Int. J. Anal. Appl. (2025), 23:36

International Journal of Analysis and Applications

Robin Boundary Value Problems With Natural Growth Term in Variable Exponent Space

Boulaye Yira[1,2,*], **Ibrahime Konaté**[2], **Mamadou Diop**[3]

[1]*LAboratoire de Mathématiques d'Informatique et Applications (LaMIA); Université Nazi BONI, 01 BP 1091 Bobo 01, Bobo Dioulasso, Burkina Faso*

[2]*Département de Mathématiques, Laboratoire de Sciences et Technologie (LaST), Université Thomas SANKARA, 12 BP 417 Ouagadougou 12, Burkina Faso*

[3]*Laboratoire Eco-Matériaux et Habitats Durables (LEMHaD), Institut International d'Ingénierie de l'Eau et de l'Environnement (Institut 2iE), Rue de la Science, Ouagadougou 01 BP 594, Burkina Faso*

[*]*Corresponding author:* ibrakonat@yahoo.fr

Abstract. The main purpose of this paper is to investigate a nonlinear elliptic problem with a natural growth term under Robin boundary conditions. Using approximation techniques and surjectivity criteria of an operator mapping from a Banach space into its dual, we prove the existence of a sequence of weakly approximated solutions and take its limit to establish the existence of a renormalized or entropy solution for the initial problem.

1. Introduction

In this paper we consider the following nonlinear Robin boundary value problem

$$(\mathcal{P})\begin{cases} \beta(u) - div\, a(x,u,\nabla u) + H(x,u,\nabla u) \ni f & \text{in } \Omega \\ \\ a(x,u,\nabla u)\cdot \nu = -|u|^{p(x)-2}u & \text{on } \partial\Omega. \end{cases}$$

Here we suppose that Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $f \in L^1(\Omega)$ and ν represents the outer unit normal vector on $\partial\Omega$. The function $p(.) : \overline{\Omega} \longrightarrow \mathbb{R}^+$ is continuous and satisfies the following conditions

$$1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty.$$

Received: Dec. 12, 2024.

2020 *Mathematics Subject Classification.* 35J15, 35J20, 35J60, 35J67.

Key words and phrases. Sobolev spaces; variable exponent; Leray-Lions operator; maximal monotone graph; renormalized solution; entropy solution.

https://doi.org/10.28924/2291-8639-23-2025-36

ISSN: 2291-8639

The function β is a maximal monotone graph such that $0 \in \beta(0)$ and $\mathrm{int}(\mathrm{dom}\beta) = (m, M)$ with $-\infty \leq m \leq 0 \leq M \leq \infty$.
We assume that the $p(.)$-Leray-Lions type operator $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions for all $\xi, \eta \in \mathbb{R}^N$, $s \in \mathbb{R}$, and for a.e. $x \in \Omega$

$$a(x,s,\xi).\xi \geq C_1|\xi|^{p(x)}, \tag{1.1}$$

$$|a(x,s,\xi)| \leq C_2(k(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{1.2}$$

$$(a(x,s,\xi) - a(x,s,\eta))(\xi - \eta) > 0 \text{ if } \xi \neq \eta, \tag{1.3}$$

and

$$a(x,s,0) = 0, \tag{1.4}$$

where $C_1 > 0,\ C_2 > 0$ and $k(.)$ is a given nonnegative function in $L^{p'(.)}(\Omega)$. Note that $p'(.)$ stands for the conjugate exponent of $p(.)$.
The Carathéodory function $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a nonlinear term having natural growth of order $p(.)$ with respect to $|\nabla u|$ and fullfils a sign condition :

$$|H(x,s,\xi)| \leq b(|s|)(c(x) + |\xi|^{p(x)}), \text{ for a.e. } x \in \Omega,\ s \in \mathbb{R},\ \xi \in \mathbb{R}^N \tag{1.5}$$

and

$$H(x,s,\xi)s \geq 0, \tag{1.6}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function, $c(.)$ a given nonnegative function in $L^1(\Omega)$.
We emphasize that the Robin problems exhibiting a natural growth term have been less analyzed when the $p(.)$-Leray-Lions type operator depends simultaneously on the spatial variable x, the solution u, and its gradient ∇u. However, assuming $g \equiv 0$, certain authors addressed the problem $(\mathcal{P})$ when the operator a depends only on x and the gradient of the solution. Indeed, in the framework of classical Sobolev spaces constant exponents p, Motreanu et *al.* [31] have recently employed a sub-supersolution approach to establish the existence of solution for the following Robin boundary value problems with full gradient dependence

$$(\mathcal{P}) \begin{cases} div\, a(x,\nabla u) + \alpha(x)|u|^{p-2} = f(x,u,\nabla u) \ \text{ in } \ \Omega \\ \\ a(x,\nabla u)\cdot \nu + \eta(x)|u|^{p-2}u = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

We also refer the reader to the references cited therein [31] for motivation regarding Robin boundary conditions. As far as the Robin boundary problems are concerned, some results have been extended from classical Sobolev spaces to Sobolev spaces with variable exponents. In [36], by considering the $p(.)$-Leray-Lions type operator $a(x,\nabla u)$ and H identically zero, Ouaro et *al.* (see also [34, 35, 38]) proved the existence and uniqueness of a renormalized and an entropy solution of a Robin boundary problem when the source term is an integrable function or a measure data. Further, they studied in [37, 42] the problem $(\mathcal{P})$ without the natural growth term in the context of the

$p(x,u)-$Laplacian operator (i.e., the exponent $p(.)$ depends on the solution u). In [39], Alain prignet analyzed the existence of solutions to the following elliptic problem

$$div\, a(x,u,\nabla u) = f \text{ in } \mathcal{D}' \tag{1.8}$$

where $f \in (C(\overline{\Omega}))'$ and with non-homogeneous boundaries condition of types Neumann (i.e $a(x,u,\nabla u).\eta = g$ on $\partial\Omega$), Fourier (i.e $a(x,u,\nabla u).\eta + \lambda u = g$ on $\partial\Omega$), Dirichlet ($u = g$ on $\partial\Omega$). We also recall that nonlinear elliptic problems involving the general Leray-lions operator are most studied under the Dirichlet boundary conditions (see [7,9,11,13–16,19,26,45]) and Neumann boundary conditions ([7,12,33]) Besides the mathematical interest in solving partial differential equations in Sobolev spaces with variable exponents, researchers are motivated by their applications across various disciplines, including physical and mechanical processes, electro-rheological fluids, as well as stationary thermo-rheological viscous flows of non-Newtonian fluids (see [6,20,25,40,41] for more details). Another great progress in their application was the modeling of image processing ([18]).

Recently, in [3], the authors analyzed the existence of solutions for the problem $(\mathcal{P})$ under Dirichlet boundary conditions in the framework of constant exponent. Then, in [28], the second author and collaborators extended their results in the context of $p(.)$ variable exponents and measure data.

In this paper, we use the theory of maximal monotone operators in Banach spaces and approximation techniques to establish the existence of renormalized and entropy solutions for a broad class of multivalued nonlinear elliptic problems involving Robin boundary conditions. We stress that the concept of renormalized solutions is an alternative approach to solving partial differential equations when the data are not smooth (L^1-function or a measure), and classical methods might fail. This type of solutions was initially introduced by Lions and Diperna [21] (see also [1,2,26,28]) to tackle elliptic equations when the right-hand side data belong to L^1 or are represented as a measure.

Since we only have weak-* convergence of the Yosida approximation β_ϵ of β (see below) in the space of bounded measures, a measure arises that must be properly handled when passing to the limit. To overcome this difficulty, we proceed as in [28,32,34–36] by taking into account the measure that appears in the definition of solution. Another obstruction is that the Poincaré inequality and the Poincaré-Wirtinger inequality with variable exponents cannot be applied. However, we overcome this by using the Poincaré-Wirtinger inequality with a constant exponent. In contrast to the Dirichlet case, the solution of the Robin boundary problem is not null on the boundary. We must handle the solution on the boundary. To overcome this difficulty, we proceed as in [38] by defining a new space that accounts for the boundary condition (see the proof of Proposition 3.1).

The added value of our work in the literature is the resolution of multivalued problems involving simultaneously a general nonlinear operator (i.e., $-div(a(x,u,\nabla u))$), a natural growth term and Robin boundary condition. Furthermore, this work is a generalization of results from constant to variable exponent Sobolev spaces.

