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A New Computational Method Based on the Method of Lines and Adomian Decomposition Method for Burgers' Equation and Coupled System of Burgers' Equations

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Abstract. This study proposes a new computational scheme for the solution of the class of one-dimensional Burgers' equations, comprising mainly the classical Burgers' equation, and the system of coupled Burgers' equations. This method is based upon coupling the Method of Lines (MOL) and the prominent Adomian Decomposition Method (ADM) for the reliable computational examination of dissimilar initial-boundary value problems of Burgers' equations. Certainly, MOL helps with the spatial semi-discretization of the governing problem to a system of nonlinear Ordinary Differential Equations (ODEs); while the ADM contributes to the efficient semi-analytical solution of the resulting nonlinear ODEs. Moreover, the computational accuracy of the new approach has been demonstrated on certain test models and further evaluated using L_2 and L_∞ norms. Indeed, the method produces better results with minimal errors than many existing computational approaches as successfully reported in various supportive figures and tables.

1. Introduction

Various forms of nonlinear evolution equations exist in the open literature with a variety of applications in the contemporary fields of science and technological processes. In light of this, the class of Burgers' equations is one of the famous evolution equations with immense relevance in the study of fluid flow in the field of fluid mechanics, and also it plays a vital role in the field of nonlinear waves ([1, 2]), optics, wave scattering, shocks, and shallow water waves to list a few. A coupled variant of Burger's equation was devised as the coupled viscous Burger's equation,

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which was utilized in modeling the transmission of poly-dispersion in sedimen tation processes; read Esipov [3]. Moreover, the importance of this class of nonlinear equations alongside the lack of enough exact analytical solutions for computational validations is what results in various scientists devising several computational methods to study the equations. Computationally, many proficient numerical approaches for the study of nonlinear Partial Differential Equations (PDEs) are available. In this regard, we make mention of the Method of Lines (MOL) [4], an interesting numerical method that goes by discretizing the spatial coordinates, thereby yielding an approximate system of coupled Ordinary Differential Equations (ODEs) in the temporal variable t. In addition, optimal computational MOL solutions are attained upon choosing positive spatial discretization points alongside a type of spatial discretization technique. The next approach of much concern is the Adomian Decomposition Method (ADM) [5]. The ADM is a semi-analytical approach, which in the same way multiples as a computational method that swiftly convergences to the accessible exact analytical solution of the problem.

However, this study is motivated by the competency of both the MOL [4] and the ADM [5] in handling diverse classes of nonlinear PDEs and thus feels the imperativeness of coupling these eminent approaches to yet come up with a new approach that combines the salient features of MOL and ADM. Besides, the study is set to make use of an optimal spatial-discretization technique together with relatively advanced spatial discretization points in MOL, for the solution numerical treatment of the governing PDEs, while the ADM is set to approximately solve the resulting temporal-based system of nonlinear ODEs. Indeed, this new computational approach is referred to as the Decomposition Method of Lines (DMOL), being the combo of MOL and ADM, which is suitable for both the classes of real-valued and complex-valued evolution equations, with the governing class of Burgers' equations inclusive. In addition, various other computational approaches have been used in the past and present times to numerically examine this class of equations, including, for instance, the differential transformation technique [6], local RBF method [7], cubic Hermite collocation approach [8], finite element method [9], cubic-B spline basis functions approach [10], quintic-B spline functions approach [11], and a lot of other approaches, including Chebyshev collocation method, Chebyshev-Legendre Pseudo-spectral approach and the differential quadrature technique among others, read also ([12–14]) and the references therewith. Additionally, the efficiency of the proposed DMOL scheme will be computationally assessed with the help of L_2 and L_∞ error norms for various fixed involving parameters, and different time levels. Equally, the implicit linearization approach by Mukundan and Awasthi [15] for the solution of the governing model will be deployed for comparative study. Lastly, the manuscript is organized in the following manner: Section 2 gives the models of scrutiny. Section 3 is explains the ADM and its application in solving nonlinear ODEs. Section 4 outlines the DMOL for the two Burgers' equations of concern. Section 6 analytically derive the error bounds for the DMOL method and demonstrate its convergence properties. Section 5 is dedicated to the linearization of the two models. Section 7 demonstrates the application of the devised DMOL scheme on certain test models of concern; while Section 8 gives some concluding notes.

2. Governing Equations

The current section presents the governing equations of curiosity in this study. Indeed, this study intends to establish some new reliable computational schemes for the solution of certain nonlinear evolution equations. Precisely, the study considers the one-dimensional Burgers' equation, and the system of coupled one-dimensional Burgers' equation, as given in what follows.

2.1. **Burgers' equation.** The standard form of the one-dimensional Burgers' equation takes the following representation [1]

$$w_t = v w_{xx} - w w_x, \tag{2.1}$$

where $w = w(x,t)$, is the real-valued function, denoting the resulting wave field or the corresponding fluid flow, depending on the examining scenario, in the spatial x and temporal t variables. In addition, v is the kinematic viscosity parameter, defined by $v = \frac{1}{Ra}$; with Ra representing the involving Reynolds number in the flow. Certainly, this nonlinear evolution equation has a rich literature, with vast applications in the study of fluid flow in the field of fluid mechanics and also it plays a vital role in the field of nonlinear waves, which are governed by evolution equations, portraying the dynamics of solitary wave propagation in various media.

2.2. **System of coupled of Burgers' equations.** Equally, among the most widely utilized system of coupled one-dimensional Burgers' equations is the following system [2]

$$\begin{aligned} w_{1_t} &= v_1 w_{1_{xx}} + \eta_1 w_1 w_{1_x} + \alpha(w_1 w_{2_x} + w_{1_x} w_2), \\ w_{2_t} &= v_2 w_{2_{xx}} + \eta_2 w_2 w_{2_x} + \beta(w_1 w_{2_x} + w_{2_x} w_2), \end{aligned} \tag{2.2}$$

where $w_1 = w_1(x,t)$, and $w_2 = w_2(x,t)$ are the resulting real-valued functions for the respective waves in each equation; of course, they can equally denote the respective fluid flows in the concerning control volumes - depending on the application, with the spatial x and temporal t variables. In addition, v_j are the associated kinematic viscosities such that $v_j = \frac{1}{Ra}$, for $j = 1,2$ with Ra, denoting the Reynolds number; while α, β and η_j are non-zero real constants all for $j = 1,2$. This coupled Burgers' model have been examined extensively in the literature, including the dynamicity of the fractional-order derivative in the model, one may read [2] and the cited references therein.

3. Adomian Decomposition Method (ADM)

George Adomian first presented the Adomian decomposition method in 1984 ([16,17]). Since then, applied mathematics in general and initial value and boundary value problems in particular have given this method a lot of attention. The decomposition method yields highly accurate numerical approximations and shows rapid convergence of the solution. Without resorting to linearization, perturbation, or the physical behavior of the physical model under study, the technique tackles applicable problems directly and simply. In the literature, numerous scholars have addressed and extensively employed the method ([18–21]). On the other hand, the Adomian approach reduces the amount of computing work, while some common methods call for large computations [22].

According to this approach, the solution to a functional equation is equivalent to the sum of an infinite series that eventually converges. Applying this approach to a class of linear and nonlinear partial differential equations has been the subject of numerous research projects lately ([23–25]). Since the approach avoids using needless limiting techniques and presumptions like linearization and perturbations, which could significantly alter the situation, it is ideally suited to physical problems. We only go over the essentials of this approach here; [26] has further information.

We first consider the equation

$$L(w) + R(w) + N(w) = g(x,t), \tag{3.1}$$

where L is the highest-order linear differential operator (invertible), R represents a linear differential operator of order lesser than L, and N represents the nonlinear terms and $g(x,t)$ is the source term and w is function in (x,t).

Unknown function $w(x,t)$ is represented as an infinite series:

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t), \tag{3.2}$$

Applying L^{-1} on the both sides of Equation 3.1 yields

$$w(x,t) = f(x,t) - L^{-1}R\left(w(x,t)\right) - L^{-1}N(w(x,t)), \tag{3.3}$$

where $f(x,t) = w(x,0) + L^{-1}(g(x,t))$ and the function $f(x,t)$ represents the terms arising after integrating the source term $g(x,t)$ and by using the given conditions.

