Int. J. Anal. Appl. (2025), 23:58

International Journal of Analysis and Applications

Three Step Hybrid Block Method with Two Generalized Off-step Points for Directly Solving Third Order Ordinary Differential Equations

Raft Abdelrahim, Abdelgabar Adam Hassan*, Hamdy M. Barakat, Mohammad S. Hijazi

Department of Mathematics, College of Science, Jouf University, P.O. Box 2014, Sakaka 72388, Saudi Arabia

**Corresponding author:* aahassan@ju.edu.sa

Abstract. This article presents a hybrid block method with steplength $k = 3$ and two hybrid points chosen within grid intervals $[x_n, x_{n+1}]$ and $[x_{n+1}, x_{n+2}]$. The block method is applied for the numerical solution of third order ordinary differential equations (ODEs) directly. An improved methodology is introduced where the generalized form to develop any three-step hybrid block method is given for solving third order ODEs irrespective of the value of the hybrid point with respect to the chosen intervals within the grid. The properties of the block method were investigated with the method exhibiting convergence following from satisfying the properties of zero-stability and consistency. To further validate the new hybrid block method, certain numerical examples were considered and the results show improved accuracy in terms of error comparison when compared with previously existing literature.

1. Introduction

Ordinary Differential Equations (ODEs) of the form

$$y^{'''} = f(x, y, y^{'}, y^{\prime\prime}),\ y(a) = \delta_0,\ y^{'}(a) = \delta_1, y^{''}(a) = \delta_2.\ x \in [a, b]. \tag{1.1}$$

are conventionally referred to as third order ODEs having initial conditions imposed or third order initial value problems (IVPs). These class of equations have been seen to adequately model real life physical problems ranging from electric circuits model, fluid flows, mechanics, vibrations and likewise applications in dynamics of systems.

Numerical methods for the approximate solution of (1) above are being developed everyday and hence the motivation of the article; to develop a new numerical method with better accuracy than previously existing methods. This article introduces a generalized hybrid block method for the numerical solution of (1) above with the aim of performing better than previously existing

Received: Dec. 17, 2024.

2020 *Mathematics Subject Classification.* 65L05, 65L06, 65L20.

Key words and phrases. normalsize block method; hybrid method; third order; initial value problem; collocation and interpolation.

https://doi.org/10.28924/2291-8639-23-2025-58
ISSN: 2291-8639

methods in literature. Authors who have developed block methods generally to numerically approximate (1) include Abdelrahim, Omar (2016), Yap, Ismail and Senu (2014), Adesanya, Udoh and Ajileye (2013), Mechee et al (2013), Langkah et al (2012), Mechee and Mshachal (2019), amongst others. In their separate works, these authors presented novel numerical methods for numerically approximating (1) above with impressive accuracy in comparison to the authors compared with in their respective work. However, it is strongly believed that the method with better accuracy can still be introduced which will also bypass the rigour of reduction to a system of first order IVPs as seen in Mechee et al (2013). Hence, the need for a new hybrid method with improved accuracy which this article presents.

2. Methodology

This section shows the derivation of a three-step hybrid block method with two generalized off-step points; x_{n+s} and x_{n+r} for solving (1.1).

Consider the power series polynomial of the form:

$$y(x) = \sum_{i=0}^{q+d-1} a_i \left(\frac{x - x_n}{h}\right)^i. \tag{2.1}$$

as an approximate solution of (1.1), where

i- $x \in [x_n, x_{n+1}]$ for $n = 0, 1, 2, ..., N-1$,

ii- q is the number of interpolation points which is same as the order of the differential equation; which is 3 in this case,

iii- d is the number of collocation points ,

iv- $h = x_n - x_{n-1}$ is the step size of partitioning the interval $[a, b]$ given by $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$.

The next step in the derivation of the hybrid block method involves differentiating (2.1) twice to give

$$y'''(x) = f(x, y, y', y'') = \sum_{i=2}^{q+d-1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x - x_n}{h}\right)^{i-3}. \tag{2.2}$$

Interpolating the approximate solution (2.1) at $x_{n+k}, k = 0, 1, 2$ and collocating the differential system (2.2) at all points in the selected interval produces nine equations. These equations can be written as a system in matrix form $AX = B$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{(24s)}{h^3} & \frac{(60s^2)}{h^3} & \frac{(120s^3)}{h^3} & \frac{(210s^4)}{h^3} & \frac{(336s^5)}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} & \frac{336}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{(24r)}{h^3} & \frac{(60r^2)}{h^3} & \frac{(120r^3)}{h^3} & \frac{(210r^4)}{h^3} & \frac{(336r^5)}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{48}{h^3} & \frac{240}{h^3} & \frac{960}{h^3} & \frac{3360}{h^3} & \frac{10752}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{72}{h^3} & \frac{540}{h^3} & \frac{3240}{h^3} & \frac{17010}{h^3} & \frac{81648}{h^3} \end{pmatrix}, X = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} \text{and } B = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+s} \\ f_{n+1} \\ f_{n+r} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{2.3}$$