The remainder of this paper is structured as follows. Section 2 introduces key definitions and properties of Sobolev spaces with variable exponents. In Section 3, we demonstrate the existence of a renormalized solution. Finally, Section 4 concludes by proving the existence of entropy solutions.

2. Preliminaries

In the following, we provide definitions and outline the fundamental properties of Lebesgue and Sobolev spaces with variable exponents.

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, and let $p(.) : \overline{\Omega} \longrightarrow \mathbb{R}^+$ be a continuous function with

$$1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty. \tag{2.1}$$

We denote

$$C_+(\overline{\Omega}) = \Big\{ p(.) : \overline{\Omega} \longrightarrow (1,\infty) \text{ continuous such that } 1 < p^- \leq p^+ < \infty \Big\}.$$

From this point onward, we assume that $p \in C_+(\overline{\Omega})$ and define the variable exponent Lebesgue space as follows:

$$L^{p(.)}(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u|^{p(x)} dx < \infty \Big\}.$$

Let $p(.)$ be in $\in C_+(\overline{\Omega})$. Then, the expression

$$\|u\|_{p(.)} := \inf\Big\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \Big\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm. Moreover, $(L^{p(.)}(\Omega), \|.\|_{p(.)})$ is a separable, reflexive and uniformly convex Banach space. Hence its dual space is isomorphic to $L^{p'(.)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ in Ω.

We call the $p(.)$-modular of $L^{p(.)}(\Omega)$ the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$

Let $u \in L^{p(.)}(\Omega)$, then, we have (see [22], [23]):

$$\min\Big\{ \|u\|_{p(.)}^{p^-} ; \|u\|_{p(.)}^{p^+} \Big\} \leq \rho_{p(.)}(u) \leq \max\Big\{ \|u\|_{p(.)}^{p^-} ; \|u\|_{p(.)}^{p^+} \Big\}. \tag{2.2}$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$, we have the Hölder type inequality (see [29]):

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{p(.)} \|v\|_{p'(.)}. \tag{2.3}$$

Let $p_1,\ p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ for any $x \in \Omega$. Then the embedding $L^{p_2(.)}(\Omega) \hookrightarrow L^{p_1(.)}(\Omega)$ is continuous (see [29]).

Proposition 2.1. *[29] Let $u_n, u \in L^{p(x)}(\Omega)$. Then, the following properties hold true:*

(i) $\|u\|_{p(.)} < 1$ *(resp,* $= 1,\ > 1$*) if and only if* $\rho_{p(.)}(u) < 1$ *(resp,* $= 1,\ > 1$*);*
(ii) $\|u\|_{p(.)} > 1$ *imply* $\|u\|_{p(.)}^{p_-} \leq \rho_{p(.)}(u) \leq \|u\|_{p(.)}^{p_+}$*, and* $\|u\|_{p(.)} < 1$ *imply* $\|u\|_{p(.)}^{p_+} \leq \rho_{p(.)}(u) \leq \|u\|_{p(.)}^{p_-}$*;*
(iii) $\|u_n\|_{p(.)} \to 0$ *if and only if* $\rho_{p(.)}(u_n) \to 0$*, and* $\|u_n\|_{p(.)} \to \infty$ *if and only* $\rho_{p(.)}(u_n) \to \infty$*.*

Now, we define the variable exponent Sobolev space by

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) :\ |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

with the norm

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

For a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we introduce the following notation

$$\rho_{1,p(.)}(u) := \int_\Omega |u|^{p(x)} dx + \int_\Omega |\nabla u|^{p(x)} dx.$$

Proposition 2.2. *(see [43, 44]) If* $u \in W^{1,p(.)}(\Omega)$*, the following properties hold true:*
(i) $\|u\|_{1,p(.)} > 1 \Rightarrow \|u\|_{1,p(.)}^{p^-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p^+}$*;*
(ii) $\|u\|_{1,p(.)} < 1 \Rightarrow \|u\|_{1,p(.)}^{p^+} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p^-}$*;*
(iii) $\|u\|_{1,p(.)} < 1$ *(respectively,* $= 1,\ > 1$*)* $\Longleftrightarrow \rho_{1,p(.)}(u) < 1$ *(respectively,* $= 1,\ > 1$*).*

By setting

$$p^\partial(x) := (p(x))^\partial := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N, \end{cases}$$

we have the following result.

Proposition 2.3. *(see [44]) Let* $p \in C_+(\overline{\Omega})$*. If* $q \in C(\partial\Omega)$ *such that*

$$1 < q(x) < p^\partial(x) \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$*. In particular, there is a compact embedding* $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$*.*

We denote by $W_0^{1,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \geq N$.

Lemma 2.1. *[9] Let* $u, u_n \in L^{p(.)}(\Omega)$ *such that* $\|u_n\|_{p(.)} \leq C$*. If* $u_n(.) \to u(.)$ *a.e. in* Ω*, then* $u_n \rightharpoonup u$ *in* $L^{p(.)}(\Omega)$*.*

Theorem 2.1. *[23, 27]*
(i) *The space* $W^{1,p(.)}(\Omega)$ *is a separable and reflexive Banach space.*
(ii) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \Omega$*, then the embedding* $W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega)$ *is continuous and compact.*
(iii) *Poincaré inequality : there exists a constant* $C > 0$*, such that*

$$\|u\|_{p(.)} \leq C\|\nabla u\|_{p(.)},\ \forall u \in W_0^{1,p(.)}(\Omega).$$

(iv) *Sobolev-Poincaré inequality : there exists a constant* $C > 0$*, such that*

$$\|u\|_{p^*(.)} \leq C\|\nabla u\|_{p(.)}, \ \forall u \in W_0^{1,p(.)}(\Omega).$$

Remark 2.1. *In view of (iii) in Theorem 2.1, we deduce that* $\|\nabla u\|_{p(.)}$ *and* $\|u\|_{1,p(.)}$ *are equivalent norms in* $W_0^{1,p(.)}(\Omega)$.

Lemma 2.2. *[39] There exists a constant* $C > 0$ *such that for any* $u \in W^{1,1}(\Omega)$*, one has*

$$\int_\Omega |u|dx \leq C\Big(\int_\Omega |\nabla u|dx + \int_{\partial\Omega} |u|d\sigma\Big) \tag{2.4}$$

and there exists a constant $C' > 0$ *such that for any* $u \in W^{1,q}(\Omega),\ 1 < q < N$*, one has*

$$\Big(\int_\Omega |u|^{q^*}dx\Big)^{\frac{q}{q^*}} \leq C'\Big(\int_\Omega |\nabla u|^q dx + \Big(\int_{\partial\Omega} |u|d\sigma\Big)^q\Big), \tag{2.5}$$

where $q^* = \frac{Nq}{N-q}$.

Remark 2.2. *It is clear that for* $1 \leq p^- \leq p^+ < \infty$*, one has*

$$(a+b)^{p(x)} \leq 2^{p^+-1}(a^{p(x)} + b^{p(x)}).$$

Throughout the paper, we will use the truncation function T_k of level $k > 0$ defined by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.6}$$

One can easily see that $\lim\limits_{k\to\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.
$\mathcal{T}^{1,p(.)}(\Omega) := \{u : \Omega \to \mathbb{R}$ measurable function such that $T_k(u) \in W^{1,p(.)}(\Omega)\}$.
Let us introduce some functions that will be frequently used in this paper.
For $r \in \mathbb{R}$, let $r^+ := \max(r, 0)$ and sign_0^+ be the function defined by

$$sign_0^+(r) = \begin{cases} 1 \text{ if } r > 0, \\ 0 \text{ if } r \leq 0. \end{cases}$$

For any s and k in $\mathbb{R}$, with $k \geq 0$, we defined the function $G_k(s) = s - T_k(s)$.
Let β be a maximal monotone operator defined on $\mathbb{R}$. Then, we define its main section β^0 by

$$\beta^0(s) = \begin{cases} \text{minimal absolute value of } \beta(s) & \text{if } \beta(s) \neq \emptyset, \\ \infty & \text{if } [s,\infty) \cap D(\beta) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap D(\beta) = \emptyset. \end{cases}$$

For a maximal monotone graph β in $\mathbb{R} \times \mathbb{R}$, for any $\epsilon \in (0,1]$, the Yosida approximation β_ϵ of β (see [4,5,]) is given by $\beta_\epsilon = \dfrac{1}{\epsilon}(I - (I + \epsilon\beta)^{-1})$, which a non-decreasing and Lipschitz-continuous function.
If $s \in Dom(\beta)$, $|\beta_\epsilon(s)| \leq |\beta^0(s)|$ and $\beta_\epsilon(s) \longrightarrow \beta^0(s)$, as $\epsilon \to 0$, and if $s \notin Dom(\beta)$, $|\beta_\epsilon(s)| \longrightarrow \infty$, as $\epsilon \to 0$.
We also use the following useful convergence result (see [32]).

