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COMPLEX OSCILLATION OF SOLUTIONS AND THEIR DERIVATIVES OF NON-HOMOGENOUS LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC

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Abstract. In this paper, we study the complex oscillation of solutions and their derivatives of the differential equation

$$f'' + A(z) f' + B(z) f = F(z),$$

where $A(z), B(z) (\not\equiv 0)$ and $F(z) (\not\equiv 0)$ are meromorphic functions of finite iterated p-order in the unit disc $\Delta = \{z : |z| < 1\}$.

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory on the complex plane and in the unit disc $\Delta = \{z : |z| < 1\}$ (see [11], [12], [15], [16], [18]). We need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (see [9, 10]).

Definition 1.1 [9, 10] Let f be a meromorphic function in Δ, and

$$D(f) = \limsup_{r\to 1^-} \frac{T(r, f)}{\log \frac{1}{1-r}} = b.$$

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If $b<\infty$, then we say that f is of finite b degree (or is non-admissible). If $b=\infty$, then we say that f is of infinite degree (or is admissible), both defined by characteristic function $T(r,f)$.

Definition 1.2 [9,10] Let f be an analytic function in Δ, and

$$D_M(f)=\limsup_{r\to 1^-}\frac{\log^+ M(r,f)}{\log\frac{1}{1-r}}=a<\infty\ \ (\text{or } a=\infty).$$

Then we say that f is a function of finite a degree (or of infinite degree) defined by maximum modulus function $M(r,f)=\max_{|z|=r}|f(z)|$.

Moreover, for $F\subset[0,1)$, the upper and lower densities of F are defined by

$$\overline{dens}_\Delta F=\limsup_{r\to 1^-}\frac{m(F\cap[0,r))}{m([0,r))},\ \ \underline{dens}_\Delta F=\liminf_{r\to 1^-}\frac{m(F\cap[0,r))}{m([0,r))}$$

respectively, where $m(G)=\int_G\frac{dt}{1-t}$ for $G\subset[0,1)$.

Now we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$, see [4], [14], [15]. Let us define inductively, for $r\in[0,1)$, $\exp_1 r=e^r$ and $\exp_{p+1} r=\exp(\exp_p r)$, $p\in\mathbb{N}$. We also define for all r sufficiently large in $(0,1)$, $\log_1 r=\log r$ and $\log_{p+1} r=\log(\log_p r), p\in\mathbb{N}$. Moreover, we denote by $\exp_0 r=r$, $\log_0 r=r, \exp_{-1} r=\log_1 r, \log_{-1} r=\exp_1 r$.

Definition 1.3 [5] The iterated p-order of a meromorphic function f in Δ is defined by

$$\rho_p(f)=\limsup_{r\to 1^-}\frac{\log_p^+ T(r,f)}{\log\frac{1}{1-r}}\ \ (p\geq 1).$$

For an analytic function f in Δ, we also define

$$\rho_{M,p}(f)=\limsup_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log\frac{1}{1-r}}\ \ (p\geq 1).$$

Remark 1.1 It follows by M. Tsuji in [18] that if f is an analytic function in Δ, then

$$\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1.$$

However, it follows by Proposition 2.2.2 in [15] that

$$\rho_{M,p}(f) = \rho_p(f) \quad (p \geq 2).$$

Definition 1.4 [5] The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{p \in \mathbb{N}, \rho_p(f) < \infty\}, & \text{if } f \text{ is admissible,} \\ \infty, & \text{if } \rho_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

For an analytic function$f$ in Δ, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{p \in \mathbb{N}, \rho_{M,p}(f) < \infty\}, & \text{if } f \text{ is admissible,} \\ \infty, & \text{if } \rho_{M,p}(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Definition 1.5 [6, 7] Let f be a meromorphic function in Δ. Then the iterated $p-$exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| < r\}$. Similarly, the iterated p-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ \overline{N}\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| < r\}$.