The unknown values of $a_i's, i = 0(1)8$ in (2.3) can be obtained by matrix inverse approach, where $X = A^{-1}B$. The values obtained are then substituted back into equation (2.1) to produce a continuous implicit scheme of the form

$$y(x) = \sum_{i=0}^{2} \alpha_i(x) y_{n+i} + \sum_{i=0}^{3} \beta_i(x) f_{n+i} + \beta_s(x) f_{n+s} + \beta_r(x) f_{n+r} \tag{2.4}$$

It is necessary to also obtain the corresponding first and second derivative schemes of equation (2.4), and these are obtained as given below

$$y'(x) = \sum_{i=0}^{2} \alpha_i'(x) y_{n+i} + \sum_{i=0}^{3} \beta_i'(x) f_{n+i} + \beta_s'(x) f_{n+s} + \beta_r'(x) f_{n+r} \tag{2.5}$$

$$y''(x) = \sum_{i=0}^{2} \alpha_i''(x) y_{n+i} + \sum_{i=0}^{3} \beta_i''(x) f_{n+i} + \beta_s''(x) f_{n+s} + \beta_r''(x) f_{n+r} \tag{2.6}$$

where

$$\alpha_0 = \frac{(h - x + x_n)(2h - x + x_n)}{(2h^2)}$$

$$\alpha_1 = \frac{(x - x_n)(2h - x + x_n)}{h^2}$$

$$\alpha_2 = \frac{-(x - x_n)(h - x + x_n)}{(2h^2)}$$

$$\begin{aligned}
\beta_0 = {} & \frac{(x - x_n)(h - x + x_n)}{(10080h^5 sr)}(2h - x + x_n)(50x^2x_n^3 - 50x^3x_n^2 - 116h^5r - 116h^5s + 33hx^4 \\
& -9h^4x + 33hx_n^4 + 9h^4x_n - 25xx_n^4 + 25x^4x_n + 39h^5 - 5x^5 + 5x_n^5 - 45h^2x^3 - 33h^3x^2 \\
& +45h^2x_n^3 - 33h^3x_n^2 + 336h^5rs + 8hrx^4 + 36h^4rx + 8hrx_n^4 - 36h^4rx_n + 8hsx^4 \\
& +36h^4sx + 8hsx_n^4 - 36h^4sx_n - 132hxx_n^3 - 132hx^3x_n + 66h^3xx_n - 60h^2rx^3 \\
& +112h^3rx^2 + 60h^2rx_n^3 + 112h^3rx_n^2 - 60h^2sx^3 + 112h^3sx^2 + 60h^2sx_n^3 + 112h^3sx_n^2 \\
& +198hx^2x_n^2 - 135h^2xx_n^2 + 135h^2x^2x_n - 364h^4rsx + 364h^4rsx_n - 32hrxx_n^3 - 32hrx^3x_n \\
& -224h^3rxx_n - 32hsxx_n^3 - 32hsx^3x_n - 224h^3sxx_n - 14h^2rsx^3 + 126h^3rsx^2 \\
& +14h^2rsx_n^3 + 126h^3rsx_n^2 + 48hrx^2x_n^2 - 180h^2rxx_n^2 + 180h^2rx^2x_n + 48hsx^2x_n^2 \\
& -180h^2sxx_n^2 + 180h^2sx^2x_n - 42h^2rsxx_n^2 + 42h^2rsx^2x_n - 252h^3rsxx_n)
\end{aligned}$$

$$\begin{aligned}
\beta_s = {} & \frac{-(x - x_n)(h - x + x_n)(2h - x + x_n)}{(1680h^5 s(s-3)(s-1)(s-2)(r-s))}(50x^3x_n^2 - 50x^2x_n^3 + 116h^5r - 33hx^4 + 9h^4x \\
& -33hx_n^4 - 9h^4x_n + 25xx_n^4 - 25x^4x_n - 39h^5 + 5x^5 - 5x_n^5 + 45h^2x^3 + 33h^3x^2 \\
& -45h^2x_n^3 + 33h^3x_n^2 - 8hrx^4 - 36h^4rx - 8hrx_n^4 + 36h^4rx_n + 132hxx_n^3 + 132hx^3x_n \\
& -66h^3xx_n + 60h^2rx^3 - 112h^3rx^2 - 60h^2rx_n^3 - 112h^3rx_n^2 - 198hx^2x_n^2
\end{aligned}$$

$$+135h^2xx_n^2 - 135h^2x^2x_n + 32hrxx_n^3 + 32hrx^3x_n + 224h^3rxx_n - 48hrx^2x_n^2$$
$$+180h^2rxx_n^2 - 180h^2rx^2x_n)$$