Lemma 2.3. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x,y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$. We suppose that: $\forall n \geq 1, w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then, $z \in dom(\beta)$.*

Proposition 2.4. *[] Let $u \in \mathcal{T}^{1,p(.)}(\Omega)$. Then there exists a unique measurable function $v : \Omega \longrightarrow \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u. Moreover if $u \in W^{1,p(.)}(\Omega)$ then $v \in (L^{p(.)}(\Omega))^N$ and $v = \nabla u$ in the usual sense.*

As in [5], we introduce $\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:
(i) $u_n \longrightarrow u$ a.e. in Ω.
(ii) $\nabla T_k(u_n) \longrightarrow \nabla T_k(u)$ in $(L^1(\Omega))^N$ for any $k > 0$.
(iii) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. in $\partial\Omega$.
The function v is the trace of u in the generalized sense introduced in [4,5].

Lemma 2.4. *[9] Let $u \in L^{p(.)}(\Omega)$ and $u_n \in L^{p(.)}(\Omega)$ such that $\|u_n\|_{L^{p(.)}(\Omega)} \leq C$. If $u_n \rightharpoonup u$ a.e. in Ω then, $u_n \rightharpoonup u$ in $L^{p(.)}(\Omega)$.*

In the sequel, the following lemma which proof follows the same lines as in [9] will be useful.

Lemma 2.5. *Assuming that* (1.1)-(1.3) *hold and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $W^{1,p(.)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p(.)}(\Omega)$ and*

$$\int_\Omega \Big[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big]\nabla(u_n - u)dx \to 0. \tag{2.7}$$

Then, $u_n \to u$ in $W^{1,p(.)}(\Omega)$.

3. Existence of renormalized solutions

Definition 3.1. *We say that a pair $(u,b) \in \mathcal{T}_{tr}^{1,p(.)}(\Omega) \times L^1(\Omega)$ is a renormalized solution of problem $(\mathcal{P})$ if it satisfies the following conditions:*

(P_1) *$u \in dom(\beta)$ $\mathcal{L}^N$- a.e. in Ω, $b \in \beta(u)$ $\mathcal{L}^N$- a.e. in Ω, $|u|^{p(x)-2}u \in L^1(\partial\Omega)$,*

(P_2) *there exists $\nu \in \mathcal{M}_b^{p(.)}(\Omega)$ such that $\nu \perp \mathcal{L}^N$, and the positive part ν^+ and negative part ν^- of ν satisfy the following conditions*

$$\nu^+ \text{ is concentrated on } [u = M] \cap [u \neq \infty],\ \nu^- \text{ is concentrated on } [u = m] \cap [u \neq -\infty],$$

(P_3) *for any $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and for any $S \in C_c(\mathbb{R})$,*

$$\int_\Omega a(x,u,\nabla u)\nabla(S(u)v)dx + \int_\Omega H(x,u,\nabla u)S(u)vdx + \int_\Omega bS(u)vdx$$
$$+ \int_\Omega S(u)vd\nu + \int_{\partial\Omega} |u|^{p(x)-2}uS(u)vd\sigma = \int_\Omega fS(u)vdx. \tag{3.1}$$

Moreover,

$$\lim_{l\to+\infty}\int_{\{l\leq|u|\leq l+1\}}|\nabla u|^{p(x)}dx=0. \tag{3.2}$$

Theorem 3.1. *Let $f\in L^1(\Omega)$. Then, the problem $(\mathcal{P})$ has at least one renormalized solution.*

Proof.

step 1 Approximate problem

Our first step is to consider the following approximate problem

$$(\mathcal{P}_\epsilon)\begin{cases}\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))-diva(x,u_\epsilon,\nabla u_\epsilon)+H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)+\epsilon|u_\epsilon|^{p(x)-2}u_\epsilon=f_\epsilon & \text{in }\Omega\\ \\ a(x,u_\epsilon,\nabla u_\epsilon)\cdot\eta=T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon)) & \text{on }\partial\Omega,\end{cases}$$

where $\gamma(u_\epsilon)=|u_\epsilon|^{p(x)-2}u_\epsilon$, $\beta_\epsilon:\mathbb{R}\to\mathbb{R}$ is the Yosida approximation of β and $H_\epsilon(x,s,\xi)=\dfrac{H(x,s,\xi)}{1+\epsilon|H(x,s,\xi)|}$, for any $\epsilon\in(0,1]$. For all $u\in W^{1,p(.)}(\Omega)$, remark that

$$\langle\beta_\epsilon(u),u\rangle\geq 0,\ |\beta_\epsilon(u)|\leq\frac{1}{\epsilon}|u|\ \text{ and }\ \lim_{\epsilon\to 0}\beta_\epsilon(u)=\beta(u).$$

One also has

$$H_\epsilon(x,s,\xi)s\geq 0,\ \ |H_\epsilon(x,s,\xi)|\leq|g(x,s,\xi)|\,,\ |H_\epsilon(x,s,\xi)|\leq\frac{1}{\epsilon}$$

and

$$|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|\leq\frac{1}{\epsilon^2}.$$

Furthermore, for any $\epsilon>0$, $f_\epsilon=T_{\frac{1}{\epsilon}}(f)$ is a sequence of bounded functions which converge to $f\in L^1(\Omega)$ as $\epsilon\to 0$.

Moreover, one has

$$\|f_\epsilon\|_{L^1(\Omega)}\leq\|f\|_{L^1(\Omega)} \tag{3.3}$$

Proposition 3.1. *For any $f\in(W^{1,p(.)}(\Omega))^*$, the problem $(\mathcal{P}_\epsilon)$ admits at least one weak solution $u_\epsilon\in W^{1,p(.)}(\Omega)$. Namely,*

$$\int_\Omega\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi dx+\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla\varphi dx+\int_\Omega H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)\varphi dx+\epsilon\int_\Omega|u_\epsilon|^{p(x)-2}u_\epsilon\varphi dx$$

$$\int_{\partial\Omega}T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))\varphi d\sigma=\int_\Omega f_\epsilon\varphi dx, \tag{3.4}$$

for all $\varphi\in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$.

Proof. We define the reflexive space

$$E:=W^{1,p(.)}(\Omega)\times L^{p(.)}(\partial\Omega)$$

and X_0 the subspace of E defined by

$$X_0=\{(u,v)\in E:v=\tau(u)\},$$

where $\tau(u)$ is the trace of $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ in the usual sense, since $u \in W^{1,p(.)}(\Omega)$. In the sequel, we will identify an element $(u,v) \in X_0$ with its representative $u \in W^{1,p(.)}(\Omega)$.
Let us define the operator $A_\epsilon : W^{1,p(.)}(\Omega) \to (W^{1,p(.)}(\Omega))^*$ as follows:
$\forall\ u, \varphi\ \in W^{1,p(.)}(\Omega)$,

$$\langle A_\epsilon(u), \varphi\rangle\ = \langle Au, \varphi\rangle + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi dx + \int_\Omega H_\epsilon(x,u,\nabla u)\varphi dx + \epsilon \int_\Omega |u|^{p(x)-2}u\varphi dx,$$

where $\langle Au, \varphi\rangle = \displaystyle\int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u))\varphi d\sigma.$

Lemma 3.1. *The operator A_ϵ is pseudo-monotone and bounded. Moreover, A_ϵ is coercive in the following sense*

$$\frac{\langle A_\epsilon(u), u\rangle}{\|u\|_{1,p(.)}} \to +\infty \text{ as } \|u\|_{1,p(.)} \to \infty.$$

Proof. There exists a constant $C_3 > 0$ such that (see [45])

$$\left|\int_\Omega H_\epsilon(x,u,\nabla u)\varphi dx\right| \le C_3\|\varphi\|_{1,p(.)}. \tag{3.5}$$

There exists a constant $C_4 > 0$ such that

$$\left|\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi dx\right| \le C_4\|\varphi\|_{1,p(.)}. \tag{3.6}$$

