [9], the Modified Kudryashov Method [10], the Galerkin Spectral Method [11], He's semi-inverse method, the Riccati-Bernoulli sub-ODE method [12], and Difference Methods [13]. Recent years have seen a surge in research focused on solving stochastic FPDEs (SFPDEs), employing techniques such as conformable fractional derivatives and Caputo fractional derivatives, among others ([14] – [20]).

The space-time Fractional ZKE (FZKE) is defined as follows [21]:

$$D_t^{\alpha}M + \lambda M D_x^{\kappa}M + \beta D_x^{\kappa}(D_x^{\kappa\kappa}M + D_y^{\upsilon\upsilon}M) = 0. \tag{1}$$

Here, M is a function in x, y and t, and D_x^{α}, D_y^{κ} , D_t^{ν} for $(0 < \alpha, \kappa, \nu \le 1)$ represent first conformable fractional derivatives with respect to space and time, $D_x^{\kappa\kappa}(M)$and $D_y^{\upsilon\upsilon}M = D_y^{\upsilon}(D_y^{\upsilon}M)$ are the second conformable fractional derivatives for $x$and y, λ and β are constants. If $\alpha = \kappa = \nu = 1$, Eq. (1) reduces to the standard ZKE. Consequently, the stochastic fractional ZKE (SFZKE) is described as follows [22]:

$$D_t^{\alpha}M + \lambda M D_x^{\alpha}M + \beta D_x^{\alpha}(D_x^{\alpha\alpha}M + D_y^{\alpha\alpha}M) = \sigma M dW(t). \tag{2}$$

In this context, $W(t)$ is a random variable known as standard Brownian motion (SBM), and σ represents the noise strength. SBM introduced by Norbert Wiener [23], has played a critical role in developing stochastic process theory. It is characterized by the following conditions:

$W(0) = 0$.

$W(t)$ is continuous function of t.

$W(t)$ has independent increments.

$W(t)$ - $W(\tau) \sim N(0, t-\tau)$ for $0 \le \tau \le t$.

In this context, $N(0, t-\tau)$ is normal distribution with expect 0 and variance $t-\tau$.

This research seeks to address Eq. (2) utilizing the conformable fractional derivative (CFD) introduced by Khalil et al. [16]. The CFD marks a significant innovation in fractional calculus. Its derivative possesses essential characteristics that have broad applications in numerous scientific fields. Defined by an order α, where $0 < \alpha \le 1$, the CFD is articulated in terms of the independent variables. The mathematical representation can be denoted as:

$D^{\alpha}M(s) = \lim\limits_{\tau \to 0} \frac{M(s+\tau s^{1-\alpha}) - M(s)}{\varepsilon\tau} \forall t > 0, \alpha \in (0,1]$.

$M^{(\alpha)}(0) = \lim\limits_{s \to 0^+} M^{(\alpha)}(s)$.

When α is set to 1 in the previous equations, the non-integer differential transforms into the widely recognized integer differential. The characteristics of CFD are detailed in Khalil et al. [16]. In light of the benefits offered by CFD, this study utilizes the Tanh-coth method (Malfliet [24]) to derive traveling wave solutions relevant to Eq. (2).

The Tanh-coth method is extensively employed to construct solutions for various NPDEs ([25]-[29]. However, its application to the SFZKE has not been adequately investigated. Therefore, this study's novelty lies in applying the Tanh-coth method to the SFZKE, given its limited use in relation to this specific equation. The primary objective is to employ the Tanh-Coth method to derive soliton, wave, and periodic solutions for SFZKE under varying conditions of space-time white noise and fractional orders. A more comprehensive range of exact analytical solutions for the SFZKE equation has been obtained. Graphical illustrations are included to elucidate the physical characteristics of the acquired solutions, demonstrating the effects fractional order and stochastic term. The structure of this paper is organized as follows: Section 1 presents the introduction to the study. Section 2 details the Tanh-Coth method. Section 3 explores the application of the SFZKE. Section 4 illustrates the physical characteristics of the SFZKE solutions with corresponding graphs. Lastly, Section 5 concludes with a summary of the findings.

2. Description of the Tanh-coth Method

This study utilizes the tanh-coth method as structured by Malfliet [24] and Wazwaz [30]. The method assumes that traveling wave solutions can be represented via the tanh function and involves the following primary steps.