$$\beta_1 = \frac{(x-x_n)(h-x+x_n)(2h-x+x_n)}{(3360h^5(s-1)(r-1))}(50x^3x_n^2 - 50x^2x_n^3 - 374h^5r - 374h^5s - 25hx^4$$
$$+143h^4x - 25hx_n^4 - 143h^4x_n + 25xx_n^4 - 25x^4x_n + 335h^5 + 5x^5 - 5x_n^5 - h^2x^3 + 47h^3x^2$$
$$+h^2x_n^3 + 47h^3x_n^2 + 490h^5rs - 8hrx^4 - 134h^4rx - 8hrx_n^4 + 134h^4rx_n - 8hsx^4$$
$$-134h^4sx - 8hsx_n^4 + 134h^4sx_n + 100hxx_n^3 + 100hx^3x_n - 94h^3xx_n + 46h^2rx^3$$
$$-14h^3rx^2 - 46h^2rx_n^3 - 14h^3rx_n^2 + 46h^2sx^3 - 14h^3sx^2 - 46h^2sx_n^3 - 14h^3sx_n^2$$
$$-150hx^2x_n^2 - 3h^2xx_n^2 + 3h^2x^2x_n + 98h^4rsx - 98h^4rsx_n + 32hrxx_n^3 + 32hrx^3x_n$$
$$+28h^3rxx_n + 32hsxx_n^3 + 32hsx^3x_n + 28h^3sxx_n + 14h^2rsx^3 - 98h^3rsx^2 - 14h^2rsx_n^3$$
$$-98h^3rsx_n^2 - 48hrx^2x_n^2 + 138h^2rxx_n^2 - 138h^2rx^2x_n - 48hsx^2x_n^2 + 138h^2sxx_n^2$$
$$-138h^2sx^2x_n + 42h^2rsxx_n^2 - 42h^2rsx^2x_n + 196h^3rsxx_n)$$

$$\beta_r = \frac{(x-x_n)(h-x+x_n)(2h-x+x_n)}{(1680h^5r(r-3)(r-1)(r-2)(r-s))}(50x^3x_n^2 - 50x^2x_n^3 + 116h^5s - 33hx^4 + 9h^4x$$
$$-33hx_n^4 - 9h^4x_n + 25xx_n^4 - 25x^4x_n - 39h^5 + 5x^5 - 5x_n^5 + 45h^2x^3 + 33h^3x^2$$
$$-45h^2x_n^3 + 33h^3x_n^2 - 8hsx^4 - 36h^4sx - 8hsx_n^4 + 36h^4sx_n + 132hxx_n^3 + 132hx^3x_n$$
$$-66h^3xx_n + 60h^2sx^3 - 112h^3sx^2 - 60h^2sx_n^3 - 112h^3sx_n^2 - 198hx^2x_n^2 + 135h^2xx_n^2$$
$$-135h^2x^2x_n + 32hsxx_n^3 + 32hsx^3x_n + 224h^3sxx_n - 48hsx^2x_n^2 + 180h^2sxx_n^2$$
$$-180h^2sx^2x_n)$$

$$\beta_2 = \frac{-(x-x_n)(h-x+x_n)(2h-x+x_n)}{(3360h^5(s-2)(r-2))}(50x^3x_n^2 - 50x^2x_n^3 + 4h^5r + 4h^5s - 17hx^4$$
$$-31h^4x - 17hx_n^4 + 31h^4x_n + 25xx_n^4 - 25x^4x_n - 47h^5 + 5x^5 - 5x_n^5 - 19h^2x^3$$
$$-23h^3x^2 + 19h^2x_n^3 - 23h^3x_n^2 + 56h^5rs - 8hrx^4 + 20h^4rx - 8hrx_n^4 - 20h^4rx_n$$
$$-8hsx^4 + 20h^4sx - 8hsx_n^4 - 20h^4sx_n + 68hxx_n^3 + 68hx^3x_n + 46h^3xx_n + 32h^2rx^3$$
$$+28h^3rx^2 - 32h^2rx_n^3 + 28h^3rx_n^2 + 32h^2sx^3 + 28h^3sx^2 - 32h^2sx_n^3 + 28h^3sx_n^2$$
$$-102hx^2x_n^2 - 57h^2xx_n^2 + 57h^2x^2x_n - 28h^4rsx + 28h^4rsx_n + 32hrxx_n^3 + 32hrx^3x_n$$
$$-56h^3rxx_n + 32hsxx_n^3 + 32hsx^3x_n - 56h^3sxx_n + 14h^2rsx^3 - 70h^3rsx^2 - 14h^2rsx_n^3$$
$$-70h^3rsx_n^2 - 48hrx^2x_n^2 + 96h^2rxx_n^2 - 96h^2rx^2x_n - 48hsx^2x_n^2 + 96h^2sxx_n^2$$
$$-96h^2sx^2x_n + 42h^2rsxx_n^2 - 42h^2rsx^2x_n + 140h^3rsxx_n)$$