By using again Hölder type inequality, one has

$$\begin{aligned}
\left|\epsilon\int_\Omega |u|^{p(x)-2}u\varphi dx\right| &\le \epsilon\int_\Omega |u|^{p(x)-1}|\varphi| dx\\
&\le \epsilon\left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right)\||u|^{p(x)-1}\|_{p'(.)}\|\varphi\|_{p(.)}\\
&\le \epsilon\left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right)\||u|^{p(x)-1}\|_{p'(.)}(\|\varphi\|_{p(.)} + \|\nabla\varphi\|_{p(.)})\\
&\le C_5\|\varphi\|_{1,p(.)},
\end{aligned}$$

where $C_5 = \epsilon\left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right)\||u|^{p(x)-1}\|_{p'(.)}$.

$$\left|\int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u))\varphi d\sigma\right| \le \frac{1}{\epsilon}\int_{\partial\Omega} |\varphi| d\sigma. \tag{3.7}$$

Using the Hölder type inequality and the growth condition (1.5), we can show that A is bounded. From this, together with (1.2), (3.5) and (3.6), we deduce that A_ϵ is also bounded.
In order to prove the coercivity of A_ϵ, we set

$$\alpha = \begin{cases} p_+ \text{ if } \|u\|_{1,p(.)} \le 1,\\ \\ p_- \text{ if } \|u\|_{1,p(.)} > 1; \end{cases}$$

then, for all $u \in W^{1,p(.)}(\Omega)$, one has

$$\begin{aligned}
\frac{\langle A_\epsilon(u), u\rangle}{\|u\|_{1,p(.)}} &= \frac{\int_\Omega a(x,u,\nabla u)udx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))udx + \int_\Omega H_\epsilon(x,u,\nabla u)udx}{\|u\|_{1,p(.)}} \\
&+ \frac{\epsilon \int_\Omega |u|^{p(x)-2}udx + \int_{\partial\Omega} |u|^{p(x)-2}ud\sigma}{\|u\|_{1,p(.)}} \\
&\geq \frac{\int_\Omega a(x,u,\nabla u)\nabla udx + \epsilon \int_\Omega |u|^{p(x)-2}udx}{\|u\|_{1,p(.)}} \qquad \text{(by neglecting positive terms)} \\
&\geq \frac{\lambda \int_\Omega |\nabla u|^{p(x)}dx + \epsilon \int_\Omega |u|^{p(x)-2}udx}{\|u\|_{1,p(.)}} \\
&\geq \frac{\min(\epsilon,\lambda)\left(\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{p(x)}dx\right)}{\|u\|_{1,p(.)}} \\
&\geq \frac{\min(\epsilon,\lambda)\rho_{1,p(.)}(u)}{\|u\|_{1,p(.)}} \\
&\geq \min(\epsilon,\lambda)\frac{\|u\|^\alpha_{1,p(.)}}{\|u\|_{1,p(.)}} \\
&\geq \min(\epsilon,\lambda)\|u\|^{\alpha-1}_{1,p(.)} \to \infty \text{ as } \|u\|_{1,p(.)} \to \infty \text{ (since } 1 < p_- \leq p_+).
\end{aligned}$$

Thus, A_ϵ is coercive.

It remains to show that A_ϵ is pseudo-monotone. For that we consider a sequence $(u_k)_{k\in\mathbb{N}}$ in $W^{1,p(.)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \text{ in } X_0, \\ A_\epsilon u_k \rightharpoonup \chi \text{ in } X_0', \\ \limsup_{k\to\infty} \langle A_\epsilon u_k, u_k\rangle \leq \langle \chi_\epsilon, u\rangle. \end{cases} \tag{3.8}$$

Now, we aim to prove that

$$\langle A_\epsilon u_k, u_k\rangle \longrightarrow \langle \chi, u\rangle \text{ as } k \longrightarrow \infty \text{ with } \chi = A_\epsilon u.$$

By the compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$, there exists a subsequence, still denoted $(u_k)_{k\in\mathbb{N}}$, such that $u_k \to u$ in $L^{p(.)}(\Omega)$ as $k \to \infty$.

Since $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,p(.)}(\Omega)$, using the growth condition it follows, that $(a(x,u_k,\nabla u_k))_{k\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$. Therefore, there exists a function $\varphi \in (L^{p'(.)}(\Omega))^N$ such that

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } k \to \infty. \tag{3.9}$$

Since $(H_\epsilon(x,u_k,\nabla u_k))_{k\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$, it follows in the same manner that there exists a function $\psi_\epsilon \in (L^{p'(.)}(\Omega))^N$ such that

$$H_\epsilon(x,u_k,\nabla u_k) \rightharpoonup \psi_\epsilon \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } k\to\infty. \tag{3.10}$$

Given the inequality $|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k))| \le \frac{1}{\epsilon^2}$ and the convergence $u_k \longrightarrow u$ a.e. in Ω, as $k\to\infty$, one can apply the Lebesgue dominated convergence theorem and deduce that

$$\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k)) \longrightarrow \beta_\epsilon(T_{\frac{1}{\epsilon}}(u)) \text{ in } L^{(p_-)'}(\Omega),\ \text{ as } k\to\infty. \tag{3.11}$$

It can also be stated that

$$|u_k|^{p(x)-2}u_k \longrightarrow |u|^{p(x)-2}u \text{ in } L^{p(.)}(\partial\Omega)\ \text{ as } k\to\infty \tag{3.12}$$

and

$$\epsilon|u_k|^{p(x)-2}u_k \longrightarrow \epsilon|u|^{p(x)-2}u\ \text{ strongly in } L^{p'(.)}(\Omega),\ \text{as } k\to\infty. \tag{3.13}$$

Thus, for any $v\in W^{1,p(.)}(\Omega)$,

$$\begin{aligned}\langle\chi_\epsilon,v\rangle &= \lim_{k\to\infty}\langle A_\epsilon u_k,v\rangle\\ &= \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla v dx + \lim_{k\to\infty}\int_\Omega H_\epsilon(x,u_k,\nabla u_k)v dx\\ &+ \lim_{k\to\infty}\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k))v dx + \epsilon\lim_{k\to\infty}\int_\Omega |u_k|^{p(x)-2}u_k v dx + \lim_{k\to\infty}\int_{\partial\Omega}|u_k|^{p(x)-2}u_k v d\sigma\end{aligned}$$

$$= \int_\Omega \varphi\nabla v dx + \int_\Omega \psi_\epsilon v dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))v dx + \epsilon\int_\Omega |u|^{p(x)-2}uv dx + \int_{\partial\Omega}|u|^{p(x)-2}uv d\sigma. \tag{3.14}$$

Based on (3.8) and (3.14), one obtains

$$\begin{aligned}\limsup_{k\to\infty}\langle A_\epsilon u_k,u_k\rangle &= \limsup_{k\to\infty}\Big(\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx + \epsilon\int_\Omega |u_k|^{p(x)}dx\\ &+ \int_\Omega H_\epsilon(x,u_k,\nabla u_k)u_k dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k))u_k dx\\ &+ \int_{\partial\Omega}|u_k|^{p(x)}d\sigma\Big)\\ &\le \int_\Omega \varphi\nabla u dx + \int_\Omega \psi_\epsilon u dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))u dx + \int_{\partial\Omega}|u|^{p(x)}d\sigma\\ &+\epsilon\int_\Omega |u|^{p(x)}dx.\end{aligned} \tag{3.15}$$

Combining (3.10)-(3.13), as $k\to\infty$, we obtain

$$\int_\Omega H_\epsilon(x,u_k,\nabla u_k)u_k dx \longrightarrow \int_\Omega \psi_\epsilon u dx, \tag{3.16}$$

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_k))u_k dx \longrightarrow \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))u dx, \tag{3.17}$$

$$\epsilon \int_{\Omega} |u_k|^{p(x)-2} u_k \longrightarrow \epsilon \int_{\Omega} |u|^{p(x)-2} u dx \tag{3.18}$$

and

$$\int_{\partial\Omega} |u_k|^{p(x)-2} u_k d\sigma \longrightarrow \int_{\partial\Omega} |u|^{p(x)-2} u d\sigma. \tag{3.19}$$

It follows that

$$\lim_{k\to+\infty} \sup \int_{\Omega} a(x, u_k, \nabla u_k) \nabla u_k dx \le \int_{\Omega} \varphi \nabla u dx. \tag{3.20}$$

According to (1.3), we have

$$\int_{\Omega} (a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u))(\nabla u_k - \nabla u) dx \ge 0.$$

Then,

$$\int_{\Omega} a(x, u_k, \nabla u_k) \nabla u_k dx \ge \int_{\Omega} a(x, u_k, \nabla u_k) \nabla u dx + \int_{\Omega} a(x, u_k, \nabla u)(\nabla u_k - \nabla u) dx.$$