Step 1: Consider the nonlinear fractional partial differential equation as shown

$$P\left(M, D_t^{\alpha} M, D_x^{\alpha} M, D_y^{\alpha} M, D_x^{\alpha}(D_y^{\alpha} u), D_x^{\alpha\alpha} M, D_y^{\alpha\alpha} M, \dots\right) = 0, \tag{3}$$

where $M(x,,\mathrm{y},t)$ is a function of the spatial variable x, y and time variable t.

Step 2: Solutions to Eq. (3) are obtained using the traveling wave transformation by setting $M(x,y) = N(\xi)$, with $\xi = \frac{1}{\alpha}(ax^{\alpha} + \mathrm{b}y^{\alpha} - ct^{\alpha})$, which transforms Eq. (3) into ordinary differential equation (ODE)

$$Q(N, N', N'', \mathrm{N}''', \dots) = 0, \tag{4}$$

where primes denote derivatives with respect to ξ.

Step 3: Introduce a new independent variable

$$X = tanh(\mu\, \xi), \tag{5}$$

that leads to the transformation of derivatives

$$\frac{d}{d\xi} = \mu(1 - X^2)\frac{d}{dX},$$
$$\frac{d^2}{d\xi^2} = -2\mu^2(1 - X^2)\frac{d}{dX} + \mu^2(1 - X^2)^2\frac{d^2}{dX},$$
$$\frac{d^3}{d\xi^3} = -2\mu^3(1 - X^2)(3X^2 - 1)\frac{d}{dX} - 6\mu^3(1 - X^2)^2\frac{d^2}{dX^2} + \mu^3(1 - X^2)^3\frac{d^3}{dX^3}. \quad (6)$$
$$\vdots$$

Other derivatives can be derived similarly.

Step 4: Subsequently, propose the function $N(x, \mathrm{y}, t)$ through an expansion given by:

$$N(\xi) = S(X) = \sum_{k=0}^{m} \rho_k X^k + \sum_{k=1}^{m} \rho_{-k} X^{-k}. \quad (7)$$

To determine the parameter m, where m is a positive integer, we typically align the highest-order linear terms in Eq. (4) with the highest-power nonlinear terms.

Step 5: Once m has been determined, substitute Eq. (7) into Eq. (4). The resulting ODE expressed in powers of X will yield coefficients that must be set to zero. This leads to a system of algebraic equations involving the parameters $\rho_k(\mathrm{k} = 0, \pm1, \pm2, \cdots, m)$ and μ. By solving for these parameters and utilizing Eq. (7), we obtain the analytic solution.

3. Application of Eq. (2)

Now, the Tanh-coth method is employed to derive wave solutions for Eq. (2).

$$D_t^\alpha M + \lambda M D_x^\alpha M + \beta D_x^\alpha\left(D_x^{\alpha\alpha} M + D_y^{\alpha\alpha} M\right) = \sigma M dW(t).$$

Consider the following traveling wave transformation

$$M(x, y, t) = N(\xi)e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}, \xi = \frac{1}{\alpha}(ax^\alpha + \mathrm{b}y^\alpha - ct^\alpha), \quad (8)$$

where N is a deterministic real function, and a, b, c are nonzero constants. Utilizing the definition of CFD, the traveling wave transformation, and properties of SBM, we obtain

$$D_t^\alpha M = (-cN' + \sigma N\, dW(t))e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \quad (9)$$

$$D_x^\alpha M = aN' e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \quad (10)$$

$$D_x^{2\alpha} M = a^2 N'' e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \quad (11)$$

$$D_x^{3\alpha} M = a^3 N''' e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \quad (12)$$

$$D_y^\alpha M = bN' e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \quad (13)$$

$$D_y^{2\alpha} M = b^2 N'' e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \quad (14)$$

$$D_x^\alpha\left(D_y^{\alpha\alpha} M\right) = ab^2 N'' e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \quad (15)$$

By substituting equations (9), (10), (11), (12), (13), (14), and (15) into Eq. (2), we have

$$-c\,N' + a\lambda N N' e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)} + k\,\beta N''' = 0, \tag{16}$$

where the constant k is defined as: $k = a^3 + ab^2$.