$$\beta_3 = \frac{(x-x_n)(h-x+x_n)(2h-x+x_n)}{(10080h^5(s-3)(r-3))}(50x^3x_n^2 - 50x^2x_n^3 - 10h^5r - 10h^5s - 9hx^4$$
$$-9h^4x - 9hx_n^4 + 9h^4x_n + 25xx_n^4 - 25x^4x_n - 9h^5 + 5x^5 - 5x_n^5 - 9h^2x^3 - 9h^3x^2$$
$$+9h^2x_n^3 - 9h^3x_n^2 + 42h^5rs - 8hrx^4 + 6h^4rx - 8hrx_n^4 - 6h^4rx_n - 8hsx^4 + 6h^4sx$$
$$-8hsx_n^4 - 6h^4sx_n + 36hxx_n^3 + 36hx^3x_n + 18h^3xx_n + 18h^2rx^3 + 14h^3rx^2 - 18h^2rx_n^3$$
$$+14h^3rx_n^2 + 18h^2sx^3 + 14h^3sx^2 - 18h^2sx_n^3 + 14h^3sx_n^2 - 54hx^2x_n^2 - 27h^2xx_n^2$$
$$+27h^2x^2x_n - 14h^4rsx + 14h^4rsx_n + 32hrxx_n^3 + 32hrx^3x_n - 28h^3rxx_n + 32hsxx_n^3$$
$$+32hsx^3x_n - 28h^3sxx_n + 14h^2rsx^3 - 42h^3rsx^2 - 14h^2rsx_n^3 - 42h^3rsx_n^2 - 48hrx^2x_n^2$$
$$+54h^2rxx_n^2 - 54h^2rx^2x_n - 48hsx^2x_n^2 + 54h^2sxx_n^2 - 54h^2sx^2x_n + 42h^2rsxx_n^2$$
$$-42h^2rsx^2x_n + 84h^3rsxx_n)$$

Equation (2.4) is evaluated at the non-interpolating points x_{n+r}, x_{n+s} and x_{n+3} while Equation (2.6) is evaluated at all points to give the discrete schemes and its derivative. The discrete scheme and its derivatives are combined in a matrix form as below

$$AY_M = BR_1 + h^3\left[CR_2 + DR_3\right] \tag{2.7}$$

$$A = \begin{bmatrix}
1 & s(s-2) & 0 & \frac{-(s(s-1))}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & r(r-2) & 1 & \frac{-(r(r-1))}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 3 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-2}{h} & 0 & \frac{1}{(2h)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{(2s-2)}{h} & 0 & \frac{-(2s-1)}{(2h)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{-1}{(2h)} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{(2r-2)}{h} & 0 & \frac{-(2r-1)}{(2h)} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{2}{h} & 0 & \frac{-3}{(2h)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{4}{h} & 0 & \frac{-5}{(2h)} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$B = \begin{bmatrix} \frac{((s-1)(s-2))}{2} & 0 & 0 \\ \frac{((r-1)(r-2))}{2} & 0 & 0 \\ 1 & 0 & 0 \\ \frac{-3}{(2h)} & -1 & 0 \\ \frac{(2s-3)}{(2h)} & 0 & 0 \\ \frac{-1}{(2h)} & 0 & 0 \\ \frac{(2r-3)}{(2h)} & 0 & 0 \\ \frac{1}{(2h)} & 0 & 0 \\ \frac{3}{(2h)} & 0 & 0 \\ \frac{1}{h^2} & 0 & -1 \\ \frac{1}{h^2} & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 \end{bmatrix}, \; Y_M = \begin{bmatrix} y_{n+s} \\ y_{n+1} \\ y_{n+r} \\ y_{n+2} \\ y_{n+3} \\ y'_{n+s} \\ y'_{n+1} \\ y'_{n+r} \\ y'_{n+2} \\ y'_{n+3} \\ y''_{n+s} \\ y''_{n+1} \\ y''_{n+r} \\ y''_{n+2} \\ y''_{n+3} \end{bmatrix}, \; R_1 = \begin{bmatrix} y_n \\ y'_n \end{bmatrix}, \; F = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \\ D_{11} \\ D_{12} \\ D_{13} \\ D_{14} \\ D_{15} \end{bmatrix},$$

$$R_2 = \begin{bmatrix} f_n \end{bmatrix}, \; D = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} \\ D_{71} & D_{72} & D_{73} & D_{74} & D_{75} \\ D_{81} & D_{82} & D_{83} & D_{84} & D_{85} \\ D_{91} & D_{92} & D_{93} & D_{94} & D_{95} \\ D_{101} & D_{102} & D_{103} & D_{104} & D_{105} \\ D_{111} & D_{112} & D_{113} & D_{114} & D_{115} \\ D_{121} & D_{122} & D_{123} & D_{124} & D_{125} \\ D_{131} & D_{132} & D_{133} & D_{134} & D_{135} \\ D_{141} & D_{142} & D_{143} & D_{144} & D_{145} \\ D_{151} & D_{152} & D_{153} & D_{154} & D_{155} \end{bmatrix}, \; R_3 = \begin{bmatrix} f_{n+s} \\ f_{n+1} \\ f_{n+r} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}$$