Definition 1.6 [8] The growth index of the iterated convergence exponent of the sequence of zeros of $f(z)$ in Δ is defined by

$$i_{\lambda}(f)=\begin{cases} 0, & \text{if } N\left(r,\frac{1}{f}\right)=O\left(\log\frac{1}{1-r}\right), \\ \min\{p\in\mathbb{N},\lambda_p(f)<\infty\}, & \text{if some } p\in\mathbb{N} \text{ with } \lambda_p(f)<\infty, \\ \infty, & \text{if } \lambda_p(f)=\infty \text{ for all } p\in\mathbb{N}. \end{cases}$$

The complex oscillation theory of solutions of linear differential equations in the complex plane $\mathbb{C}$ was started by Bank and Laine in 1982 ([1]). After their well-known work, many important results have been obtained on the growth and the complex oscillation theory of solutions of linear differential equations in the unit disc $\Delta=\{z:|z|<1\}$, (see $[2,3,5,6,7,8,9,10,12,13,16,20]$). Recently, the second author (see, [2]) extended some results of $[6,20]$ to the case of higher order linear differential equations with analytic coefficients. He investigated the relation between solutions and their derivatives of the differential equation

$$f^{(k)}+A(z)f=0 \tag{1.1}$$

and analytic functions of finite iterated p-order, and obtained the following results:

Theorem A [2] *Let H be a set of complex numbers satisfying*

$$\overline{dens}_{\Delta}\{|z|:z\in H\subset\Delta\}>0,$$

and let $A(z)\not\equiv 0$ be an analytic function in Δ such that $\rho_{M,p}(A)=\sigma<\infty$ and for real number $\alpha>0$, we have for all $\varepsilon>0$ sufficiently small,

$$|A(z)|\geq \exp_p\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

as $|z|\to 1^{-}$ for $z\in H$. If $\varphi(z)$ is an analytic function in Δ such that $\varphi^{(k-j)}(z)\not\equiv 0$ $(j=0,\cdots,k)$ with finite iterated $p-$order $\rho_p(\varphi)<\infty$, then every solution $f\not\equiv 0$ of (1.1), satisfies

$$\overline{\lambda}_p\left(f^{(j)}-\varphi\right)=\lambda_p\left(f^{(j)}-\varphi\right)=\rho_p(f)=\infty\ \ (j=0,\cdots,k),$$

$$\overline{\lambda}_{p+1}\left(f^{(j)}-\varphi\right)=\lambda_{p+1}\left(f^{(j)}-\varphi\right)=\rho_{p+1}(f)=\rho_{M,p}(A)\ \ (j=0,\cdots,k).$$

Theorem B [2] *Let H be a set of complex numbers satisfying*

$$\overline{dens}_{\Delta}\left\{|z| : z \in H \subset \Delta\right\} > 0,$$

and let $A(z) \not\equiv 0$ be an analytic function in Δ such that $\rho_p(A) = \sigma < \infty$ and for real number $\alpha > 0$, we have for all $\varepsilon > 0$ sufficiently small,

$$|A(z)| \geq \exp_{p-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

as $|z| \to 1^-$ for $z \in H$. If $\varphi(z)$ is an analytic function in Δ such that $\varphi^{(k-j)}(z) \not\equiv 0$ $(j = 0, \cdots, k)$ with finite iterated $p-$order $\rho_p(\varphi) < \infty$, then every solution $f \not\equiv 0$ of (1.1), satisfies

$$\overline{\lambda}_p\left(f^{(j)} - \varphi\right) = \lambda_p\left(f^{(j)} - \varphi\right) = \rho_p(f) = \infty \quad (j = 0, \cdots, k),$$

$$\sigma \leq \overline{\lambda}_{p+1}\left(f^{(j)} - \varphi\right) = \lambda_{p+1}\left(f^{(j)} - \varphi\right) = \rho_{p+1}(f) \leq \rho_{M,p}(A) \quad (j = 0, \cdots, k).$$

In this paper we consider the oscillation problem of solutions and their derivatives of second order non-homogenous linear differential equation

$$f'' + A(z) f' + B(z) f = F(z), \tag{1.2}$$

where $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ are meromorphic functions of finite iterated p-order in Δ. It is a natural to ask what about the exponent of convergence of zeros of $f^{(j)}(z)$ $(j = 0, 1, 2, \cdots)$, where f is a solution of (1.2). For some related papers in the complex plane on the usual order see, $[17, 19]$. The main purpose of this paper is give an answer to this question. Before we state our results we need to define the following notations

$$A_j(z) = A_{j-1}(z) - \frac{B'_{j-1}(z)}{B_{j-1}(z)} \text{ for } j = 1, 2, 3, \cdots, \tag{1.3}$$

$$B_j(z) = A'_{j-1}(z) - A_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} + B_{j-1}(z) \text{ for } j = 1, 2, 3, \cdots \tag{1.4}$$

and

$$F_j(z) = F'_{j-1}(z) - F_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} \text{ for } j = 1, 2, 3, \cdots, \tag{1.5}$$

where $A_0(z) = A(z)$, $B_0(z) = B(z)$ and $F_0(z) = F(z)$. We obtain the following results.