By taking the expectation on both sides of Eq. (16), $E[e^{\sigma W(t)}] = e^{\left(\frac{1}{2}\sigma^2 t\right)}$ and integrating with a zero constant, we obtain:

$$-c\,N + \frac{1}{2}a\lambda N^2 + \kappa\,\beta N'' = 0. \tag{17}$$

Balancing the nonlinear term N^2 with the highest order derivative N'', we find that $m = 2$. Consequently,

$$N(\xi) = \rho_{-2}X^{-2} + \rho_{-1}X^{-1} + \rho_0 + \rho_1 X + \rho_2 X^2. \tag{18}$$

Substitute Eq. (18) into Eq. (17), collect the terms involving powers of X, equate each power to zero, and use Maple to solve the resulting system of algebraic equations to determine the solutions.

Case 1:

$$\rho_0 = -\frac{c}{a\lambda},\ \rho_1 = 0,\ \rho_2 = \frac{3c}{a\lambda},\ \rho_{-1} = 0,\ \rho_{-2} = 0,\ \mu = \frac{1}{2}\sqrt{\frac{c}{\beta\,k}}. \tag{19}$$

Case 2:

$$\rho_0 = \frac{c}{2a\lambda},\ \rho_1 = 0,\ \rho_2 = \frac{3c}{4a\lambda},\ \rho_{-1} = 0,\ \rho_{-2} = \frac{3c}{4a\lambda},\ \mu = \frac{1}{4}\sqrt{\frac{c}{\beta\,k}}. \tag{20}$$

Case 3:

$$\rho_0 = -\frac{c}{a\lambda},\ \rho_1 = 0,\ \rho_2 = 0,\ \rho_{-1} = 0,\ \rho_{-2} = \frac{3c}{a\lambda},\ \mu = \frac{1}{2}\sqrt{\frac{c}{\beta\,k}}. \tag{21}$$

Case 4:

$$\rho_0 = \frac{3c}{a\lambda},\ \rho_1 = 0,\ \rho_2 = 0,\ \rho_{-1} = 0,\ \rho_{-2} = -\frac{3c}{a\lambda},\ \mu = \frac{1}{2}\sqrt{-\frac{c}{\beta\,k}}. \tag{22}$$

Case 5:

$$\rho_0 = \frac{3c}{a\lambda},\ \rho_1 = 0,\ \rho_2 = -\frac{3c}{a\lambda},\ \rho_{-1} = 0,\ \rho_{-2} = 0,\ \mu = \frac{1}{2}\sqrt{-\frac{c}{\beta\,k}}. \tag{23}$$

Case 6:

$$\rho_0 = \frac{3c}{2a\lambda},\ \rho_1 = 0,\ \rho_2 = -\frac{3c}{4a\lambda},\ \rho_{-1} = 0,\ \rho_{-2} = -\frac{3c}{4a\lambda},\ \mu = \frac{1}{4}\sqrt{-\frac{c}{\beta\,k}}. \tag{24}$$

For $\frac{c}{\beta k} > 0$, by substituting equations (19), (20), or (21) into Eq. (18), and subsequently integrating the resulting equation into Eq. (8), we derive the soliton solution

$$M_1(x,y,t) = -\frac{c}{a\lambda}\left[1 - 3\tanh^2\left(\frac{1}{2\alpha}\sqrt{\frac{c}{\beta\, k}}(ax^\alpha + \mathrm{by}^\alpha - ct^\alpha)\right)\right]e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}, \tag{25}$$

and travelling wave solution is then given by

$$M_2(x,y,t) = \begin{cases} \frac{c}{4a\lambda}\left[\mathbf{2} + 3tanh^2\left(\frac{1}{4\alpha}\sqrt{\frac{c}{\beta\, k}}(ax^\alpha + by^\alpha - ct^\alpha)\right)\right. \\ \left. +3coth^2\left(\frac{1}{4\alpha}\sqrt{\frac{c}{\beta\, k}}(ax^\alpha + \mathrm{by}^\alpha - ct^\alpha)\right)\right]e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)} \end{cases}, \tag{26}$$

and

$$M_3(x,y,t) = -\frac{c}{a\lambda}\left[1 - 3\coth^2\left(\frac{1}{2\alpha}\sqrt{\frac{c}{\beta\, k}}(ax^\alpha + \mathrm{by}^\alpha - ct^\alpha)\right)\right]e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \tag{27}$$