Substituting the infinite series of components Eq.(4.6) into both sides of Equation 3.3 yields

$$\sum_{n=0}^{\infty} w_n(x,t) = f(x,t) - L^{-1}R\left(\sum_{n=0}^{\infty} w_n(x,t)\right) - L^{-1}N\left(\sum_{n=0}^{\infty} w_n(x,t)\right), \tag{3.4}$$

The decomposition method suggests that the zeroth component w_0 is usually defined by all terms not included under the inverse operator L^{-1}, which arise from the initial data and from integrating the inhomogeneous term. This in turn gives the formal recursive relation

$$\begin{aligned} w_0(x,t) &= f(x,t), \\ w_{k+1}(x,t) &= -L^{-1}R\left(w_k(x,t)\right) - L^{-1}N\left(w_k(x,t)\right), \quad k = 0,1,2,.... \end{aligned} \tag{3.5}$$

More, the nonlinear term $N(w)$ is then expressed by an infinite series of Adomian polynomials as follows

$$N(w) = \sum_{n=0}^{\infty} A_n, \tag{3.6}$$

where A_n are the Adomian polynomials expressed using the following algorithm ([26–28])

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i w_i(x,t)\right)\right]_{\lambda=0}, \quad n = 0,1,2,... \tag{3.7}$$

The general formula Equation 4.11 can be simplified as follows

$$\begin{aligned} A_0 &= N(w_0), \\ A_1 &= w_1 N'(w_0), \\ A_2 &= w_1 N'(w_0) + \frac{1}{2!}w_1^2 N''(w_0), \\ A_3 &= w_3 N'(w_0) + w_1 w_2 N''(w_0) + \frac{1}{3!}w_1^2 N'''(w_0). \end{aligned} \tag{3.8}$$

It is noted from the above equation that A_0 depends only on w_0, A_1 on w_0 and w_1, and continuously in that manner.

Finally, after determined these Components $w_0(x,t), w_1(x,t), w_0(x,t), ...$, we then substitute the obtained component into Equation (3.1), to obtain the solution in a series form.

The determined series may converge very rapidly to a closed-form solution if an exact solution exists. For concrete problems, where a closed-form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. A few terms of the truncated series give an approximation with a high degree of accuracy. In [29], the Adomian decomposition method (ADM) was applied to the coupled system of Burgers' equation.

4. Derivation for the Computational Scheme

This section derives the proposed Decomposition Method of Lines (DMOL) schemes for the computational treating of one-dimensional Burgers' equation, and the system of coupled one-dimensional Burgers' equations as presented in the above section. In light of this, the procedure starts with the MOL process by first converting the partial differential equations under consideration into a coupled system first-order Ordinary Differential Equations (ODEs) through approximating the spatial derivatives in the governing models with the non-central difference 5-point scheme. Thereafter, the resulting coupled of system first-order ODEs is then solved with the help of the standard ADM.

4.1. **DMOL scheme for Burgers' equation.** To derive the DMOL computational scheme for Burgers' equation, let us consider an initial-boundary value problem for the one-dimensional Burgers' equation earlier expressed in Equation 2.1 as follows [8]

$$w_t = v w_{xx} - w w_x, \qquad a < x < b, \tag{4.1}$$

subject to the following initial-boundary data

$$w(x,0) = f(x), \quad a \le x \le b, \tag{4.2}$$

$$w(a,t) = g_1(t), \quad w(b,t) = g_2(t), \quad 0 \le t \le T, \tag{4.3}$$

where v is as explained; while the functions f, g are known nice functions.

Accordingly, to begin with, we portray the spatial discretization by considering the spatial variable x, and discretize it into $N+1$ uniformly spaced grid points as follows

$$x_i = x_{i-1} + \Delta x, \quad i = 1, 2, ..., N,$$

where $\Delta x = x_i - x_{i-1} = 1/N$ is a constant spacing, x_0 and x_N are the two end-points (specifically, the boundary points), while $x_i' s$ for $i = 2, 3, 4, ..., (N-1)$ are the interior points.

Now, to approximate the first and second-order spatial derivatives using non-central difference 5-point scheme [4], we define Y and Λ are $N \times N$ matrices that represent approximate first and second derivatives, respectively, as follows

$$\mathrm{Y} = \begin{pmatrix} -25 & 48 & -36 & 16 & -3 & 0 & 0 & 0 & \dots & 0 \\ 0 & -3 & -10 & 18 & -6 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -8 & 0 & 8 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & -8 & 0 & 8 & -1 & 0 & \dots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & -8 & 0 & 8 & -1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 6 & -18 & 10 & 3 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 3 & -16 & 36 & -48 & 25 \end{pmatrix}, \tag{4.4}$$

and

$$\Lambda = \begin{pmatrix} 35 & -104 & 114 & -56 & 11 & 0 & 0 & 0 & \dots & 0 \\ 0 & 11 & -20 & 6 & 4 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & -16 & -30 & 16 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 16 & -30 & 16 & -1 & 0 & \dots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \cdots & 0 & -1 & 16 & -30 & 16 & -1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 4 & 6 & -20 & 11 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 11 & -56 & 114 & -104 & 35 \end{pmatrix}. \tag{4.5}$$

Furthermore, if we discretize in space and leave time continuous, a system of ODEs is obtained. Next, we solve the resulting system of ODEs by ADM [5]. Indeed, the decomposition method decomposes the unknown function $w(x,t)$ as follows

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t), \tag{4.6}$$

where the component $w_n, n \geq 1$ are determined recurrently. Further, upon utilizing the differential operator of the expression $L_t = \partial/\partial t$ into Equation 4.1, the equation is then expressed as follows

$$L_t(w) = v\Lambda w - \frac{1}{2}\Upsilon(w^2), \tag{4.7}$$

where the nonlinear term appearing in Equation 4.7 is expressed component-wise as follows

$$\begin{aligned}
A_0 &= w_0 w_{0_x},\\
A_1 &= \frac{1}{2}L_x(2w_0w_1) = w_{0_x}w_1 + w_0w_{1_x},\\
A_2 &= \frac{1}{2}L_x(2w_0w_2 + w_1^2) = w_{0_x}w_2 + w_{1_x}w_1 + w_{2_x}w_0,\\
A_3 &= \frac{1}{2}L_x(2w_0w_3 + 2w_1w_2) = w_{0_x}w_3 + w_{1_x}w_2 + w_{2_x}w_1 + w_{3_x}w_0,\\
&\vdots
\end{aligned} \tag{4.8}$$

and so on. Also, for the sake of numerical computation, the $(N-1)$-term approximant is considered; while the overall expected recursive scheme is obtained as follows

$$\begin{aligned}
w_1 &= f(x_0) + L_t^{-1}v\,L_{xx}(w_0) - L_t^{-1}A_0,\\
w_2 &= f(x_1) + L_t^{-1}v\,L_{xx}(w_1) - L_t^{-1}A_1,\\
&\vdots\\
w_{N-1} &= f(x_{N-2}) + L_t^{-1}v\,L_{xx}(w_{N-2}) - L_t^{-1}A_{N-2},
\end{aligned} \tag{4.9}$$

where $L_t^{-1} = \int_0^t (.)dt$ is the inverse operator that is applied on both sides of Equation 4.7. Notably, it is worth mentioning that the generation of the overall recurrent scheme is that both the zeroth term $w(x,0)$ and $w(N,t)$ must always be defined from the prescribed initial and boundary data; while the remaining component $w_{N-1}(x,t)$ can then be successfully computed recurrently.

Now, upon using the approximations and Adomian polynomials in Equation 4.1, one gets the first-order coupled system of ODEs as follows

$$\dot{w}_i = \frac{v}{4!(\Delta x)^2}\left[-w_{i-2} + 16w_{i-1} - 30w_i + 16w_{i+1} - w_{i+2}\right] - L_t^{-1}A_i, \quad \text{for} \quad i = 3(1)N-3. \tag{4.10}$$

where A_i's are the resulting Adomian polynomials for the nonlinear term, which are computed for any given nonlinearity based on the Adomian's scheme as follows ([26–28]).