Multiplying Equation (2.7) by the inverse of A gives

$$IY_m = \bar{B}R_1 + h^2 [\bar{C}R_2 + \bar{D}R_3] \tag{2.8}$$

where I is 8×8 identity matrix and

$$\bar{B} = \begin{bmatrix} 1 & hs & \frac{h^2s^2)}{2} \\ 1 & h & \frac{h^2}{2} \\ 1 & hr & \frac{h^2r^2}{2} \\ 1 & 2h & 2h^2 \\ 1 & 3h & \frac{9h^2}{2} \\ 0 & 1 & hs \\ 0 & 1 & h \\ 0 & 1 & hr \\ 0 & 1 & 2h \\ 0 & 1 & 3h \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \bar{C} = \begin{bmatrix} -\frac{h^3s^3(252s-1260r+462rs-84rs^2+6rs^3-154s^2+36s^3-3s^4)}{10080r} \\ \frac{h^3(1064rs-188s-188r+57)}{10080rs} \\ \frac{h^3r^3(1260s-252r-462rs+84r^2s-6r^3s+154r^2-36r^3+3r^4)}{10080s} \\ \frac{h^3(175rs-38s-38r+12)}{315rs} \\ \frac{27h^3(56rs-12s-12r+3)}{1120rs} \\ -\frac{h^2s^2(210s-840r+385rs-84rs^2+7rs^3-154s^2+42s^3-4s^4)}{2520r} \\ \frac{h^2(679rs-147s-147r+53)}{2520rs} \\ \frac{h^2r^2(840s-210r-385rs+84r^2s-7r^3s+154r^2-42r^3+4r^4)}{2520s} \\ \frac{2h^2(98rs-21s-21r+4)}{315rs} \\ \frac{3h^2(91rs-21s-21r+9)}{280rs} \\ -\frac{hs(60s-180r+110rs-30rs^2+3rs^3-55s^2+18s^3-2s^4)}{360r} \\ \frac{h(135rs-38s-38r+17)}{360rs} \\ \frac{hr(180s-60r-110rs+30r^2s-3r^3s+55r^2-18r^3+2r^4)}{360s)} \\ \frac{h(15rs-2s-2r-2)}{45rs)} \\ \frac{3h(5rs-2s-2r+3)}{40rs)} \end{bmatrix},$$

$$\bar{D} = \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{13} & \bar{D}_{14} & \bar{D}_{15} \\ \bar{D}_{21} & \bar{D}_{22} & \bar{D}_{23} & \bar{D}_{24} & \bar{D}_{25} \\ \bar{D}_{31} & \bar{D}_{32} & \bar{D}_{33} & \bar{D}_{34} & \bar{D}_{35} \\ \bar{D}_{41} & \bar{D}_{42} & \bar{D}_{43} & \bar{D}_{44} & \bar{D}_{45} \\ \bar{D}_{51} & \bar{D}_{52} & \bar{D}_{53} & \bar{D}_{54} & \bar{D}_{55} \\ \bar{D}_{61} & \bar{D}_{62} & \bar{D}_{63} & \bar{D}_{64} & \bar{D}_{65} \\ \bar{D}_{71} & \bar{D}_{72} & \bar{D}_{73} & \bar{D}_{74} & \bar{D}_{75} \\ \bar{D}_{81} & \bar{D}_{82} & \bar{D}_{83} & \bar{D}_{84} & \bar{D}_{85} \\ \bar{D}_{91} & \bar{D}_{92} & \bar{D}_{93} & \bar{D}_{94} & \bar{D}_{95} \\ \bar{D}_{101} & \bar{D}_{102} & \bar{D}_{103} & \bar{D}_{104} & \bar{D}_{105} \\ \bar{D}_{111} & \bar{D}_{112} & \bar{D}_{113} & \bar{D}_{114} & \bar{D}_{115} \\ \bar{D}_{121} & \bar{D}_{122} & \bar{D}_{123} & \bar{D}_{124} & \bar{D}_{125} \\ \bar{D}_{131} & \bar{D}_{132} & \bar{D}_{133} & \bar{D}_{134} & \bar{D}_{135} \\ \bar{D}_{141} & \bar{D}_{142} & \bar{D}_{143} & \bar{D}_{144} & \bar{D}_{145} \\ \bar{D}_{151} & \bar{D}_{152} & \bar{D}_{153} & \bar{D}_{154} & \bar{D}_{155} \end{bmatrix}$$

3. Analysis of the Method

3.1. **Order of the Methhod.** The linear difference operator L associated with (2.8) is defined as

$$L[y(x);h] = IY_M - \bar{B}R_1 - h^3\left[\bar{C}R_2 + \bar{D}R_3\right] \tag{3.1}$$

where $y(x)$ is an arbitrary test function continuously differentiable on [a,b]. Y_M and R_3 component's are expanded in Taylor's series respectively and its terms are collected in powers of h to give

$$L[y(x),h] = \bar{C}_0 y(x) + \bar{C}_1 h y^{'}(x) + \bar{C}_2 h y^{''}(x) + \cdots \tag{3.2}$$

Definition 3.1 Hybrid block method (2.8) and associated linear operator (3.1)are said to be of order p, if $\bar{C}_0 = \bar{C}_1 = \bar{_2} = \cdots = \bar{C}_{p+2} = 0$ and $\bar{C}_{p+3} \neq 0$ with error vector constants $\bar{C}_{p+2}$.