With respect to $\frac{\mathrm{c}}{\beta k} < 0$, by substituting any of the equations (19), (20), or (21) into Eq. (18), and subsequently substituting the resulting equation into Eq. (8) respectively, we obtain the periodic solution:

$$M_4(x,y,t) = -\frac{c}{a\lambda}\left[1 + 3\tan^2\left(\frac{1}{2\alpha}\sqrt{-\frac{c}{\beta\, k}}(ax^\alpha + \mathrm{by}^\alpha - ct^\alpha)\right)\right]e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}, \tag{28}$$

and

$$M_5(x,y,t) = \begin{cases} \frac{c}{4a\lambda}\left[\mathbf{2} - 3tan^2\left(\frac{1}{4\alpha}\sqrt{-\frac{c}{\beta\, k}}(ax^\alpha + by^\alpha - ct^\alpha)\right)\right. \\ \left. -3cot^2\left(\frac{1}{4\alpha}\sqrt{-\frac{c}{\beta\, k}}(ax^\alpha + \mathrm{by}^\alpha - ct^\alpha)\right)\right]e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)} \end{cases}, \tag{29}$$

and

$$M_6(x,y,t) = -\frac{c}{a\lambda}\left[1 + 3\cot^2\left(\frac{1}{2\alpha}\sqrt{-\frac{c}{\beta\, k}}(ax^\alpha + \mathrm{by}^\alpha - ct^\alpha)\right)\right]e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}. \tag{30}$$

Concerning $\frac{\mathrm{c}}{\beta k} > 0$, by inserting any of the equations (22), (23), or (24) into Eq. (18), and then substituting the resulting equation into Eq. (8) respectively, we derive the periodic solution

$$M_7(x,y,t) = \frac{3c}{a\lambda}\left[1 + \cot^2\left(\frac{1}{2\alpha}\sqrt{\frac{c}{\beta\, k}}(ax^\alpha + \mathrm{by}^\alpha - ct^\alpha)\right)\right]e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}, \tag{31}$$

and

$$M_8(x,y,t) = \frac{3c}{a\lambda}\left[1 + \tan^2\left(\frac{1}{2\alpha}\sqrt{\frac{c}{\beta\, k}}(ax^\alpha + \mathrm{by}^\alpha - ct^\alpha)\right)\right]e^{\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)}, \tag{32}$$

and

$$M_9(x,y,t)=\begin{cases}\frac{3c}{4a\lambda}\left[\mathbf{2}-tan^2\left(\frac{1}{4\alpha}\sqrt{\frac{c}{\beta\,k}}(ax^\alpha+by^\alpha-ct^\alpha)\right)\right.\\ \left.+cot^2\left(\frac{1}{4\alpha}\sqrt{\frac{c}{\beta\,k}}(ax^\alpha+\mathrm{b}y^\alpha-ct^\alpha)\right)\right]e^{\left(\sigma W(t)-\frac{1}{2}\sigma^2t\right)}\end{cases}. \quad (33)$$

In terms of $\frac{c}{\beta k}<0$, by substituting any of the equations (22), (23), or (24) into Eq. (18) and subsequently incorporating the resultant equation into Eq. (8), respectively, we derive the traveling wave solution:

$$M_{10}(x,y,t)=\frac{3c}{a\lambda}\left[1-\coth^2\left(\frac{1}{2\alpha}\sqrt{-\frac{c}{\beta\,k}}(ax^\alpha+\mathrm{b}y^\alpha-ct^\alpha)\right)\right]e^{\left(\sigma W(t)-\frac{1}{2}\sigma^2t\right)}. \quad (34)$$

and soliton solution

$$M_{11}(x,y,t)=\frac{3c}{a\lambda}\left[1-\tanh^2\left(\frac{1}{2\alpha}\sqrt{-\frac{c}{\beta\,k}}(ax^\alpha+\mathrm{b}y^\alpha-ct^\alpha)\right)\right]e^{\left(\sigma W(t)-\frac{1}{2}\sigma^2t\right)}. \quad (35)$$