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i w_i\right)\right]_{\lambda=0}, \qquad n = 0,1,2,... \tag{4.11}$$

Lastly, with the above development, the components in Equation 4.10 are acquired, which when further substituted into Equation 4.6 gives the explicit expressions for $w_1, w_2, w_i, ..., w_{N-1}$. Indeed, one obtains the approximate computational solution for the governing equation in Equation 4.1. Besides, this study utilizes only m-term of the Adomian polynomials.

4.2. **DMOL scheme for system of coupled Burgers' equations.** Similarly, we use a non-central difference 5-point scheme [4] to approximate the first and second-order spatial derivatives of the system of coupled Burgers' equation as carried out in the preceding case. For more convenience, we re-express the governing model in Equation 2.2 in the form of an initial-boundary conditions as follows ([30–32])

$$w_{1_t} = v_1 w_{1_{xx}} + \eta_1 w_1 w_{1_x} + \alpha(w_1 w_{2_x} + w_{1_x} w_2), \quad a < x < b, \tag{4.12}$$

$$w_{2_t} = v_2 w_{2_{xx}} + \eta_2 w_2 w_{2_x} + \beta(w_1 w_{2_x} + w_{2_x} w_2), \quad a < x < b, \tag{4.13}$$

$$w_1(x,0) = f_1(x), \quad w_2(x,0) = g_1(x), \quad a \le x \le b,$$

$$w_1(a,t) = f_2(t), \quad w_2(a,t) = g_2(t), \quad 0 \le t \le T,$$

$$w_1(b,t) = f_3(t), \quad w_2(b,t) = g_3(t), \quad 0 \le t \le T,$$

where η_j and v_j are real constants, for $j = 1,2$ and α and β are as explained; while the functions f_1, f_2, f_3, g_1, g_2 and g_3 are known nice functions.

Now, as in the previous section, we first portray the spatial discretization by considering the spatial variable x, and discretizing it into $N+1$ equal-sized grid points as $x_i = x_{i-1} + \Delta x$, for $i = 1,2,...,N$. Indeed, $\Delta x = x_i - x_{i-1} = 1/N$ is a constant, spacing, while x_0 and x_N are the two end-points (specifically, the boundary points), together with $x_i's$ for $i = 2,3,...,(N-1)$ as interior points.

Next, to approximate the first and second-order spatial derivatives using the non-central difference 5-point scheme as matrices Υ, Λ in Equations 4.4 and 4.5 that represent approximate first and second-order derivatives, respectively. We obtained a coupled system of ODEs when discretizes in space and leaves time continuous. Thus, the system can be solved by the ADM [5]. Accordingly, the decomposition method decomposes the unknown functions $w_1(x,t)$, and $w_2(x,t)$ as follows

$$w_1(x,t) = \sum_{n=0}^{\infty} w_{1_n}(x,t), \tag{4.14}$$

$$w_2(x,t) = \sum_{n=0}^{\infty} w_{2_n}(x,t), \tag{4.15}$$

where the components $w_{1_n}, w_{2_n}, n \ge 1$ are determined recurrently.

Further, with the help of a differential operator $L_t = \partial/\partial t$, Equations 4.12 - 4.13 are now expressed as follows

$$L_t(w_{1_t}) = v_1 \Lambda w_1 + \eta_1 \frac{1}{2} \Upsilon (w_1)^2 + \alpha(w_1 \Upsilon(w_2) + \Upsilon(w_1) w_2), \tag{4.16}$$

$$L_t(w_{2_t}) = v_2 \Lambda w_2 + \eta_2 \frac{1}{2} \Upsilon (w_2)^2 + \beta(w_1 \Upsilon(w_2) + \Upsilon(w_1) w_2). \tag{4.17}$$

Equally, the nonlinear expressions in Equations 4.16-4.17 can be denoted as follows

$$N(w) = w_1 w_{1_x} = \sum_{n=0}^{\infty} A_n, \quad M(w) = w_2 w_{2_x} = \sum_{n=0}^{\infty} B_n, \tag{4.18}$$

$$C(w) = w_1 w_{2_x} = \sum_{n=0}^{\infty} C_n, \quad D(w) = w_{1_x} w_2 = \sum_{n=0}^{\infty} D_n, \tag{4.19}$$

where A_n, B_n, C_n, D_n are related Adomian's polynomials to be determined with the help of the Adomian algorithm ([26–28]), which will then be substituted into the first derivative in Equation 4.4. More precisely, for the sake of numerical computation, the $(N-1)$-term approximant is considered as the expected recursive scheme is obtained as follows

$$\begin{aligned}
w_{1_1} &= f_1(x_0) + L_t^{-1} v_1 L_{xx}(w_1) + L_t^{-1}\mu_1(A_0) + L_t^{-1}\lambda_1(C_0 + D_0),\\
w_{2_1} &= g_1(x_0) + L_t^{-1} v_2 L_{xx}(w_2) + L_t^{-1}\mu_2(B_0) + L_t^{-1}\lambda_2(C_0 + D_0),\\
\\
w_{1_2} &= f_1(x_1) + L_t^{-1} v_1 L_{xx}(w_1) + L_t^{-1}\mu_1(A_1) + L_t^{-1}\lambda_1(C_1 + D_1),\\
w_{2_2} &= g_1(x_1) + L_t^{-1} v_2 L_{xx}(w_2) + L_t^{-1}\mu_2(B_1) + L_t^{-1}\lambda_2(C_1 + D_1),\\
&\ \vdots\\
w_{1_{N-1}} &= f_1(x_{N-2}) + L_t^{-1} v_1 L_{xx}(w_1) + L_t^{-1}\mu_1(A_{N-2}) + L_t^{-1}\lambda_1(C_{N-2} + D_{N-2}),\\
w_{2_{N-1}} &= g_2(x_{N-2}) + L_t^{-1} v_2 L_{xx}(w_2) + L_t^{-1}\mu_2(B_{N-2}) + L_t^{-1}\lambda_2(C_{N-2} + D_{N-2}),
\end{aligned} \tag{4.20}$$

where $L_t^{-1} = \int_0^t (.)dt$ is the inverse operator that is applied on both sides of Equations (4.16)-(4.17).

Moreover, it is thus pertinent to notice that the resultant recurrent scheme is formed based on the zeroth components $w_1(x,0)$ and $w_2(x,0)$ of the imposed initial data; while the remaining components $w_{1_{N-1}}(x,t)$ and $w_{2_{N-1}}(x,t)$ are constructed in such a way that each term follows recurrently.

Therefore, the resulting first-order coupled system of ODEs is thus determined as follows

$$\begin{aligned}
\dot{w}_{1_i} &= \frac{v_1}{4!(\Delta x)^2}\left[-w_{1_{i-2}} + 16 w_{1_{i-1}} - 30 w_{1_i} + 16 w_{1_{i+1}} - w_{1_{i+2}}\right] - \eta_1(L_t^{-1}A_i)\\
&\quad - \alpha(L_t^{-1}C_i - L_t^{-1}D_i), \qquad \text{for} \quad i = 3(1)N-3,\\
\dot{w}_{2_i} &= \frac{v_2}{4!(\Delta x)^2}\left[-w_{2_{i-2}} + 16 w_{2_{i-1}} - 30 w_{2_i} + 16 w_{2_{i+1}} - w_{2_{i+2}}\right] - \eta_2(L_t^{-1}B_i),\\
&\quad - \beta(L_t^{-1}C_i - L_t^{-1}D_i), \qquad \text{for} \quad i = 3(1)N-3.
\end{aligned} \tag{4.21}$$

Accordingly, the m-term Adomian polynomial will be utilized computationally, to determine the components from Equation 4.21. Additionally, upon substituting these components into Equations 4.14-4.15, one obtains the explicit expressions for $w_{j_1}, w_{j_2}, w_{j_i}, ..., w_{j_{N-1}}$, for $j = 1, 2$, which yields the approximate computational solution for the governing coupled model in Equations 4.12- 4.13.

5. Linearization Method

In the current section, we illustrate the solution of the one-dimensional Burgers' equation, together with that of the coupled system of one-dimensional Burgers' equations using the linearization method, which was introduced in [15].

5.1. Burger's equation.

By applying the standard MOL procedure, we get the following nonlinear system

$$\frac{\partial \mathrm{w_i}}{\partial t} = \mathrm{v}\,\Lambda \mathrm{w} - \frac{1}{2}\Upsilon(\mathrm{w})^2, \quad i = 1..N, \tag{5.1}$$

where, $\mathrm{w} = (\mathrm{w}_0, \mathrm{w}_1, \mathrm{w}_2, \cdots, \mathrm{w}_N)^T$, Υ and Λ are presented in Equations 4.5 and 4.4.