Given that $\nabla u_k \rightharpoonup \nabla u$ in $L^{p(.)}(\Omega)$, and using (3.41) one obtains

$$\lim_{k\to\infty} \inf \int_{\Omega} (a(x, u_k, \nabla u_k) \nabla u_k dx \ge \int_{\Omega} \varphi \nabla u dx.$$

From (3.20), one can deduce that

$$\lim_{k\to\infty} \int_{\Omega} a(x, u_k, \nabla u_k) \nabla u_k dx = \int_{\Omega} \varphi \nabla u dx. \tag{3.21}$$

Combining (3.16), (3.17), (3.18) and (3.21), we obtain

$$\langle A_\epsilon u_k, u_k \rangle \to \langle \chi_\epsilon, u \rangle \text{ as } k \to \infty.$$

It remain to prove that $a(x, u_k, \nabla u_k) \rightharpoonup a(x, u, \nabla u)$ in $(L^{p'(.)}(\Omega))^N$ and

$$H_\epsilon(x, u_k, \nabla u_k) \rightharpoonup H_\epsilon(x, u, \nabla u) \text{ in } L^{p'(.)}(\Omega) \text{ as } k \to \infty.$$

From (3.21), we can prove that

$$\lim_{k\to\infty} \int_{\Omega} (a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u))(\nabla u_k - \nabla u) dx = 0.$$

As stated in Lemma 2.5, one obtains

$$u_k \to u \text{ in } W^{1,p(.)}(\Omega) \text{ and } \nabla u_k \to \nabla u \text{ a.e. in } \Omega, \text{ as } k \to \infty.$$

Then, we have

$$a(x, u_k, \nabla u_k) \rightharpoonup a(x, u, \nabla u) \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } k \to \infty \tag{3.22}$$

and

$$H_\epsilon(x, u_k, \nabla u_k) \rightharpoonup H_\epsilon(x, u, \nabla u) \text{ in } L^{p'(.)}(\Omega) \text{ as } k \to \infty. \tag{3.23}$$

Thus, we can express $\chi_\epsilon = A_\epsilon u$, which completes the proof of Lemma 3.1. □

Since A_ϵ is bounded, coercive and pseudo-monotone, according to Theorem 2.7 in [30], A_ϵ is surjective.

Therefore, for any $f \in (W^{1,p(.)}(\Omega))^*$, there exists at least one solution $u_\epsilon \in W^{1,p(.)}(\Omega)$ to $(\mathcal{P}_\epsilon)$, which concludes the proof of Proposition 3.1. □

Step 2 The a priori estimate

Lemma 3.2. *Let u_ϵ be a weak solution of $(\mathcal{P}_\epsilon)$. Then, for any $k > 0$ one has,*

$$\int_\Omega |\nabla T_k(u_\epsilon)|^{p(x)}dx \leq k\frac{\|f\|_{L^1(\Omega)}}{C_3}, \tag{3.24}$$

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_k(u_\epsilon)dx \leq k\|f\|_{L^1(\Omega)} \tag{3.25}$$

and

$$\|T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))\|_{L^1(\partial\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{3.26}$$

Proof. Taking $T_k(u_\epsilon)$ as test function in (3.4), one obtains

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_k(u_\epsilon)dx + \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)dx + \int_\Omega H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)T_k(u_\epsilon)dx$$

$$+\epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon T_k(u_\epsilon)dx + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))T_k(u_\epsilon)d\sigma = \int_\Omega f_\epsilon T_k(u_\epsilon)dx. \tag{3.27}$$

Let us remark that

$$\int_\Omega f_\epsilon T_k(u_\epsilon)dx \leq k\|f\|_{L^1(\Omega)}.$$

Having in mind (1.1), (1.6) and the fact that T_k, $\beta_\epsilon, s \mapsto |s|^{r(.)-2}s$ are non-decreasing functions with $\beta_\epsilon(0) = T_k(0) = 0$, then all the integrals in (3.27) are nonnegative. Therefore, by disregarding the positive terms, it follows that

$$\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)dx \leq k\|f\|_{L^1(\Omega)}, \tag{3.28}$$

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_k(u_\epsilon)dx \leq k\|f\|_{L^1(\Omega)}, \tag{3.29}$$

$$\int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))T_k(u_\epsilon)d\sigma \leq k\|f\|_{L^1(\Omega)}. \tag{3.30}$$

Applying (1.1) in (3.28), one obtains (3.24).

After dividing (3.29) by $k > 0$ and letting $k \to 0$, one arrives at (3.25).

Similarly, dividing (3.30) by $k > 0$, one gets

$$\int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))\frac{1}{k}T_k(u_\epsilon)d\sigma \leq \|f\|_{L^1(\Omega)}.$$

Then, (3.26) follows by applying Lebesgue dominated convergence theorem. □

Remark 3.1. *We claim that* $|u|^{p(x)-2}u \in L^1(\partial\Omega)$.
Indeed, from (3.26), *we have*

$$\int_{\partial\Omega} |T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))|dx \le \|f\|_{L^1(\Omega)}. \tag{3.31}$$

By applying Fatou's Lemma, we obtain

$$\int_{\partial\Omega} |\gamma(u)|dx \le \liminf_{\epsilon\to 0} \int_{\partial\Omega} |T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))|dx \le \|f\|_{L^1(\Omega)}.$$

Lemma 3.3. *The sequences* $(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))_{\epsilon>0}$ *and* $(\beta_\epsilon(T_{\frac{1}{\epsilon}}(T_k(u_\epsilon))))_{\epsilon>0}$ *are uniformly bounded in* $L^1(\Omega)$.

Proof. Indeed, based on (3.29), one has

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\frac{1}{k}T_k(u_\epsilon)dx \le \|f\|_{L^1(\Omega)}.$$

Passing to the limit as $k \to 0$, one gets

$$\int_\Omega |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|dx \le \|f\|_{L^1(\Omega)}.$$

Thus, the sequence $(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$.
Using the following inequality

$$\int_\Omega |\beta_\epsilon(T_{\frac{1}{\epsilon}}(T_k(u_\epsilon)))|dx \le \int_\Omega |\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))|dx,$$

we deduce that $(\beta_\epsilon(T_{\frac{1}{\epsilon}}(T_k(u_\epsilon))))_{\epsilon>0}$ is also bounded in $L^1(\Omega)$. □

Lemma 3.4. *[36] Let* u_ϵ *be a solution* $(\mathcal{P}_\epsilon)$ *with* $k > 0$*. Then,*

$$\int_\Omega |\nabla T_k(u_\epsilon)|^{p^-}dx \le C(k+1) \tag{3.32}$$

and

$$\int_{\partial\Omega} |T_k(u_\epsilon)|^{p^-}d\sigma \le (k+1)\|f\|_{L^1(\Omega)}. \tag{3.33}$$

The following estimates follows from Lemmas 2.2 and 3.4.

Proposition 3.2. *[36] Let* u_ϵ *be a weak solution of* $(\mathcal{P}_\epsilon)$ *with* $k > 0$ *large enough. Then,*

$$meas\{|u_\epsilon| > k\} \le \frac{Const(f, p^-, (p^-)^*, \Omega)}{k^\alpha} \tag{3.34}$$

and

$$meas\Big\{|\nabla u_\epsilon| > k\Big\} \le \frac{const(f,\Omega)(k+1)}{k^{p^-}} + \frac{Const(f, p^-, (p^-)^*, \Omega)}{k^\alpha} \tag{3.35}$$

where $\alpha = (p^-)^*(1-\frac{1}{p^-})$ *and* $(p^-)^* = \begin{cases} \frac{Np^-}{N-p^-} \text{ if } p^- < N, \\ \\ \text{any element in } [N,\infty) \text{ if } p^- = N. \end{cases}$

Step 3 Convergence results.

Proposition 3.3. *[36] Let* u_ϵ *be a weak solution of* $(\mathcal{P}_\epsilon)$ *and* $k > 0$*. Then, we have*

(i) $T_k(u_\epsilon) \longrightarrow T_k(u)$ *in* $L^{p^-}(\Omega)$ *and a.e. in* Ω, *as* $\epsilon \to 0$.

(ii) $T_k(u_\epsilon) \longrightarrow T_k(u)$ *in* $L^{p^-}(\partial\Omega)$ *and a.e. on* $\partial\Omega$.

(iiI) *There exists* $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ *such that* $u \in dom(\beta)$ *a.e. in* Ω *and*

$$u_\epsilon \to u \text{ in measure and a.e. in } \Omega, \text{ as } \epsilon \to 0.$$

(iv) u_ϵ *converges a.e. on* $\partial\Omega$ *to some function* u.