Additionally, the traveling wave solution is given by

$$M_{12}(x,y,t)=\begin{cases}\frac{3c}{4a\lambda}\left[\mathbf{2}-tanh^2\left(\frac{1}{4\alpha}\sqrt{-\frac{c}{\beta\,k}}(ax^\alpha+by^\alpha-ct^\alpha)\right)\right.\\ \left.-coth^2\left(\frac{1}{4\alpha}\sqrt{-\frac{c}{\beta\,k}}(ax^\alpha+\mathrm{b}y^\alpha-ct^\alpha)\right)\right]e^{\left(\sigma W(t)-\frac{1}{2}\sigma^2t\right)}\end{cases}. \quad (36)$$

4. Discussion of the Graphical Representation

Visual representations of the derived solutions were created using Maple software to showcase their behaviors and characteristics. By manipulating the free parameters' values, we can modulate these solutions' behaviors, thereby altering the resulting graphs' nature. To illustrate the impact of the stochastic term and the fractional order on the graphical representation, we will fix the following parameters: $a=1,\mathrm{b}=1,c=1,\lambda=1,\beta=1,k=2,y=3,$ with $x\in[-10,10]$ and $t\in[0,10]$. It is crucial to note that the Brownian process, being a Gaussian process, can be represented as a linear combination of independent normal random variables.

4.1 Examination of stochastic term impact

The 3D graphs presented in Figures 1 and 2 illustrate various types of solutions, including soliton, wave, and periodic solutions, under different values of the noise term σ. Specifically, solutions are depicted for $\sigma=0,\sigma=0.5,\ \sigma=1$ and $\sigma=2$. As the visualizations demonstrate, the noise term significantly affects SFZKE solutions, leading to their instability around zero when increasing the noise term. By adjusting the noise term, it is possible to tune the nature and stability of the solutions effectively.

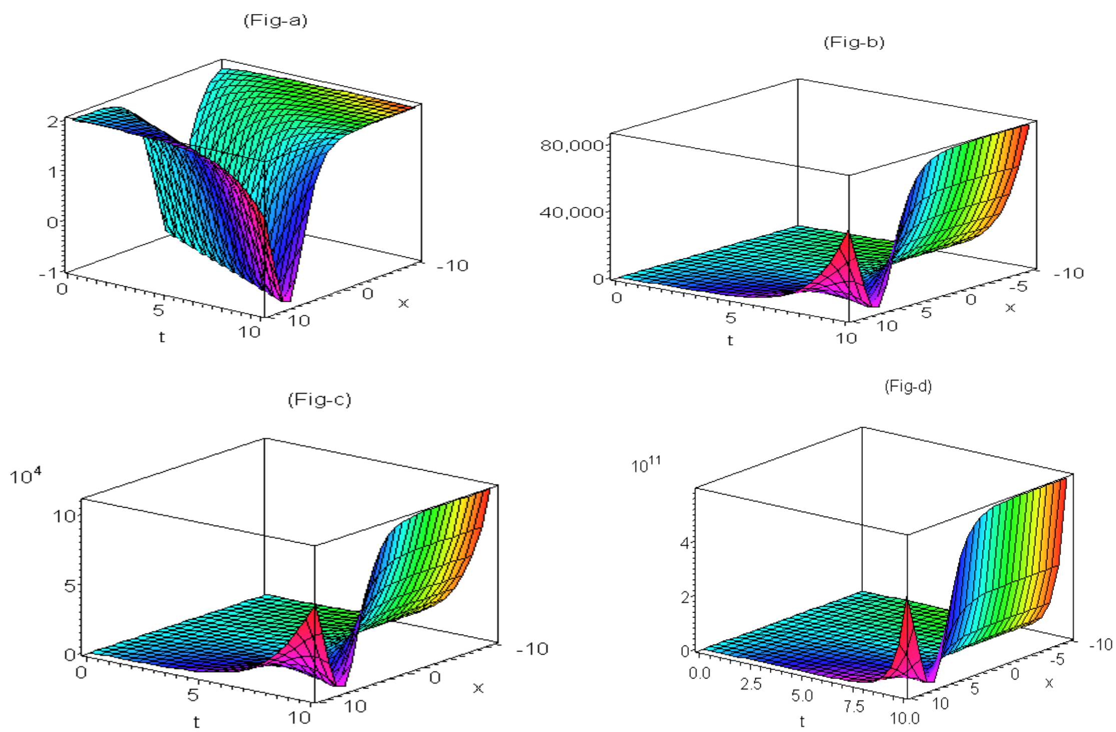

Figure 1. For $M_1(x, y, t)$, (a–d) with $a = 1, \mathrm{b} = 1, c = 1, \lambda = 1, \alpha = 1$, $\beta = 1, k = 2, \mathrm{y} = 3$, $x \in [-10,10]$ $t \in [0,10]$ and 3D graphs (a) $\sigma = 0$ (b) $\sigma = 0.5$ (c) $\sigma = 1$ (d) $\sigma = 2$.