Then, in order to solve the system in Equation 5.1, one converts it into a linear system by using the linearization technique [15]. Firstly, the system in Equation 5.1 can be rewritten as

$$\frac{dW}{dt} = H(W^n), \quad W(0) = W_0, \tag{5.2}$$

where $W_i(t) = \mathrm{w}_i$, and $H(W) = \mathrm{v}\,\Lambda \mathrm{W} - \frac{1}{2}\Upsilon(\mathrm{W})^2$. Further, integrating Equation 5.2 using implicit method, we have

$$\begin{aligned}
\int_{t=t_n}^{t=t_{n+1}} dW &= \int_{t=t_n}^{t=t_{n+1}} H(W)dt, \\
\frac{W^{n+1} - W^n}{\triangle\, t} &= \frac{[H(W^{n+1}) + H(W^n)]}{2}, \\
W^{n+1} &= W^n + \frac{\triangle\, t[H(W^{n+1}) + H(W^n)]}{2},
\end{aligned} \tag{5.3}$$

or equally when linearized by Taylor series reveals

$$H(W^{n+1}) = H(W^n) + J_H^n(W^{n+1} - W^n) + O(\triangle t^2), \tag{5.4}$$

where J_H^n is a Jacobian matrix at the n^{th} time level. Now, when one substitutes Equation 5.4 into Equation 5.3 yields

$$W^{n+1} = W^n + \left(I - \frac{\triangle t}{2} J_H^n\right)^{-1} \frac{\triangle t}{2}\,[2\,H(W^n)]. \tag{5.5}$$

Hence, the formula in Equation 5.5 is linearized, upon which at each time step one is needed to solve the resulting system of linear algebraic equations only.

5.2. System of coupled Burgers' equations.

Similarly, we make use of the linearization technique to solve the governing system of coupled Burgers' equations by primarily the system in Equation 2.2 as follows

$$\frac{dW_1}{dt} = S_1(W_1^n, W_2^n), \quad W_1(0) = W_{1_0} \quad \text{and} \quad W_2(0) = W_{2_0}, \tag{5.6}$$

$$\frac{dW_2}{dt} = S_2(W_1^n, W_2^n), \quad W_1(0) = W_{1_0} \quad \text{and} \quad W_2(0) = W_{2_0}, \tag{5.7}$$

where,

$$S_1(W_1, W_2) = \mathrm{v}_1 \Lambda W_1 + \frac{\eta_1}{2}\Upsilon(W_1)^2 + \alpha\,(W_1\,\Upsilon W_2 + W_2\,(\Upsilon W_1)),$$

and

$$S_2(W_1, W_2) = \mathrm{v}_2 \Lambda W_2 + \frac{\eta_2}{2} \Upsilon (W_2)^2 + \beta \left(W_1 \, \Upsilon W_2 + W_2 \left(\Upsilon W_1 \right) \right),$$

Υ while Λ are presented in Equations 4.5 and 4.4. Further, integrating Equations 5.6 and 5.7, one gets

$$\begin{aligned} W_1^{n+1} &= W_1^n + \frac{\triangle\, t[S_1(W_1^{n+1}, W_2^{n+1}) + S_1((W_1^n, W_2^n)]}{2}, &(5.8)\\ W_2^{n+1} &= W_2^n + \frac{\triangle\, t[S_2(W_1^{n+1}, W_2^{n+1}) + S_1((W_1^n, W_2^n)]}{2}, &(5.9)\end{aligned}$$

or equally when linearized via the use of Taylor series yields

$$\begin{aligned} S_1(W_1^{n+1}, W_2^{n+1}) &= S_1(W_1^n, W_2^n) + J_{S_{1,1}}^n (W_1^{n+1} - W_1^n) + J_{S_{1,2}}^n (W_2^{n+1} - W_2^n) + O(\triangle t^2),\\ S_2(W_1^{n+1}, W_2^{n+1}) &= S_2(W_1^n, W_2^n) + J_{S_{2,1}}^n (W_1^{n+1} - W_1^n) + J_{S_{2,2}}^n (W_2^{n+1} - W_2^n) + O(\triangle t^2),\end{aligned} \tag{5.10}$$

where, $S_1 = (s_1^1, s_1^2, ..., s_1^N)$, $S_2 = (s_2^1, s_2^2, ..., s_2^N)$, $z_1 = W_1$, $z_2 = W_2$, while $J_{S_{1,j}}$ and $J_{S_{2,j}}$ are given as follows

$$J_{S_{1,j}} = \begin{pmatrix} \left(\frac{\partial s_1^1}{\partial z_{j,1}}\right)^n & \left(\frac{\partial s_1^1}{\partial z_{j,2}}\right)^n & \cdots & \left(\frac{\partial s_1^1}{\partial z_{j,N}}\right)^n \\ \left(\frac{\partial s_1^2}{\partial z_{j,1}}\right)^n & \left(\frac{\partial s_1^2}{\partial z_{j,2}}\right)^n & \cdots & \left(\frac{\partial s_1^2}{\partial z_{j,N}}\right)^n \\ \vdots & & & \\ \left(\frac{\partial s_1^N}{\partial z_{j,1}}\right)^n & \left(\frac{\partial s_1^N}{\partial z_{j,2}}\right)^n & \cdots & \left(\frac{\partial s_1^N}{\partial z_{j,N}}\right)^n \end{pmatrix}, \text{ and } J_{S_{2,j}} = \begin{pmatrix} \left(\frac{\partial s_2^1}{\partial z_{j,1}}\right)^n & \left(\frac{\partial s_2^1}{\partial z_{j,2}}\right)^n & \cdots & \left(\frac{\partial s_2^1}{\partial z_{j,N}}\right)^n \\ \left(\frac{\partial s_2^2}{\partial z_{j,1}}\right)^n & \left(\frac{\partial s_2^2}{\partial z_{j,2}}\right)^n & \cdots & \left(\frac{\partial s_2^2}{\partial z_{j,N}}\right)^n \\ \vdots & & & \\ \left(\frac{\partial s_2^N}{\partial z_{j,1}}\right)^n & \left(\frac{\partial s_2^N}{\partial z_{j,2}}\right)^n & \cdots & \left(\frac{\partial s_2^N}{\partial z_{j,N}}\right)^n \end{pmatrix}, \tag{5.11}$$

for $j = 1, 2$. What is more, from Equation 5.10 in Equations 5.8 and 5.9, one sees that

$$\begin{aligned} W_1^{n+1} =& W_1^n + \left(I - \frac{\triangle t}{2} J_{S_{1,1}}^n\right)^{-1} \frac{\triangle t}{2} \left[2\, S_1(W_1^n, W_2^n) + J_{S_{1,2}}^n (W_2^{n+1} - W_2^n)\right],\\ W_2^{n+1} =& W_2^n + \left(I - \frac{\triangle t}{2} J_{S_{2,2}}^n\right)^{-1} \frac{\triangle t}{2} \left[2\, S_2(W_1^n, W_2^n) + J_{S_{2,1}}^n (W_1^{n+1} - W_1^n)\right].\end{aligned} \tag{5.12}$$

Consequently, since Equation 5.12 is implicit, one approximates W_j^{n+1}, for $j = 1, 2$ using Euler method. Thus, the scheme in Equation 5.12 is then linearized, upon which at each time step one is needed to solve the resultant system of linear algebraic equations only.

6. Error and Convergence Analysis

6.1. **Error Analysis.** To assess the accuracy of the Decomposition Method of Lines (DMOL), we analyze the errors arising from both spatial and temporal discretizations. The total error E can be expressed as:

$$E = E_{spatial} + E_{temporal},$$

where $E_{spatial}$ is the spatial discretization error and $E_{temporal}$ is the temporal discretization error.