The above convergences are not sufficient to pass to the limit, so we must prove the strong convergence of the sequence $(T_k(u_\epsilon))_{\epsilon>0}$.

Proposition 3.4. *Let* u_ϵ *be a solution of* $(\mathcal{P}_\epsilon)$ *and* $k > 0$. *Then, one has*

$$T_k(u_\epsilon) \to T_k(u) \text{ strongly in } W^{1,p(.)}(\Omega). \tag{3.36}$$

Proof. For all $k \geq 0$, we defined the function $\varphi(s) = se^{\alpha s^2}$ such that $\alpha = \left(\frac{b(k)}{\lambda}\right)^2$.
Let us recall that for any $s \in \mathbb{R}$, the function φ verifies the following inequality (see [13], Lemma 1)

$$\varphi'(s) - \frac{b(k)}{\lambda}|\varphi(s)| \geq \frac{1}{2}. \tag{3.37}$$

Now, we choose $\varphi_\epsilon = \varphi(T_k(u_\epsilon) - T_k(u))$ as a test function in (3.4) to obtain

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon dx + \int_\Omega a(x, u_\epsilon, \nabla u_\epsilon)\nabla\varphi_\epsilon dx + \int_\Omega H_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx$$
$$+ \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))\varphi_\epsilon d\sigma + \epsilon\int_\Omega |u_\epsilon|^{p(x)-1}|\varphi_\epsilon| dx = \int_\Omega f\varphi_\epsilon dx.$$

Due to the fact that φ_ϵ and u_ϵ have the same sign in the set $\{|u_\epsilon| > k\}$, one has

$$I_\epsilon^1 + I_\epsilon^2 + I_\epsilon^3 + I_\epsilon^4 + I_\epsilon^5 \leq I_\epsilon^6,$$

where $I_\epsilon^1 = \int_{\{|u_\epsilon|\leq k\}} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon dx$, $I_\epsilon^2 = \int_\Omega a(x, u_\epsilon, \nabla u_\epsilon)\nabla\varphi_\epsilon dx$, $I_\epsilon^3 = \int_\Omega H_\epsilon(x, u_\epsilon, \nabla u_\epsilon)\varphi_\epsilon dx$,
$I_\epsilon^4 = \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))\varphi_\epsilon d\sigma$, $I_\epsilon^5 = \epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon\varphi_\epsilon dx$, $I_\epsilon^6 = \int_\Omega f_\epsilon\varphi_\epsilon dx$.
Since $|u_\epsilon|^{p(x)-2}u_\epsilon \longrightarrow |u|^{p(x)-2}u$ in $L^{p'(.)}(\Omega)$ and $\varphi_\epsilon \longrightarrow 0$ in $L^\infty(\Omega)$ as $\epsilon \to 0$, it follows that $\lim_{\epsilon\to 0} I_\epsilon^5 = 0$.
Since u_ϵ converges to u and $\varphi_\epsilon \longrightarrow 0$ a.e. on $\partial\Omega$ as $\epsilon \to 0$, then by the continuity of γ, one has $T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))\varphi_\epsilon \to 0$ a.e. on $\partial\Omega$.
On the other hand, as $|T_k(u_\epsilon) - T_k(u)| \leq 2k$, for any $k \geq 1$, we have

$$|\varphi_\epsilon| \leq 2ke^{4\alpha k^2} \text{ and } |T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))\varphi_\epsilon| \leq 2ke^{4\alpha k^2}|\gamma(u_\epsilon)| \in L^1(\partial\Omega).$$

Using the Lebesgue generalized convergence theorem, one obtains $\lim_{\epsilon\to 0} I_\epsilon^4 = 0$.

On the set $\{|u_\epsilon| \leq k\}$, one has $|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))| \leq \max\left(\beta_\epsilon(T_{\frac{1}{\epsilon}}(-k)),\ \beta_\epsilon(T_{\frac{1}{\epsilon}}(k))\right)$ and

$$|\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi_\epsilon| \leq 2ke^{4\alpha k^2} \max\left(\beta_\epsilon(T_{\frac{1}{\epsilon}}(-k)),\ \beta_\epsilon(T_{\frac{1}{\epsilon}}(k))\right).$$

By applying the Lebesgue dominated converge theorem, it follows that $\lim_{\epsilon\to 0} I_\epsilon^1 = 0$.
Since $\varphi_\epsilon \longrightarrow 0$ in $L^\infty(\Omega)$ and $f_\epsilon \longrightarrow f$ in $L^1(\Omega)$ as $\epsilon \to 0$, one has $\lim_{\epsilon\to 0} I_\epsilon^6 = 0$.
Proceeding as in [28], we have

$$\lim_{\epsilon\to 0}\int_\Omega [a(x,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) - a(x,T_k(u_\epsilon),\nabla T_k(u))]\cdot\nabla(T_k(u_\epsilon)-T_k(u))dx = 0. \tag{3.38}$$

By employing Lemma 2.5, we obtain (3.36) □

Remark 3.2. *Letting $\epsilon \to 0$, the above results imply that*

- $\nabla u_\epsilon \to \nabla u$ *and* $\epsilon|u_\epsilon|^{p(x)-2}u_\epsilon \longrightarrow 0$ *a.e. in* Ω.
- *Since the functions* $a(x,.,.)$ *and* $H(x,.,.)$ *are continuous for a.e.* x *in* Ω, *we have*

$$a(x,u_\epsilon,\nabla u_\epsilon) \to a(x,u,\nabla u) \text{ a.e. in } \Omega \tag{3.39}$$

and

$$H_\epsilon(x,u_\epsilon,\nabla u_\epsilon) \to H(x,u,\nabla u) \text{ a.e. in } \Omega. \tag{3.40}$$

- *According to* (3.39) *and Lemma 2.1, one has*

$$a(x,u_\epsilon,\nabla u_\epsilon) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L^{p'(\cdot)}(\Omega))^N. \tag{3.41}$$

Note that the above strong convergence is not sufficient to pass to the limit in (3.4), so we need the following results.

Lemma 3.5. *Let u_ϵ be a solution of $(\mathcal{P}_\epsilon)$. Then, letting $\epsilon \to 0$, we have*

$$H_\epsilon(x,u_\epsilon,\nabla u_\epsilon) \to H(x,u,\nabla u) \text{ and } \epsilon|u_\epsilon|^{p(x)-2}u_\epsilon \longrightarrow 0 \text{ strongly in } L^1(\Omega). \tag{3.42}$$

Proof. According to Remark 3.2, we have

$$H_\epsilon(x,u_\epsilon,\nabla u_\epsilon) \to H(x,u,\nabla u) \text{ and } \epsilon|u_\epsilon|^{p(x)-2}u_\epsilon \longrightarrow 0 \text{ a.e. in } \Omega.$$

By taking $\varphi = T_1(u_\epsilon - T_n(u_\epsilon))$ as a test function in (3.4), we obtain

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon).\nabla[T_1(u_\epsilon - T_n(u_\epsilon))]dx \tag{3.43}$$

$$+\int_\Omega H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx + \epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx$$

$$+\int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))T_1(u_\epsilon - T_n(u_\epsilon))d\sigma = \int_\Omega f_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx.$$

Since $T_1(u_\epsilon - T_n(u_\epsilon))$ has the same sign with u_ϵ and $\nabla T_1(u_\epsilon - T_n(u_\epsilon)) = \nabla u_\epsilon \chi_{[n<u_\epsilon\le n+1]}$, all the terms in the left-hand side of (3.43) are nonnegative.