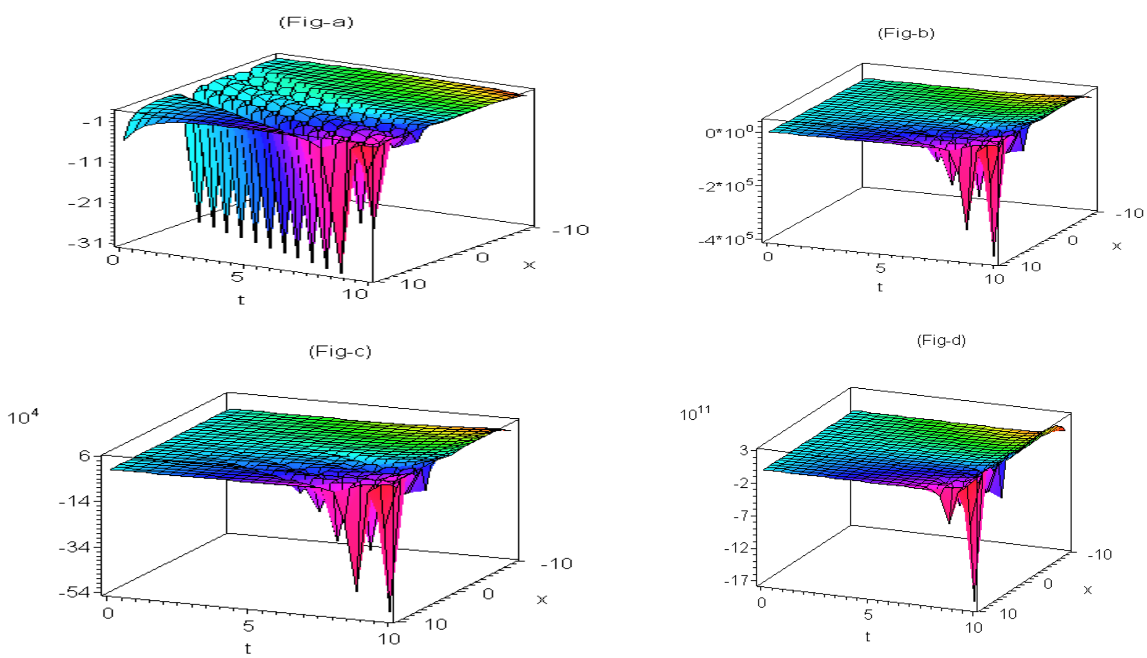

Figure 2. For $M_9(x, y, t)$, (a–d) with $a = 1, \mathrm{b} = 1, c = 1, \lambda = 1, \alpha = 1, \beta = 1, k = 2, \mathrm{y} = 3$, $x \in [-10,10]$ $t \in [0,10]$ and 3D graphs (a) $\sigma = 0$ (b) $\sigma = 0.5$ (c) $\sigma = 1$ (d) $\sigma = 2$.

4.2 Examination of fractional order impact

The 3D graphs presented in Figures 3 and 4 illustrate the behavior of the fractional order when different values are assigned, specifically when $\alpha = 1, \sigma = 0.9$ and $\sigma = 0.8$. Notably, as the fractional order decreases, the solutions of the SFZKE demonstrate a distinct rightward shift. This shift highlights the sensitivity of the SFZKE solutions to changes in fractional order, providing valuable insights into the dynamics of the system under varying conditions.