Spatial Discretization Error: The spatial discretization in the DMOL method involves approximating the partial derivatives using a non-central difference 5-point scheme. Let $w(x,t)$ be the exact solution and $w_h(x,t)$ be the numerical solution obtained by the DMOL method with a spatial step size h. The spatial discretization error Espatial can be given as:

$$E_{spatial} = w(x,t) - w_h(x,t).$$

The 5-point non-central difference scheme provides an approximation to the second-order spatial derivative w_{xx} with an error term that is $O(h^4)$. Thus, we have:

$$w_{xx} \approx \frac{-w_{i-2} + 16w_{i-1} - 30w_i + 16w_{i+1} - w_{i+2}}{12h^2} + O(h^4).$$

For the first-order spatial derivative w_x, the scheme provides an approximation with an error term that is $O(h^4)$:

$$w_x \approx \frac{-w_{i-2} + 8w_{i-1} - 8w_{i+1} - w_{i+2}}{12h} + O(h^4).$$

Thus, the spatial error for the DMOL method is $O(h^4)$.

Temporal Discretization Error: The temporal discretization involves solving the resulting system of ordinary differential equations (ODEs) using the Adomian Decomposition Method (ADM). Let $w(t)$ be the exact temporal solution and $w_{\Delta t}(t)$ be the numerical solution with a time step Δt. The temporal discretization error Etemporal is given by:

$$E_{temporal} = w(t) - w_{\Delta t}(t).$$

ADM provides a series solution that converges to the exact solution under certain conditions. The error in the ADM is dependent on the number of terms N in the decomposition series. For practical purposes, we consider a finite number of terms, leading to an error term that decreases exponentially with the number of terms included in the series.

Combining the spatial and temporal errors, the total error E for the DMOL method is:

$$E = O(h^4) + O(e^{-\lambda N}),$$

where λ is a positive constant depending on the problem and N is the number of terms in the ADM series.

6.2. **Convergence Analysis.** The convergence of the DMOL method can be demonstrated by showing that the numerical solution $w_{numerical}(x,t)$ converges to the exact solution $w(x,t)$ as $h \to 0$ and $\Delta t \to 0$.

Spatial Convergence: As $h \to 0$, the spatial discretization error $O(h^4)$ tends to zero, indicating that the DMOL method provides a spatially convergent solution.

$$\lim_{h \to 0} E_{spatial} = 0.$$

Temporal Convergence: Similarly, as the number of terms N in the ADM series increases, the exponential error term $O(e^{-\lambda N})$ tends to zero, indicating temporal convergence.

$$\lim_{N\to\infty} E_{temporal} = 0.$$

Therefore, the combined error tends to zero as $h \to 0$ and $N \to \infty$:

$$\lim_{h\to 0, N\to\infty} E = 0.$$

Numerical Validation. To validate the theoretical error bounds and convergence properties, we consider several test problems with known exact solutions and compare the numerical results obtained using the DMOL method.

Consider the one-dimensional Burgers' equation:

$$w_t + ww_x = vw_{xx}, \quad 0 \le x \le 1,\ t \ge 0$$

with the initial condition:

$$w(x,0) = 2v\frac{\pi\sin(\pi x)}{\sigma + \cos(\pi x)}, \tag{6.1}$$

and boundary conditions:

$$w(0,t) = w(1,t) = 0, \tag{6.2}$$

that satisfies the following exact analytical solution [34]

$$w(x,t) = \frac{2v\pi\exp^{-\pi^2 vt}\sin(\pi x)}{\sigma + \exp^{-\pi^2 vt}\cos(\pi x)}, \tag{6.3}$$

where σ is an arbitrary constant.

We solve this equation using the DMOL method and compute the L_2 and L_∞ norms of the error.

Results: The results for the test problem are summarized in Table 1, showing the L_2 and L_∞ norms of the error for different values of h and Δt. The errors decrease as h and Δt decrease, validating the theoretical error bounds and demonstrating the convergence of the DMOL method. Finally, the error and convergence analysis for the DMOL method demonstrates that the method is highly accurate and convergent for solving the Burgers' equation and the coupled system of Burgers' equations. The theoretical error bounds and numerical results validate the effectiveness of the DMOL method in providing reliable computational solutions with minimal errors.

h	Δt	L_2	L_∞
0.1	0.01	2.87736×10^{-5}	7.62714×10^{-5}
0.05	0.005	1.04066×10^{-5}	2.79760×10^{-5}
0.025	0.0025	4.2981×10^{-6}	7.5362×10^{-6}

TABLE 1. L_2 and L_∞ Norms of Error for Burgers' Equation

7. Numerical Examples and Results

The current section exhibits the competency of the proposed method for the computational treatment of the Burgers' equation and the system of coupled Burgers' equations. Indeed, certain Initial-Boundary Value problems (IBVPs) of the governing equations will be considered for the examination. In the same vein, the section measures the truthfulness of the proposed DMOL schemes for the considered IBVPs by deploying L_2 and L_∞ norms, expressed as

$$L_\infty(w) = \max_{1\leq i\leq N-1} \left|w_i^n - W_i^n\right|, \tag{7.1}$$

and

$$L_2(w) = \sqrt{\left[h\sum_{i=1}^{N-1} \left|w_i^n - W_i^n\right|^2\right]}, \tag{7.2}$$

respectively, as the estimators of interest.

7.1. **Burgers' equation.** This subsection makes consideration of certain IBVPs for the Burgers' equation by examining the computational efficiency of the proposed DMOL scheme on them. In addition, various IBVPs will be examined to effectively show the computational reliability of the devised method.

Problem 7.1. *Consider the IBVP for Burgers' equation as follows*

$$w_t - vw_{xx} + \alpha ww_x = 0, \tag{7.3}$$

that admits the following exact solitary wave structure [33]

$$w(x,t) = \frac{c}{\alpha} + \left(\frac{2v}{\alpha}\tanh(x-ct)\right), \qquad a \leq x \leq b, \tag{7.4}$$

where v, α and c are arbitrary constants, with c denoting the speed of the wave.

Therefore, we computationally simulate the given IBVP using the proposed DMOL scheme and further report the results in Tables 2 and 3, showing norm two and norm infinity error estimations for different values of α and v with various time levels with $a = 0$, $\Delta t = 0.01$, $c = 0.1$, and $b = 1$. Moreover, Tables 2 and 3, showing the variational effect of the kinematic viscosity v. The numerical solution obtained by the 5-point non-central DMOL and the exact solution of $w(x,t)$ for different values of h in Table 4. In addition, Figure 1, portrays the computational and the exact solutions of Problem 7.1: $N = 10$, $m = 2$, $\Delta t = 0.001$, $v = 0.0001$, $t = 0.1$, $\alpha = 1$, and $c = 0.05$.

TABLE 2. Influence of the kinematic viscosity v on the error for different values of α at different times for $w(x,t)$ with $N = 10$, $m = 2$ of Problem 7.1.

Δt	α	v	t	L_2		L_∞	
				DMOL	Linearized	DMOL	Linearized
0.01	0.1	0.01	0.05	7.9250×10^{-5}	1.3976×10^{-2}	2.5061×10^{-4}	1.9445×10^{-2}
			0.1	6.1679×10^{-5}	2.6276×10^{-4}	1.9505×10^{-4}	8.3091×10^{-4}
		0.001	0.05	8.3016×10^{-6}	4.1959×10^{-5}	2.6252×10^{-5}	1.3269×10^{-4}
			0.1	6.1677×10^{-6}	2.4747×10^{-5}	1.9504×10^{-5}	7.8258×10^{-3}
		0.0001	0.05	8.339×10^{-7}	4.178×10^{-6}	2.637×10^{-6}	1.3212×10^{-5}
			0.1	6.170×10^{-7}	2.4594×10^{-6}	1.951×10^{-6}	7.7773×10^{-6}
	1	0.01	0.05	7.9250×10^{-6}	1.7718×10^{-5}	2.5061×10^{-5}	5.6029×10^{-5}
			0.1	6.1679×10^{-6}	1.4245×10^{-5}	1.9505×10^{-5}	5.0456×10^{-5}
		0.001	0.05	8.302×10^{-7}	1.7537×10^{-6}	2.6252×10^{-6}	5.5458×10^{-6}
			0.1	6.168×10^{-7}	1.4097×10^{-6}	1.9504×10^{-6}	4.4577×10^{-6}
		0.0001	0.05	8.34×10^{-8}	1.752×10^{-7}	2.637×10^{-7}	5.540×10^{-7}
			0.1	6.17×10^{-8}	1.408×10^{-7}	1.951×10^{-7}	4.453×10^{-7}

TABLE 3. Influence of the kinematic viscosity v on the error for different values of α at different times for $w(x,t)$ with $N = 20$, $m = 2$ of Problem 7.1.