By neglecting some positive terms and recalling that $|f_\epsilon| \leq |f| \in L^1(\Omega)$ and $\text{meas}(\{|u_\epsilon| > n\}) \longrightarrow 0$ as $n \to \infty$, one obtains

$$\begin{aligned}\int_{\{|u_\epsilon|>n\}} H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx + \epsilon\int_{\{|u_\epsilon|>n\}} |u_\epsilon|^{p(x)-2}u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx \\ \leq \int_{\{|u_\epsilon|>n\}} |f_\epsilon| dx \\ \leq \int_{\{|u_\epsilon|>n\}} |f| dx \\ \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Since $\{|u_\epsilon| \geq n+1\} \subset \{|u_\epsilon| > n\}$, we deduce from the above inequality that

$$\lim_{n\to\infty}\limsup_{\epsilon\to 0}\left(\int_{\{|u_\epsilon|\geq n+1\}} |H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx + \epsilon\int_{\{|u_\epsilon|\geq n+1\}} |u_\epsilon|^{p(x)-1}dx\right) = 0.$$

Thus, for any $\rho > 0$, there exists $h(\rho) > 0$ such that

$$\int_{\{|u_\epsilon|\geq h(\rho)\}} |H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx + \epsilon\int_{\{|u_\epsilon|\geq h(\rho)\}} |u_\epsilon|^{p(x)-1}dx \leq \frac{\rho}{2}. \tag{3.44}$$

For any measurable subset $A \subset \Omega$, one has

$$\begin{aligned}\int_A |H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx + \epsilon\int_A |u_\epsilon|^{p(x)-1}dx \leq b(h(\rho))\int_A (c(x) + |\nabla T_{h(\rho)}(u_\epsilon)|^{p(x)})dx \\ + \int_{\{|u_\epsilon|\geq h(\rho)\}} |H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx \\ + \int_A |T_{h(\rho)}(u_\epsilon)|^{p(x)-1}dx \\ + \epsilon\int_{\{|u_\epsilon|\geq h(\rho)\}} |u_\epsilon|^{p(x)-1}dx.\end{aligned} \tag{3.45}$$

According to (3.36), there exists $\theta(\rho) > 0$ such that, for all $A \subseteq \Omega$ with $\text{meas}(A) \leq \theta(\rho)$,

$$b(h(\rho))\int_A (c(x) + |\nabla T_{h(\rho)}(u_\epsilon)|^{p(x)})dx + \epsilon\int_A |u_\epsilon|^{p(x)-1}dx \leq \frac{\rho}{2}. \tag{3.46}$$

Combining (3.44), (3.45) and (3.46), one has

$$\int_A |H_\epsilon(x,u_\epsilon,\nabla u_\epsilon)|dx + \epsilon\int_A |u_\epsilon|^{p(x)-1}dx \leq \rho, \tag{3.47}$$

for all $A \subseteq \Omega$ such that $\text{meas}(A) \leq \theta(\rho)$.

We conclude that the sequences $(H_\epsilon(x,u_\epsilon,\nabla u_\epsilon))_{\epsilon>0}$ and $(\epsilon|u_\epsilon|^{p(x)-2}u_\epsilon)_{\epsilon>0}$ are equi-integrable. Therefore, by Vitali's Theorem, one arrives at (3.42). □

Lemma 3.6. *(see [34]) Let u_ϵ be a solution of $(\mathcal{P}_\epsilon)$. Then, letting $\epsilon \to 0$, we have*

$$\nabla[S(u_\epsilon)\xi] \longrightarrow \nabla[S(u)\xi] \text{ strongly in } L^{p(.)}(\Omega),$$

for all $S \in C_c^1(\mathbb{R})$ and $\xi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Now, we focus on the demonstration of (3.1) and (3.2). For that, one firstly uses $S(u_\epsilon)\xi$ as test function in (3.4) to obtain

$$\int_\Omega f_\epsilon S(u_\epsilon)\xi dx - \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))S(u_\epsilon)\xi d\sigma - \int_\Omega a(x,u_\epsilon,\nabla u_\epsilon).\nabla[S(u_\epsilon)\xi]dx - \int_\Omega H(x,u_\epsilon,\nabla u_\epsilon)S(u_\epsilon)\xi dx$$

$$-\epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon S(u_\epsilon)\xi dx = \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))S(u_\epsilon)\xi dx, \tag{3.48}$$

where $\xi \in W^{1,p(.)}(\Omega)\cap L^\infty(\Omega)$ and $S \in C^1_c(\mathbb{R})$.
By applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\epsilon\to 0}\int_\Omega f_\epsilon S(u_\epsilon)\xi dx = \int_\Omega fS(u)\xi dx. \tag{3.49}$$

Due to Lemma 3.5 and the fact that $S(u_\epsilon)\xi \longrightarrow S(u)\xi$ in $L^\infty(\Omega)$, one has

$$\lim_{\epsilon\to 0}\int_\Omega H(x,u_\epsilon,\nabla u_\epsilon)S(u_\epsilon)\xi dx = \int_\Omega H(x,u,\nabla u)S(u)\xi dx \tag{3.50}$$

and

$$\lim_{\epsilon\to 0}\epsilon\int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon S(u_\epsilon)\xi dx = 0. \tag{3.51}$$

Since $|T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))S(u_\epsilon)\xi| \le C(S,\|\xi\|_\infty)|\gamma(u_\epsilon)| \in L^1(\partial\Omega)$, we apply the generalized Lebesgue dominated convergence theorem, to obtain

$$\lim_{\epsilon\to 0}\int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))S(u_\epsilon)\xi d\sigma = \lim_{\epsilon\to 0}\int_{\partial\Omega}\gamma(u)S(u)\xi d\sigma. \tag{3.52}$$

According to Lemma 3.6 and the convergence (3.41), we obtain (see [28])

$$\lim_{\epsilon\to 0}\int_\Omega a(x,u_\epsilon,\nabla u_\epsilon).\nabla[S(u_\epsilon)\varphi]dx = \int_\Omega a(x,u,\nabla u).\nabla[S(u)\varphi]dx. \tag{3.53}$$

Lemma 3.7. *Let u_ϵ be a solution of $(\mathcal{P}_\epsilon)$ with $k > 0$. Then, we have*

(i) *there exists $z \in \mathcal{M}_b(\Omega)$ such that $\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) \overset{*}{\rightharpoonup} z$, as $\epsilon \to 0$,*

(ii) $\lim_{\epsilon\to 0}\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))S(u_\epsilon)\xi dx = \int_\Omega S(u)\xi dz.$

Proof.
(i) Since $(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))_{\epsilon>0}$ is uniformly bounded in $L^1(\Omega)$ (see Lemma 3.3), then up to a subsequence still denoted by ϵ, there exists $z \in \mathcal{M}_b(\Omega)$ such that

$$\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) \overset{*}{\rightharpoonup} z, \text{ as } \epsilon \to 0.$$

(ii) Proceeding as in [42] (*see also* aso [28]), it follows (ii). □

Now, passing to the limit as $\epsilon \to 0$, and combining (3.49)-(3.53) and Lemma 3.7-(ii), we obtains

$$-\int_\Omega a(x,u,\nabla u).\nabla[S(u)\xi]dx - \int_\Omega H(x,u,\nabla u)S(u)\xi dx - \int_{\partial\Omega}\gamma(u)S(u)\xi d\sigma$$

$$+\int_\Omega fS(u)dx = \int_\Omega S(u)\xi dz. \tag{3.54}$$

Therefore, we deduce that $z \in \mathcal{M}_b^{p(.)}(\Omega)$. The Radon Nikodym decomposition of the measure z with respect to Lebesgue measure $\mathcal{L}^N$ can be expressed as follows

Lemma 3.8. *[28, 42] Let u_ϵ be a solution of $(\mathcal{P}_\epsilon)$ with $k > 0$. Then, $z = b\,\mathcal{L}^N + \nu$ such that $\nu \perp \mathcal{L}^N$ and*

$$\begin{cases} b \in \beta(u)\ \mathcal{L}^N - a.e.\ in\ \Omega,\ b \in L^1(\Omega),\ \nu \in \mathcal{M}_b^{p(.)}(\Omega), \\ \\ \nu^+ \text{ is concentrated on } [u = M] \cap [u \neq \infty], \\ \\ \nu^- \text{ is concentrated on } [u = m] \cap [u \neq -\infty]. \end{cases}$$

In view of Lemma 3.8, (3.1) follows from (3.54).
To prove (3.2), one chooses $\varphi = T_1(u_\epsilon - T_n(u_\epsilon))$ as test function in (3.4) to obtain

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_\Omega H_\epsilon(x, u_\epsilon, \nabla u_\epsilon)T_1(u_\epsilon - T_n(u_\epsilon))dx$$
$$+\epsilon \int_\Omega |u_\epsilon|^{p(x)-2}u_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx + \int_{\partial\Omega} T_{\frac{1}{\epsilon}}(\gamma(u_\epsilon))T_1(u_\epsilon - T_n(u_\epsilon))d\sigma$$
$$+ \int_\Omega a(x, u_\epsilon, \nabla u_\epsilon).\nabla[T_1(u_\epsilon - T_n(u_\epsilon))]dx = \int_\Omega f_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx. \tag{3.55}$$