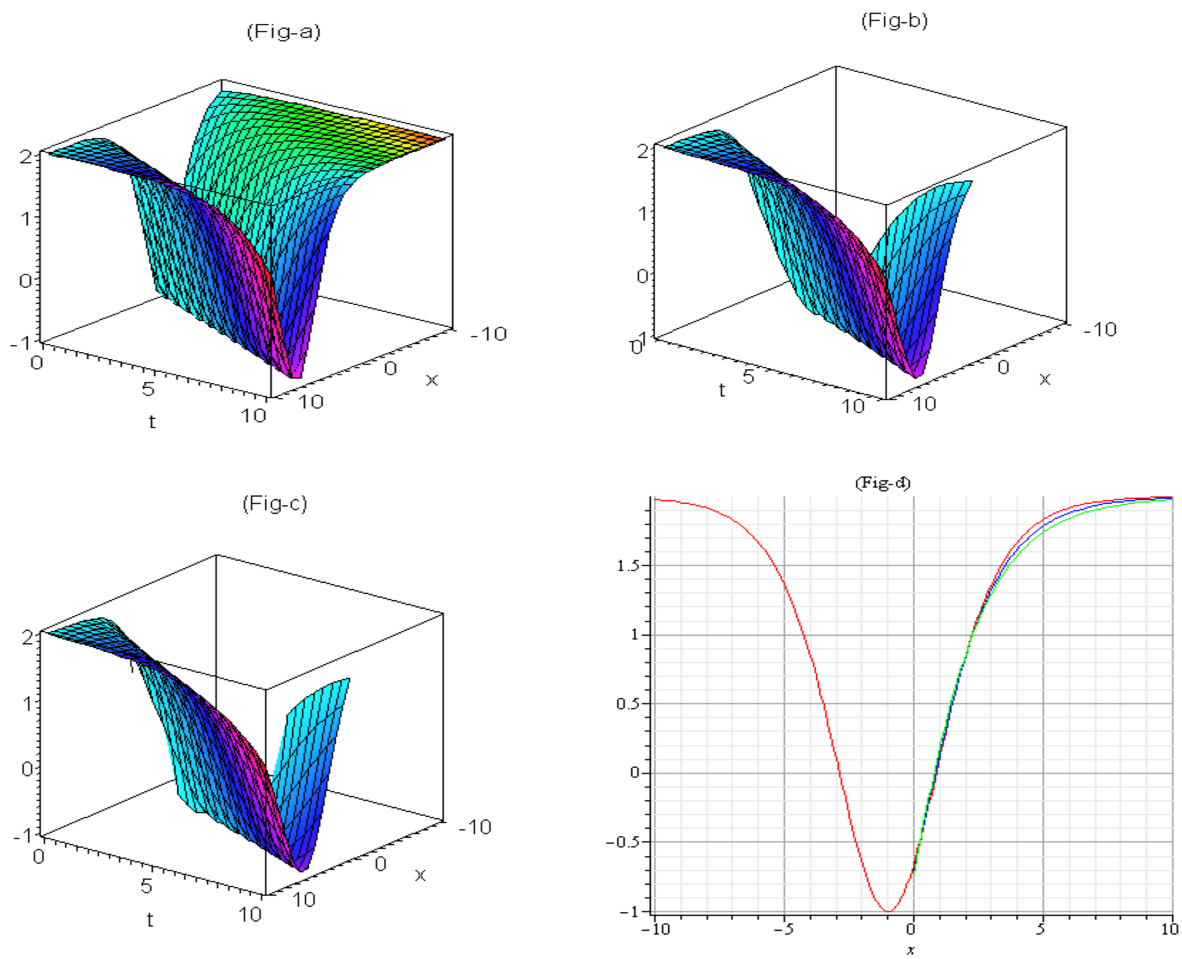

Figure 3. For $M_1(x, y, t)$, (a-c) with $a = 1, \mathrm{b} = 1, c = 1, \lambda = 1, \sigma = 0\,, \beta = 1, k = 2, \mathrm{y} = 3\,\,, x \in [-10,10]\; t \in [0,10]$ and 3D graphs (a) $\alpha = 1$ (b) $\alpha = 0.9$ (c) $\sigma = 0.8$ and (d) denotes the 2D plot with t = 2.