Δt	α	v	t	L_2		L_∞	
				DMOL	Linearized	DMOL	Linearized
0.01	0.1	0.01	0.1	5.2887×10^{-5}	3.1447×10^{-4}	2.3652×10^{-4}	1.4063×10^{-3}
			0.2	4.0552×10^{-5}	1.3699×10^{-4}	1.8136×10^{-4}	6.1266×10^{-4}
		0.001	0.1	5.5683×10^{-6}	3.0088×10^{-5}	2.4902×10^{-5}	1.3456×10^{-4}
			0.2	4.0553×10^{-6}	1.2876×10^{-5}	1.8136×10^{-5}	5.7584×10^{-5}
		0.0001	0.1	5.597×10^{-7}	2.9952×10^{-6}	2.503×10^{-6}	1.3395×10^{-5}
			0.2	4.054×10^{-7}	1.2794×10^{-6}	1.8130×10^{-6}	5.7216×10^{-6}
	1	0.01	0.1	5.2887×10^{-6}	1.3591×10^{-5}	2.3652×10^{-5}	6.0781×10^{-5}
			0.2	4.0552×10^{-6}	7.2839×10^{-6}	1.8136×10^{-5}	3.2575×10^{-5}
		0.001	0.1	5.568×10^{-7}	1.3454×10^{-6}	2.4902×10^{-6}	6.017×10^{-6}
			0.2	4.055×10^{-7}	7.204×10^{-7}	1.8136×10^{-6}	3.2219×10^{-6}
		0.0001	0.1	5.60×10^{-8}	1.344×10^{-7}	2.503×10^{-7}	6.011×10^{-7}
			0.2	4.05×10^{-8}	7.2×10^{-8}	1.813×10^{-7}	3.218×10^{-7}

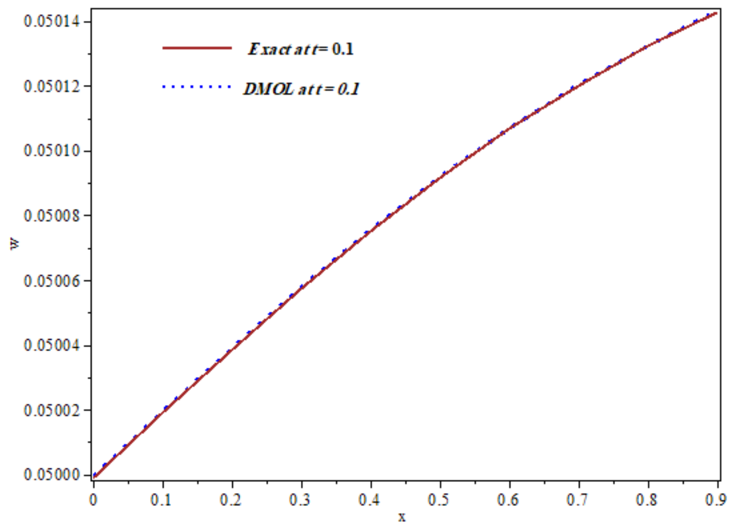

FIGURE 1. Comparison between DMOL and exact solutions of Problem 7.1 for $N = 10$, $m = 2$, $\Delta t = 0.01$, $\alpha = 1$, and $c = 0.05$.

TABLE 4. Comparing DMOL and exact solutions of Problem 7.1 when $a = 0$, $b = 1$ for $\alpha = 1$, $v = 0.01$ and $c = 0.1$ using various mesh sizes.

Δt	t	h	$w(x,t)$	
			DMOL	Exact
0.1	0.5	0.2	0.1132807354	0.1132695469
		0.1	0.1075069583	0.1067919301
		0.06666666667	0.1051102820	0.1050235024
		0.02857142857	0.1021729316	0.1020781860
		0.01538461538	0.1011272144	0.1010298576
		0.0125	0.1008974856	0.1007995736

Problem 7.2. *Consider the Burgers' equation 7.3, subject to the following initial-boundary data*

$$w(x,0) = 2v\frac{\pi \sin(\pi x)}{\sigma + \cos(\pi x)}, \tag{7.5}$$

and

$$w(0,t) = w(1,t) = 0, \tag{7.6}$$

that satisfies the following exact analytical solution [34]

$$w(x,t) = \frac{2v\pi \exp^{-\pi^2 vt} \sin(\pi x)}{\sigma + \exp^{-\pi^2 vt} \cos(\pi x)}, \tag{7.7}$$

where σ is an arbitrary constant.

Accordingly, Table 5 shows the comparison between the computational DMOL solution with that of the exact solution when $h = 1/10$ at $t = 1\times10^{-4}$. In addition, Figure 2 graphically portrays the comparison between the proposed DMOL solution and that of the available exact solution of Problem 7.2 when $m = 3$, $N = 10$, $\Delta t = 0.0001$, $v = 0.01$, $t = 1\times10^{-4}$, and $\sigma = 2$. Notably, the two solutions are found to graphically agree with each other with a precision; besides, this precision is noted to relatively reduce as x increases, see Table 5.

TABLE 5. Comparing DMOL and exact solutions of Problem 7.2 when $a = 0$, $b = 1$, and $t = 1\times10^{-4}$ for $m = 3$, $N = 10$, $v = 0.01$ and $\sigma = 2$.

Δt	x	$w(x,t)$		Absolute Error
		DMOL	Exact	
0.0001	0.1	0.0065793519	0.0065793320	1.99×10^{-8}
	0.2	0.0131474764	0.0131474381	3.83×10^{-8}
	0.3	0.0196429742	0.0196429163	5.79×10^{-8}
	0.4	0.0258795481	0.0258794685	7.97×10^{-8}
	0.5	0.0314157250	0.0314156165	1.085×10^{-7}
	0.6	0.0353381534	0.0353379931	1.603×10^{-7}
	0.7	0.0359943265	0.0359940487	2.778×10^{-7}
	0.8	0.0310096154	0.0310088594	7.560×10^{-7}
	0.9	0.0185101587	0.0185098104	3.483×10^{-7}

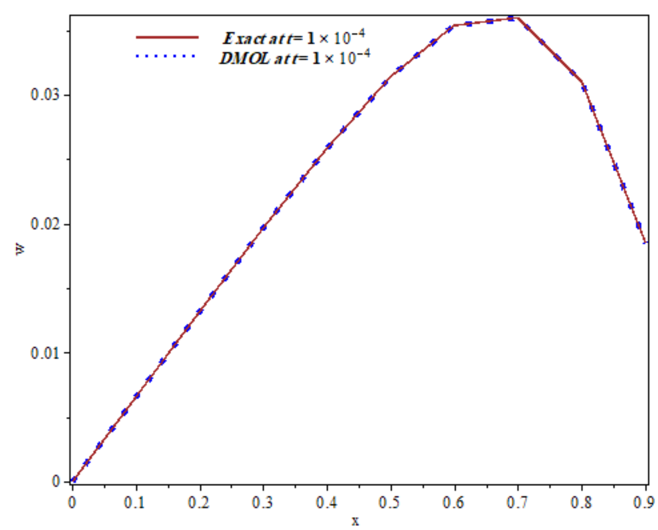

FIGURE 2. Comparison between the DMOL and exact solutions of Problem 7.2 when $m = 3$, $N = 10$, $\Delta t = 0.0001$, and $\sigma = 2$.

7.2. **System of coupled Burgers' equation.** This subsection makes consideration of certain IBVPs for the system of coupled Burgers' equations by examining the computational efficiency of the proposed DMOL scheme on them. In addition, various IBVPs will be examined in order to effectively show the computational reliability of the devised method.