Since u_ϵ and $T_1(u_\epsilon - T_n(u_\epsilon))$ have the same sign and β_ϵ is nondecreasing, the four first terms are positives. Then, one can deduce that

$$\int_{[n<|u_\epsilon|<n+1]} a(x, u_\epsilon, \nabla u_\epsilon).\nabla u_\epsilon dx \leq \int_\Omega f_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx. \tag{3.56}$$

Since $\nabla[T_1(u_\epsilon - T_n(u_\epsilon))] = \nabla u_\epsilon \chi_{[n<|u_\epsilon|<n+1]}$, using (1.1), one obtains

$$C_1 \int_{[n<|u_\epsilon|<n+1]} |\nabla u_\epsilon|^{p(x)}dx \leq \int_\Omega f_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx. \tag{3.57}$$

According to [36], one has $\lim_{n\to\infty} \lim_{\epsilon\to 0} \int_\Omega f_\epsilon T_1(u_\epsilon - T_n(u_\epsilon))dx = 0$.
Therefore, passing to the limit in (3.57) as $\epsilon \to 0$ and $n \to \infty$, one obtains (3.2). □

Remark 3.3. *We emphasize that if $M = \infty$ (resp. $m = -\infty$), then $\nu^+ \equiv 0$ (resp. $\nu^- \equiv 0$). In particular, when the domain of β is the entire $\mathbb{R}$, the above result leads to the standard reformulation of the renormalized solution as follows.*

Corollary 3.1. *Let $\mathcal{D}(\beta) = \mathbb{R}$. Then, the problem $(\mathcal{P})$ has at least one solution $(u, b) \in \mathcal{T}_{tr}^{1,p(.)}(\Omega) \times L^1(\Omega)$ such that $u \in dom(\beta)$ $\mathcal{L}^N$- a.e. in Ω, $b \in \beta(u)$ $\mathcal{L}^N$- a.e. in Ω, $|u|^{p(x)-2}u \in L^1(\partial\Omega)$ and for any $S \in C_c(\mathbb{R})$,*

$$\int_\Omega a(x, u, \nabla u)\nabla(S(u)v)dx + \int_\Omega H(x, u, \nabla u)S(u)vdx + \int_\Omega bS(u)vdx$$
$$+ \int_{\partial\Omega} |u|^{p(x)-2}uS(u)vd\sigma = \int_\Omega fS(u)vdx, \tag{3.58}$$

for any $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$. *Moreover,*

$$\lim_{n\to\infty} \int_{[n\leq u\leq n+1]} |\nabla u|^{p(x)} dx = 0. \tag{3.59}$$

Remark 3.4. *If the domain of* β *is bounded (meaning* $-\infty < m \leq 0 \leq M < \infty$*), one can choose* $h \equiv 1$. *As a result, the renormalization by the function h is not required in definition 3.1.*

Theorem 3.2. *Let* $-\infty < m \leq 0 \leq M < \infty$. *Then, the condition* (P_3) *can be expressed as follows:* (P'_3)*: for any* $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x,u,\nabla u)\nabla v dx + \int_\Omega H(x,u,\nabla u) v dx + \int_\Omega b v dx + \int_\Omega v d\nu + \int_{\partial\Omega} |u|^{p(x)-2} u v d\sigma$$
$$= \int_\Omega f v dx. \tag{3.60}$$

Proof. Let us consider the function S_l defined by $S_l(r) = \inf\{1, (l+1-|r|)^+\}$ with $l > 0$.
Now, we set $S_0 = S_{l_0}$ where $l_0 > 0$ satisfying the following conditions

$$\begin{cases} S_0 \in C_c^1(\mathbb{R}),\ S_0(r) \geq 0, \forall r \in \mathbb{R}, \\ \\ S_0(r) = 1 \text{ if } |r| \leq l_0 \text{ and } S_0(r) = 0 \text{ if } |r| \geq l_0 + 1. \end{cases}$$

As the domain of β is bounded, we can choose $l_0 > 0$ such that $[m, M] \subset [-l_0, l_0]$. Then, setting $S = S_0$ in (3.1), we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla(S_0(u)v) dx + \int_\Omega H(x,u,\nabla u) S_0(u) v dx + \int_\Omega b S_0(u) v dx,$$
$$+ \int_\Omega S_0(u) v d\nu + \int_{\partial\Omega} |u|^{p(x)-2} S_0(u) v d\sigma = \int_\Omega f S_0(u) v dx. \tag{3.61}$$

Since $u \in \text{dom}(\beta)$, one has $S_0(u) = 1$. Hence, from (3.61), it follows (P'_3). □

4. Entropy solutions

In the following result, we derive the relationship between a renormalized solution and an entropy solution.

Definition 4.1. *A couple* $(u, b) \in \mathcal{T}_{tr}^{1,p(.)}(\Omega) \times L^1(\Omega)$ *is said to be an entropy solution of the problem* $(\mathcal{P})$ *if it satisfies the condition* (P_1), (P_3) *and*

$$\int_\Omega b T_k(u-v) dx + \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v) dx + \int_\Omega H(x,u,\nabla u) T_k(u-v) dx$$
$$+ \int_{\partial\Omega} |u|^{p(x)-2} u T_k(u-v) d\sigma \leq \int_\Omega f T_k(u-v) dx, \tag{4.1}$$

for any $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ *such that* $v \in dom(\beta)$.

Theorem 4.1. *Let* $f \in L^1(\Omega)$. *Then, the problem* $(\mathcal{P})$ *has at least one entropy solution.*

Proof. For any $n > 0$, let S_n be the function defined on $\mathbb{R}$ by $S_n(r) = \inf\{1, (n+1-|r|)^+\}$ and (u, b) a renormalized solution of $(\mathcal{P})$.
Note that $S_n(u) \longrightarrow 1$ a.e. in $\mathbb{R}$ as $n \to \infty$. Setting $S = S_n$ in (3.58), one obtains

$$\int_\Omega a(x,u,\nabla u)\nabla(S_n(u)v)dx + \int_\Omega H(x,u,\nabla u)S_n(u)vdx + \int_\Omega bS_n(u)vdx + \int_\Omega S_n(u)vd\nu$$

$$+ \int_{\partial\Omega} |u|^{p(x)-2}uS_n(u)vd\sigma = \int_\Omega fS_n(u)vdx, \tag{4.2}$$

for any $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.
Since $\nabla(S_n(u)v) = S_n(u)\nabla v + S_n'(u)v\nabla u$, one can rewrite (4.2) as follows.

$$\int_\Omega S_n(u)a(x,u,\nabla u)\nabla vdx + \int_\Omega S_n'(u)va(x,u,\nabla u)\nabla udx + \int_\Omega H(x,u,\nabla u)S_n(u)vdx$$

$$+ \int_\Omega bS_n(u)vdx + \int_\Omega S_n(u)vd\nu + \int_{\partial\Omega} |u|^{p(x)-2}uS_n(u)vd\sigma = \int_\Omega fS_n(u)vdx. \tag{4.3}$$

Since, $S_n(u) \longrightarrow 1$ a.e. in Ω as $n \to \infty$, excepted the sixth term, all the terms in (4.3) pass to the limit as $n \to \infty$ (we refer to [28] for the detail). Hence, we obtain

$$\int_\Omega a(x,u,\nabla u).\nabla vdx + \int_\Omega H(x,u,\nabla u)vdx + \int_\Omega bvdx + \int_\Omega vd\nu + \lim_{n\to\infty}\int_{\partial\Omega} |u|^{p(x)-2}uS_n(u)vd\sigma = \int_\Omega fvdx. \tag{4.4}$$

By applying the Lebesgue dominated convergence Theorem, we obtain

$$\int_\Omega a(x,u,\nabla u).\nabla vdx + \int_\Omega H(x,u,\nabla u)vdx + \int_\Omega bvdx + \int_\Omega vd\nu + \int_{\partial\Omega} |u|^{p(x)-2}uvd\sigma$$

$$= \int_\Omega fvdx, \tag{4.5}$$

where $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.
Let $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$. Then, we choose $T_k(u-v)$ as test function in (4.5) to get

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega H(x,u,\nabla u)T_k(u-v)dx + \int_\Omega bT_k(u-v)dx$$

$$+ \int_\Omega T_k(u-v)d\nu + \int_{\partial\Omega} |u|^{p(x)-2}uT_k(u-v)d\sigma = \int_\Omega fT_k(u-v)dx, \tag{4.6}$$

for any $v \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.
Since $v \in \mathrm{dom}\beta$, it follows that the fourth term in (4.6) is nonnegative (see [34]), which leads to (4.1). □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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