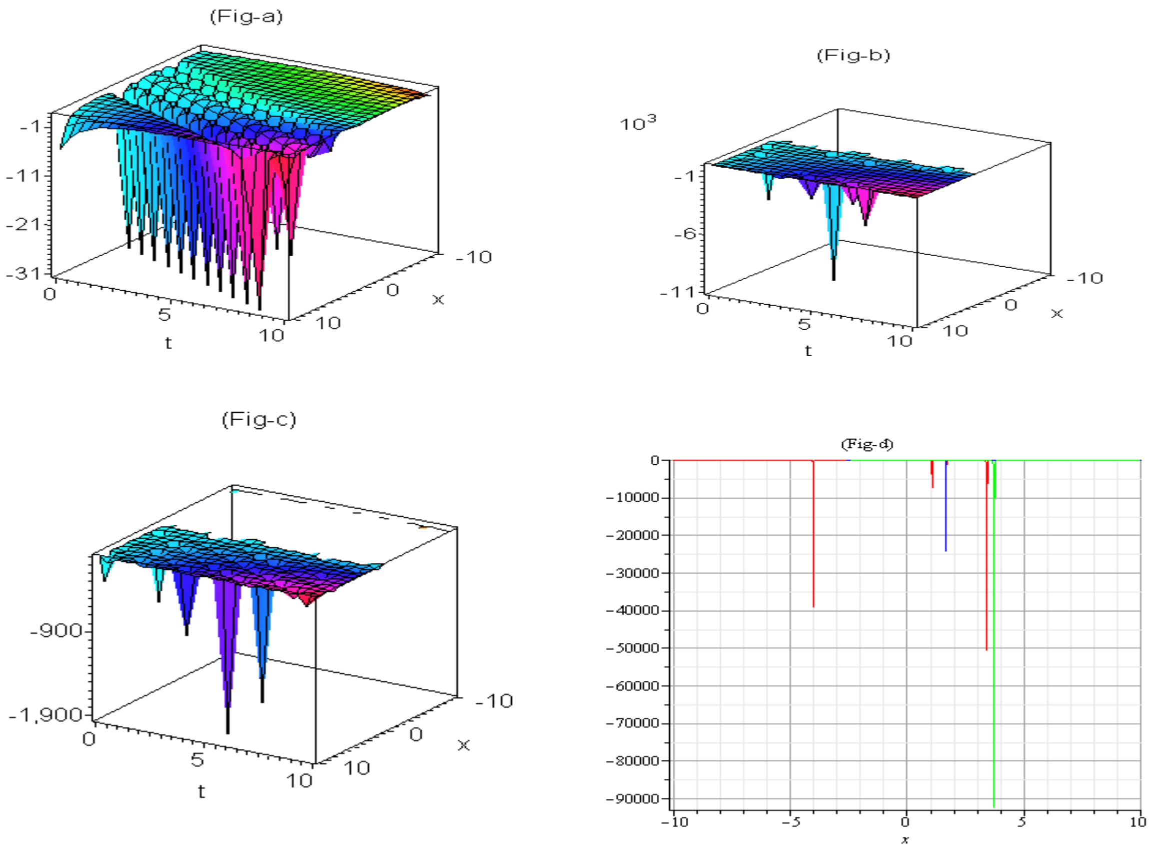

Figure 4. For $M_9(x,y,t)$, (a-c) with $a = 1, \mathrm{b} = 1, c = 1, \lambda = 1, \sigma = 0, \beta = 1, k = 2, \mathrm{y} = 3$, $x \in [-10,10]$ $t \in [0,10]$ and 3D graphs (a) $\alpha = 1$ (b) $\alpha = 0.9$ (c) $\sigma = 0.8$ and (d) denotes the 2D plot with t = 2.

5. Conclusion

This study successfully derived numerous analytical solutions of the fractional stochastic Zakharov-Kuznetsov equation (SFZKE) using the tanh-coth method with the conformable fractional derivative (CFD). The obtained solutions, including soliton, wave, and periodic forms, are crucial for comprehending various phenomena within the SFZKE framework. Our analysis demonstrated that the noise term significantly influences the SFZKE solutions, introducing instability around zero when the fractional order is constant. As the visualizations demonstrate, the noise term significantly affects SFZKE solutions, leading to their instability around zero. By adjusting the noise term, it is possible to tune the nature and stability of the solutions effectively. Additionally, as the fractional order decreases, a rightward shift in the SFZKE solutions is observed in the absence of the noise term. The physical characteristics and behaviors of these solutions were effectively illustrated through graphical representations. All computations were performed using MAPLE software, underscoring the tanh-coth method's efficacy in resolving the

SFZKE equation. These findings contribute valuable insights into the dynamics of fractional-order systems and their response to noise and order variations.

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