Problem 7.3. *Consider the coupled system of IBVPs for homogeneous Burgers' equations as follows*

$$w_{1_t} - v_1 w_{1_{xx}} + \eta_1 w_1 w_{1_x} + \alpha(w_1 w_{2_x} + w_{1_x} w_2) = 0, \quad a < x < b, \tag{7.8}$$

$$w_{2_t} - v_2 w_{2_{xx}} + \eta_2 w_2 w_{2_x} + \beta(w_1 w_{2_x} + w_{2_x} w_2) = 0, \quad a < x < b, \tag{7.9}$$

that satisfies the following exact solitary wave solution [33]

$$w_1(x,t) = a_0 - 2A\left(\frac{2\alpha-1}{4\alpha\beta-1}\right)\tanh\left[A(x-2At)\right],$$

$$w_2(x,t) = a_0\left(\frac{2\beta-1}{2\alpha-1}\right) - 2A\left(\frac{2\alpha-1}{4\alpha\beta-1}\right)\tanh\left[A(x-2At)\right],$$

where $A = a_0(1-4\alpha\beta)/2(1-2\alpha)$, *with* α, β *and* a_0, *as free arbitrary constants.*

As proceeded, Tables 6 and 7 report the resulting norm two infinity errors as the difference between the proposed DMOL and exact solutions of the present coupled model when $a_0 = 0.05$, $v_1 = v_2 = 1$, $N = 10$, and $\eta_1 = \eta_2 = 2$. Indeed, the proposed numerical results are found to be in good agreement with the available exact solution. Moreover, we have shown in Figure 3 (3a) and (3b) the graphical illustration for the acquired computational solution in comparison with the exact solution fields $w_1(x,t)$ and $w_2(x,t)$ when the following constants are fixed: $m = 2$, $N = 10$, $\Delta t = 0.1$, $t = 0.5$, $\alpha = 2$, $\beta = 3$, and $a = 0.1$.

TABLE 6. Norm two and norm infinity errors of the solution pair $w_1(x,t)$ when $a = -10$, $b = 10$, with $a_0 = 0.05$, and $m = 2$ for Problem 7.3.

N	Δt	α	β	t	L_2		L_∞	
					DMOL	Linearized	DMOL	Linearized
10	0.1	0.1	0.3	0.5	3.778×10^{-7}	9.9249×10^{-6}	2.672×10^{-7}	7.0179×10^{-6}
				1	1.0194×10^{-5}	2.9705×10^{-4}	7.2082×10^{-6}	2.1005×10^{-4}
		0.3	0.03	0.5	1.5268×10^{-5}	7.0116×10^{-4}	1.07961×10^{-5}	4.9579×10^{-4}
				1	4.1054×10^{-5}	7.2969×10^{-4}	2.9029×10^{-5}	5.1597×10^{-4}
	0.01	0.1	0.3	0.05	3.77×10^{-8}	6.2538×10^{-6}	2.66×10^{-8}	4.4221×10^{-6}
				0.1	1.0194×10^{-6}	3.0817×10^{-5}	7.208×10^{-7}	2.1790×10^{-5}
		0.3	0.03	0.05	1.5275×10^{-6}	6.9953×10^{-5}	1.0801×10^{-6}	4.9464×10^{-5}
				0.1	4.1042×10^{-6}	7.1514×10^{-5}	2.9021×10^{-6}	5.0568×10^{-5}

Table 7. Norm two and norm infinity errors of the solution pair $w_2(x,t)$ when $a=-10, b=10$, with $a_0=0.05$, and $m=2$ for Problem 7.3.

N	Δt	α	β	t	L_2		L_∞	
					DMOL	Linearized	DMOL	Linearized
10	0.1	0.1	0.3	0.5	2.9183×10^{-6}	1.0692×10^{-4}	2.0636×10^{-6}	7.5605×10^{-5}
				1	1.0194×10^{-5}	6.2583×10^{-5}	7.2082×10^{-6}	4.4253×10^{-5}
		0.3	0.03	0.5	3.1622×10^{-6}	5.2490×10^{-5}	2.2360×10^{-6}	3.7116×10^{-5}
				1	4.1054×10^{-5}	2.2508×10^{-5}	2.9029×10^{-5}	1.5915×10^{-5}
	0.01	0.1	0.3	0.05	2.919×10^{-7}	1.0679×10^{-5}	2.064×10^{-7}	7.5515×10^{-6}
				0.1	1.0194×10^{-6}	6.2551×10^{-6}	7.208×10^{-7}	4.4230×10^{-6}
		0.3	0.03	0.05	3.157×10^{-7}	5.248×10^{-6}	2.232×10^{-7}	3.7109×10^{-6}
				0.1	4.1042×10^{-6}	2.2857×10^{-6}	2.9021×10^{-6}	1.6162×10^{-6}

(A) (B)

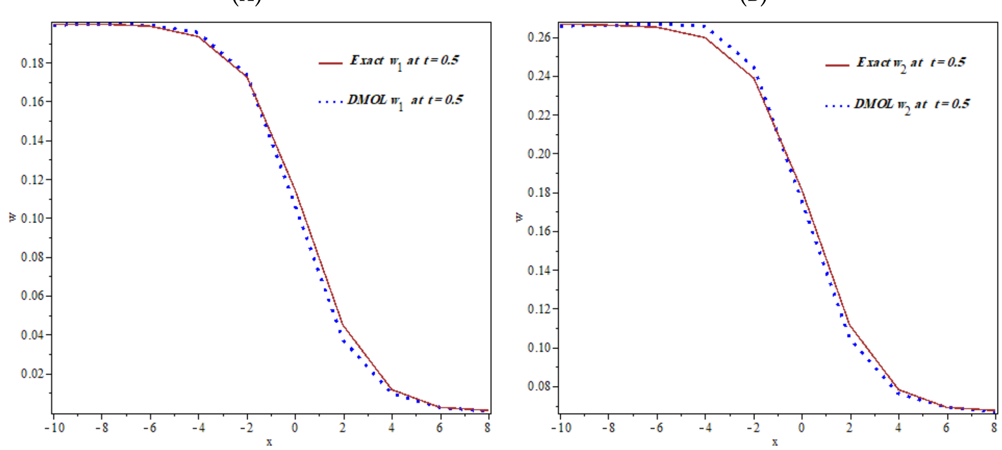

Figure 3. Comparison between DMOL and exact solutions of Problem 7.3 for $m=2$, $N=10$, $\Delta t=0.1$ and $\alpha=2, \beta=3$ at $t=0.5$, with $a_0=0.1$.

Problem 7.4. *Let us yet consider the coupled system of IBVPs for homogeneous Burgers' equations expressed in 7.8- 7.9 with* $\alpha=\beta=\frac{5}{2}$ *and* $\eta=-2$.

Moreover, this coupled IBVP is said to satisfy the following analytical exact solution [32].

$$w_1(x,t)=w_2(x,t)=\lambda\left[1-\tanh\left(\lambda(60(x-\frac{1}{2})-\frac{9}{2}\lambda t)\right)\right], \qquad x\in[0,1], \tag{7.10}$$

where λ *is an arbitrary constant, while the imposed initial and boundary data are obtained from the given analytical solution and* $t=0$*; and* $x=0$ *and* $x=1$ *correspondingly.*

This problem has large gradients moving rightward with constant velocity. In Table 8, we show the obtained results for various values of λ. It can be seen from Table 8 that by decreasing the values of λ one can achieve more accurate results. On the other hand, as time grows errors get smaller. The numerical solution obtained by DMOL and the exact solution of $w(x,t)$ for different values of h in Table 9. In Figure 4 (4a) and (4b), we plot the numerical solution for $\lambda = 0.05, 0.1$ and $\Delta t = 0.005$ at $t = 0.05$. In Figure 5, we display the numerical and the exact solutions fileds $w_1(x,t)$ and $w_2(x,t)$ values when $N = 64$ for $\lambda = 0.01, 0.05, 0.1, 0.2$ and $\Delta t = 0.005$ at $t = 0.005$. We see that for greater values of λ, large gradient regions occur in the solution.

Table 8. Comparing DMOL and exact solutions of Problem 7.4 for $\lambda = 0.05, 0.10$ and $\Delta t = 0.001$ with $N = 10$, $m = 5$, at $a = 0$, $b = 1$ for $\alpha = \beta = \frac{5}{2}$.