Expanding (2.8) in Taylor series about x_n gives

$$\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{(s)^jh^j}{j!} - y_n^j - y_n - (sh)y_n^{'} - \frac{s^2h^2}{2}y_n^{''} - \frac{h^3s^3(1260r-252s-462rs+84rs^2-6rs^3+154s^2-36s^3+3s^4)}{10080r}y_n^{'''} \\
+\frac{s^3(420r-168s-308rs+84rs^2-8rs^3+154s^2-48s^3+5s^4)}{((1680r-1680s)(s-1)(s-2)(s-3))}\sum_{j=0}^{\infty}\frac{(s)^jh^{j+3}}{j!}y_n^{j+3} \\
-\frac{s^5(252r-84s-70rs+6rs^2+30s^2-3s^3)}{((3360r-3360)(s-1))}\sum_{j=0}^{\infty}\frac{(1)^jh^{j+3}}{j!}y_n^{j+3} \\
+\frac{s^5(3s^3-36s^2+154s-252)}{(1680r(r-s)(r-1)(r-2)(r-3))}\sum_{j=0}^{\infty}\frac{(r)^jh^{j+3}}{j!}y_n^{j+2} \\
+\frac{s^5(126r-42s-56rs+6rs^2+24s^2-3s^3)}{((3360r-6720)(s-2))}\sum_{j=0}^{\infty}\frac{(2)^jh^{j+3}}{j!}y_n^{j+3} \\
-\frac{s^5(84r-28s-42rs+6rs^2+18s^2-3s^3)}{((10080r-30240)(s-3))}\sum_{j=0}^{\infty}\frac{(3)^jh^{j+3}}{j!}y_n^{j+3} \\
\\
\sum_{j=0}^{\infty}\frac{(1)^jh^j}{j!} - y_n^j - y_n - (h)y_n^{'} - \frac{h^2}{2}y_n^{''} + \frac{h^3(188r+188s-1064rs-57)}{(10080rs)}y_n^{'''} \\
+\frac{(188r-57)}{(1680s(r-s)(s-1)(s-2)(s-3))}\sum_{j=0}^{\infty}\frac{(s)^jh^{j+3}}{j!}y_n^{j+3} + \frac{(106r+106s-294rs-49)}{(3360(r-1)(s-1))}\sum_{j=0}^{\infty}\frac{(1)^jh^{j+3}}{j!}y_n^{j+3} \\
-\frac{(188s-57)}{(1680r(r-s)(r-1)(r-2)(r-3))}\sum_{j=0}^{\infty}\frac{(r)^jh^{j+3}}{j!}y_n^{j+2} - \frac{(36r+36s-112rs-15)}{((3360r-6720)(s-2))}\sum_{j=0}^{\infty}\frac{(2)^jh^{j+3}}{j!}y_n^{j+3} \\
+\frac{(22r+22s-70rs-9)}{(10080(r-3)(s-3))}\sum_{j=0}^{\infty}\frac{(3)^jh^{j+3}}{j!}y_n^{j+3} \\
\\
\sum_{j=0}^{\infty}\frac{(r)^jh^j}{j!} - y_n^j - y_n - (rh)y_n^{'} - \frac{h^2}{2}y_n^{''} + \frac{h^3r^3(252r-1260s+462rs-84r^2s+6r^3s-154r^2+36r^3-3r^4)}{(10080s)}y_n^{'''} \\
-\frac{r^5(3r^3-36r^2+154r-252)}{(1680s(r-s)(s-1)(s-2)(s-3))}\sum_{j=0}^{\infty}\frac{(s)^jh^{j+3}}{j!}y_n^{j+3} + \frac{r^5(84r-252s+70rs-6r^2s-30r^2+3r^3)}{(3360(r-1)(s-1))}\sum_{j=0}^{\infty}\frac{(1)^jh^{j+3}}{j!}y_n^{j+3} \\
+\frac{r^3(168r-420s+308rs-84r^2s+8r^3s-154r^2+48r^3-5r^4)}{(1680(r-s)(r-1)(r-2)(r-3))}\sum_{j=0}^{\infty}\frac{(r)^jh^{j+3}}{j!}y_n^{j+2} \\
-\frac{r^5(42r-126s+56rs-6r^2s-24r^2+3r^3)}{(3360(r-2)(s-2))}\sum_{j=0}^{\infty}\frac{(2)^jh^{j+3}}{j!}y_n^{j+3} \\
+\frac{r^5(28r-84s+42rs-6r^2s-18r^2+3r^3)}{(10080(r-3)(s-3))}\sum_{j=0}^{\infty}\frac{(3)^jh^{j+3}}{j!}y_n^{j+3} \\
\\
\sum_{j=0}^{\infty}\frac{(2)^jh^j}{j!} - y_n^j - y_n - (2h)y_n^{'} - 2h^2y_n^{''} + \frac{h^3(38r+38s-175rs-12)}{(315rs)}y_n^{'''} \\
+\frac{4(19r-6)}{(105s(r-s)(s-1)(s-2)(s-3))}\sum_{j=0}^{\infty}\frac{(s)^jh^{j+3}}{j!}y_n^{j+3} + \frac{2(30r+30s-49rs-24)}{(105(r-1)(s-1))}\sum_{j=0}^{\infty}\frac{(1)^jh^{j+3}}{j!}y_n^{j+3} \\
-\frac{4(19s-6)}{(105r(r-s)(r-1)(r-2)(r-3))}\sum_{j=0}^{\infty}\frac{(r)^jh^{j+3}}{j!}y_n^{j+2} - \frac{(4r+4s-21rs+4)}{((105(r-2)(s-2))}\sum_{j=0}^{\infty}\frac{(2)^jh^{j+3}}{j!}y_n^{j+3} \\
+\frac{2(2r+2s-7rs)}{(315(r-3)(s-3))}\sum_{j=0}^{\infty}\frac{(3)^jh^{j+3}}{j!}y_n^{j+3} \\
\\
\sum_{j=0}^{\infty}\frac{(3)^jh^j}{j!} - y_n^j - y_n - (3h)y_n^{'} - \frac{9h^2}{2}y_n^{''} + \frac{27h^3(12r+12s-56rs-3)}{(1120rs)}y_n^{'''} \\
+\frac{(243fnsh^3(4r-1))}{(560s(r-s)(s-1)(s-2)(s-3))}\sum_{j=0}^{\infty}\frac{(s)^jh^{j+3}}{j!}y_n^{j+3} + \frac{+(243h^3(10r+10s-14rs-9))}{(1120(r-1)(s-1))}\sum_{j=0}^{\infty}\frac{(1)^jh^{j+3}}{j!}y_n^{j+3} \\
-\frac{(243(4s-1))}{(560r(r-s)(r-1)(r-2)(r-3))}\sum_{j=0}^{\infty}\frac{(r)^jh^{j+3}}{j!}y_n^{j+2} + \frac{+(243(4r+4s-9))}{(1120(r-2)(s-2))}\sum_{j=0}^{\infty}\frac{(2)^jh^{j+3}}{j!}y_n^{j+3} \\
+\frac{(9(6r+6s-14rs-9))}{(1120(r-3)(s-3))}\sum_{j=0}^{\infty}\frac{(3)^jh^{j+3}}{j!}y_n^{j+3}
\end{bmatrix}$$