λ	t	$w_1(x,t) = w_2(x,t)$		Absolute Error
		DMOL	Exact	
0.01	0.0010	0.0129122039	0.0129131302	9.263×10^{-7}
	0.005	0.0105993289	0.0105992855	4.34×10^{-8}
	0.01	0.0076450425	0.0076450468	4.2×10^{-9}
	0.0110	0.0070868739	0.0070868780	4.1×10^{-9}
0.05	0.0010	0.0952155827	0.0952575144	4.1932×10^{-5}
	0.005	0.0645709808	0.0645661454	4.8354×10^{-6}
	0.01	0.0083172696	0.0083174412	1.716×10^{-7}
	0.0110	0.0047425873	0.0047426890	1.016×10^{-7}
0.1	0.0010	0.1993892579	0.1995055198	1.1626×10^{-4}
	0.005	0.1537208964	0.1537081587	1.2738×10^{-5}
	0.01	0.0016325142	0.0016326600	1.457×10^{-7}
	0.0110	0.0004945246	0.0004945690	4.44×10^{-8}

(A) (B)

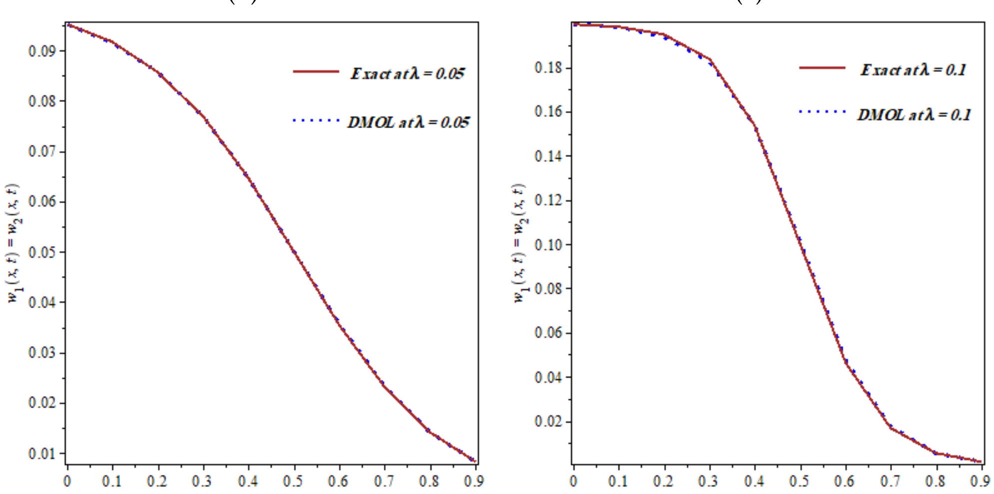

Figure 4. Comparison between DMOL and exact solutions of Problem 7.4 when $m = 5$, $N = 10$, $\Delta t = 0.001$, $t = 0.001$, $\alpha = \beta = \frac{5}{2}$, and $\lambda = 0.05, 0.1$.

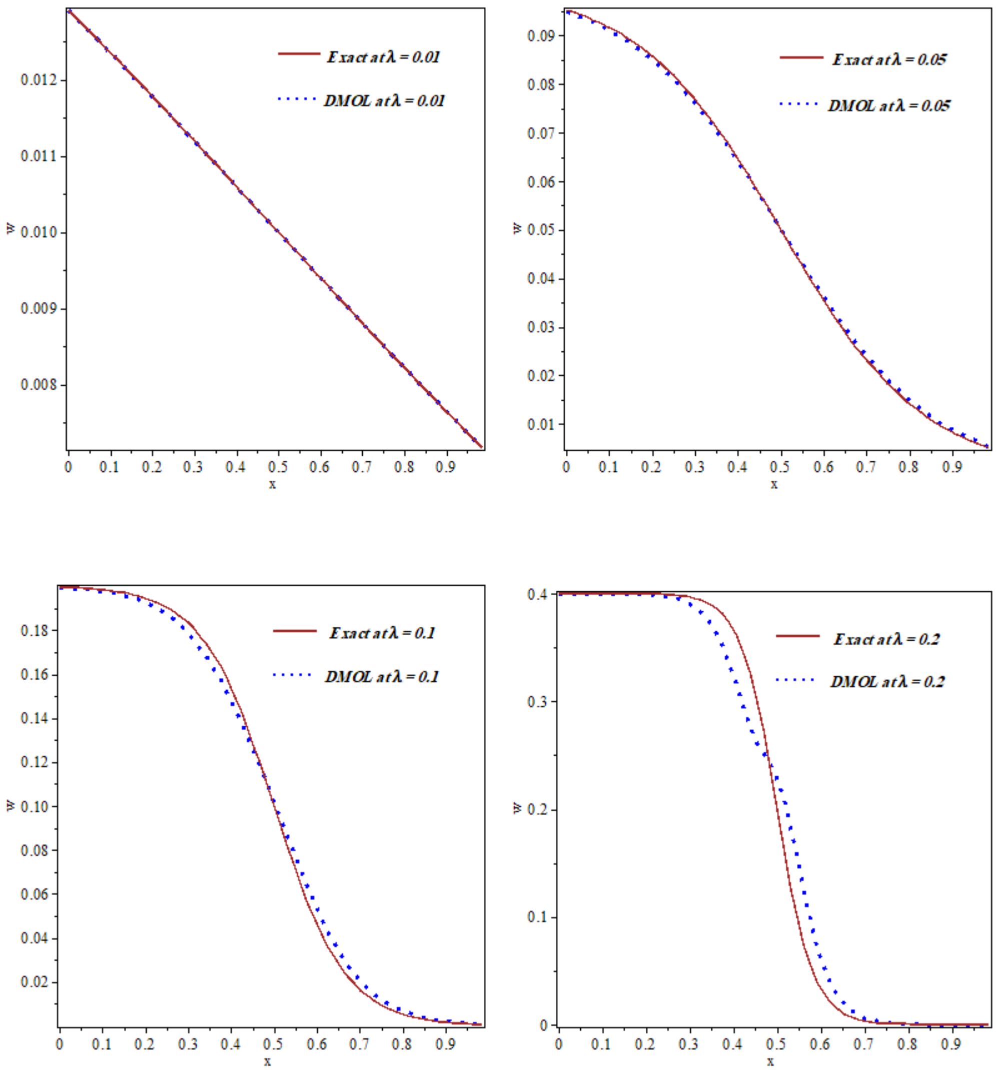

FIGURE 5. Comparison between the DMOL and exact solutions of Problem 7.4 when $m = 2$, $N = 64$, $\Delta t = 0.005$, $t = 0.005$, $\alpha = \beta = \frac{5}{2}$, and $\lambda = 0.01, 0.05, 0.1, 0.2$.

TABLE 9. Comparing DMOL and exact solutions of Problem 7.4 when $a = 0$, $b = 1$ for $\alpha = \beta = 5/2$, $t = 0.005$ using various mesh sizes.

Δt	λ	h	$w_1(x,t) = w_2(x,t)$	
			DMOL	Exact
0.001	0.05	0.1428571429	0.0396238199	0.0394472887
		0.0625	0.0816032205	0.0817577832
		0.05	0.0856575627	0.0858151674
		0.02777777778	0.0910361003	0.0911602136
		0.01960784314	0.0925106087	0.0926179683
		0.015625	0.0931462242	0.0932454726

8. Conclusions

In conclusion, the current study proposed a new computational scheme for the solution of the class of one-dimensional Burgers' equations, comprising mainly the classical Burgers' equation, and the system of coupled Burgers' equations. Certainly, this new approach is based upon coupling the semi-discretization approach, through the application of the standard MOL and on the other hand, the renowned ADM for a reliable computational treatment of the class of IBVPs under examination. In this new approach, the semi-discretization of partial differential equations in spatial variables is carried out by MOL, and thereafter, the ADM proceeds with the treatment of the resulting coupled system of ODEs. Further, the study went ahead to assess the derived scheme on several test problems, where the results obtained were compared with the available exact at different time levels. The computational errors were also evaluated using L_2 and L_∞ norms. Thus, the proposed method produces better results, as it tends to the available exact analytical solutions so rapidly. The numerical results obtained by DMOL proved that it is indeed a very accurate, efficient, and powerful way to solve IBVPs for different classes of partial differential equations. Also, the devised DMOL scheme produces better results at small values, while the linearization method produces better results at larger values. Besides, the reported figures and tables indicate that calculations are more accurate and in conformity with the analytical solutions, in addition to the fact that the proposed scheme outperformed several other computational approaches.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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