Comparing the coefficients to power h gives, $\bar{C_1} = \bar{C_2} = ... = \bar{C_{10}} = 0$, and $\bar{C_{11}} \neq 0$. This yields, the main method $\hat{B} = [y_{n+s}, y_{n+1}, y_{n+r}, y_{n+2}, y_{n+3}]^T$ having order $[8, 8, 8, 8, 8]^T$

3.2. **Zero Stability.** The hybrid block method $\hat{B}$ is said to be zero stable if the first characteristic polynomial $\pi(x)$ having roots such that $|x_z| \leq 1$, and if $|x_z| = 1$, then, the multiplicity of x_z must not exceed three. This is clear below,

$$\begin{aligned}
\Pi(x) &= |x\, I - \hat{B}| \\
&= \left| x \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= x^4(x-1)
\end{aligned}$$

which implies $x = 0, 0, 0, 0, 1$. Hence, our method is zero stable for all $s, r \in (0, 3)$

3.3. **Consistency.** The hybrid block method $\hat{B}$ is said to be consistent if its order greater than or equal one i.e. $P \geq 1$

This proves that our method is consistent for all $s, r \in (0, 3)$. Author et al. [1] proved the following theorem:

Theorem 3.1. *[Henrici, 1962] Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent*

Since the method is consistent and zero stable, it implies the method is convergent for all $s, r \in (0, 3)$.

3.4. **Numerical Results.** In finding the accuracy of our methods, the following third order ODEs are examined. The new block methods solved the same problems the existing methods solved in order to compare results in terms of error.

Problem 1: $y''' + y' = 0$, $y(0) = 1$, $y'(0) = 0, y''(0) = -2$, $h = 0.01$.
Exact solution: $y(x) = 1 - e^x$

Problem 2: $y''' = 2y'' + 3y' - 10y + 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 34$, $y(0) = 3$, $y'(0) = 0$, $y'' = 0$ $h = 0.0125$.
Exact solution: $y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right)$

Problem 3: $y''' = -e^x$, $y(0) = 1$, $y'(0) = -1$, $y'' = 3$ $h = 0.1$
Exact solution: $y(x) = 2 + 2x^2 - e^x$

Table 1. Comparison of the new method with some existing methods for solving problem 1

h	Method	Error at x=1.0
0.1	Error in new method	$9.90e^{-11}$
	Yap(2014)	$1.11e^{-10}$
	Adams (2013)	$2.76e^{-6}$
	Awoyemi(2006)	$1.07e^{-6}$
		Error at x=5.
0.025	New method	$8.58e^{-13}$
	Yap(2014)	$1.19e^{-12}$
	Adams (2013)	$9.72e^{-8}$
	Awoyemi(2003)	$3.53e^{-6}$
		Error at x=10.
0.025	New method	$2.09e^{-12}$
	Yap(2014)	$2.11e^{-12}$
	Adams	$1.83e^{-7}$
	Awoyemi(2003)	$2.25e^{-6}$

Table 2. Comparison of the new method with some existing methods for solving problem 1

h	Method	Error at x=1.0
0.0125	Error in new method	$4.56e^{-12}$
	Abdeul majid	$4.36e^{-5}$
	Adams	$3.56e^{-8}$
	Awoyemi(2003)	$7.60e^{-6}$
		Error at x=4.
0.01	New method	$3.50e^{-10}$
	Abdeul majid	$5.11e^{-3}$
	Adams (2013)	$3.56e^{-8}$
	Awoyemi(2005)	$7.60e^{-6}$

TABLE 3. Comparison of the new method with Omar and Kuboye (2015) for solving Problem 3, $h = 0.1$

x	exact solution	computed solution in new method	error in our method, $s = \frac{1}{4}$, $r = \frac{3}{2}$	errors in Omar and Kuoye(2015)
0.1	0.914829081924352305	0.914829081924358523	$6.253802e^{-15}$	$2.885470e^{-13}$
0.2	0.858597241839830216	0.858597241839952119	$1.218935e^{-13}$	$1.837197e^{-12}$
0.3	0.830141192423996976	0.830141192424161511	$1.645877e^{-13}$	$4.572231e^{-12}$
0.4	0.828175302358729937	0.828175302357143983	$1.585955e^{-12}$	$8.562928e^{-12}$
0.5	0.851278729299871806	0.851278729294348002	$5.523756e^{-12}$	$1.374012e^{-11}$

4. Conclusion

This article has presented a new hybrid block method for numerically approximating the solution of third order initial value problems. This hybrid block method has satisfied possessing properties that will confirm its convergence when applied to solve third order ODEs as seen in the numerical results above. Also, it is worth noting that the block method presented is in a generalized form and hence can take varying hybrid point values which gives room for flexibility. Hence, future research can consider the other hybrid points as this article is limited to value $s = \frac{1}{4}$ and $r = \frac{3}{2}$.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A.O. Adesanya, D.M. Udoh, A.M. Ajileye, A New Hybrid Block Method for the Solution of General Third Order Initial Value Problems of Ordinary Differential Equations, Int. J. Pure Appl. Math. 86 (2013), 365-375. https://doi.org/10.12732/ijpam.v86i2.11.

[2] L.K. Yap, F. Ismail, N. Senu, An Accurate Block Hybrid Collocation Method for Third Order Ordinary Differential Equations, J. Appl. Math. 2014 (2014), 549597. https://doi.org/10.1155/2014/549597.

[3] M. Alkasassbeh, Z. Omar, F. Mebarek-Oudina, J. Raza, A. Chamkha, Heat Transfer Study of Convective Fin with Temperature-dependent Internal Heat Generation by Hybrid Block Method, Heat Transf. Asian Res. 48 (2019), 1225–1244. https://doi.org/10.1002/htj.21428.

[4] M. Mechee, N. Senu, F. Ismail, B. Nikouravan, Z. Siri, A Three-Stage Fifth-Order Runge-Kutta Method for Directly Solving Special Third-Order Differential Equation with Application to Thin Film Flow Problem, Math. Probl. Eng. 2013 (2013), 795397. https://doi.org/10.1155/2013/795397.

[5] Z.A. Majid, N.A. Azmi, M. Suleiman, Z.B. Ibrahaim, Solving Directly General Third Order Ordinary Differential Equations Using Two-Point Four Step Block Method, Sains Malays. 41 (2012), 623-632.

[6] D. Greenspan, Discrete Variable Methods in Ordinary Differential Equations (P. Henrici), SIAM Rev. 4 (1962), 261–262. https://doi.org/10.1137/1004070.

[7] D.O. Awoyemi, A P-Stable Linear Multistep Method For Solving General Third Order Ordinary Differential Equations, Int. J. Comput. Math. 80 (2003), 985–991. https://doi.org/10.1080/0020716031000079572.

[8] D.O. Awoyemi, M.O. Udoh, A.O. Adesanya, Nonsymmetric Collocation Method for Direct Solution of General Third Order Initial Value Problems of Ordinary Differential Equations, J. Nat. Appl. Math. 7 (2006), 31–37.

[9] D.O. Awoyemi, O.M. Idowu, A Class of Hybrid Collocation Methods for Third-Order Ordinary Differential Equations, Int. J. Comput. Math. 82 (2005), 1287–1293. https://doi.org/10.1080/00207160500112902.

[10] Z. Omar, J. O. Kuboye, Developing Block Method of Order Seven for Solving Third Order Ordinary Differential Equations Directly using Multistep Collocation Approach, Int. J. Appl. Math. Stat. 53 (2015), 166173. .

[11] Z. Omar, R. Abdelrahim, Application of Single Step With Three Generalized Hybrid Points Block Method for Solving Third Order Ordinary Differential Equations, J. Nonlinear Sci. Appl. 9 (2016), 2705–2717.

[12] R. Abdelrahim, Z. Omar, Solving Third Order Ordinary Differential Equations Using Hybrid Block Method of Order Five, Int. J. Appl. Eng. Res. 24 (2015), 44307-44310.

[13] M.S. Mechee, J.K. Mshachal, Derivation of Embedded Explicit RK Type Methods for Directly Solving Class of Seventh-Order Ordinary Differential Equations, J. Interdiscip. Math. 22 (2019), 1451–1456. https://doi.org/10.1080/09720502.2019.1700936.