Theorem 1.1 *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions of finite iterated $p-$order in Δ such that $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \cdots)$. If f is a meromorphic solution in Δ of* (1.2) *with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho$, then f satisfies*

$$\overline{\lambda}_p\left(f^{(j)}\right) = \lambda_p\left(f^{(j)}\right) = \rho_p(f) = \infty \ \ (j = 0, 1, 2, \cdots)$$

and

$$\overline{\lambda}_{p+1}\left(f^{(j)}\right) = \lambda_{p+1}\left(f^{(j)}\right) = \rho_{p+1}(f) = \rho \ \ (j = 0, 1, 2, \cdots).$$

Theorem 1.2 *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions in Δ with finite iterated $p-$order such that $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \cdots)$. If f is a meromorphic solution in Δ of* (1.2) *with*

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(B), \rho_p(F)\},$$

then

$$\overline{\lambda}_p\left(f^{(j)}\right) = \lambda_p\left(f^{(j)}\right) = \rho_p(f) \ \ (j = 0, 1, 2, \cdots).$$

Remark 1.2 In Theorems 1.1, 1.2, the conditions $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \cdots)$ are necessary. For example $f(z) = \exp\left(\frac{1}{1-z}\right)^2 - 1$ satisfies (1.2) where $A(z) = \frac{-3}{1-z}$, $B(z) = -\frac{4}{(1-z)^6}$, $F(z) = \frac{4}{(1-z)^6}$ and $\rho_1(f) = 1 > \max\{\rho_1(A), \rho_1(B), \rho_1(F)\} = 0$. On the other hand, we have

$$A_1 = A - \frac{B'}{B} = -\frac{9}{1-z},$$

$$B_1 = A' - A\frac{B'}{B} + B = \frac{15}{(1-z)^2} - \frac{4}{(1-z)^6}, \ F_1 = F' - F\frac{B'}{B} \equiv 0,$$

and

$$\lambda_1(f) = 1 > \lambda_1(f') = 0.$$

Here, we give some sufficient conditions on the coefficients which guarantee $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \cdots)$, and we obtain:

Theorem 1.3 *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be analytic functions in Δ with finite iterated $p-$order such that $\beta = \rho_p(B) > \max\{\rho_p(A), \rho_p(F)\}$. Then all nontrivial solutions of* (1.2) *satisfy*

$$\rho_p(B) \leq \overline{\lambda}_{p+1}\left(f^{(j)}\right) = \lambda_{p+1}\left(f^{(j)}\right) = \rho_{p+1}(f) \leq \rho_{M,p}(B) \ \ (j = 0, 1, 2, \cdots)$$

with at most one possible exceptional solution f_0 *such that*

$$\rho_{p+1}(f_0) < \rho_p(B).$$

In the next, we set

$$\sigma_p(f) = \limsup_{r\to 1^-} \frac{\log_p m(r,f)}{\log \frac{1}{1-r}}.$$

Theorem 1.4 *Let* $A(z)$, $B(z) \not\equiv 0$ *and* $F(z) \not\equiv 0$ *be meromorphic functions in* Δ *with finite iterated p-order such that* $\sigma_p(B) > \max\{\sigma_p(A), \sigma_p(F)\}$. *If* f *is a meromorphic solution in* Δ *of* (1.2) *with* $\rho_p(f) = \infty$ *and* $\rho_{p+1}(f) = \rho$, *then* f *satisfies*

$$\overline{\lambda}_p\left(f^{(j)}\right) = \lambda_p\left(f^{(j)}\right) = \rho_p(f) = \infty \;\; (j = 0, 1, 2, \cdots)$$

and

$$\overline{\lambda}_{p+1}\left(f^{(j)}\right) = \lambda_{p+1}\left(f^{(j)}\right) = \rho_{p+1}(f) = \rho \;\; (j = 0, 1, 2, \cdots).$$

2. Some lemmas

Lemma 2.1 [2] *Let* f *be a meromorphic function in the unit disc for which* $i(f) = p \geq 1$ *and* $\rho_p(f) = \beta < \infty$, *and let* $k \in \mathbb{N}$. *Then for any* $\varepsilon > 0$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right)$$

for all r *outside a set* $E_1 \subset [0,1)$ *with* $\int_{E_1} \frac{dr}{1-r} < \infty$.

Lemma 2.2 [7] *Let* $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ *be meromorphic functions in* Δ, *and let* f *be a meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = F(z) \tag{2.1}$$

such that $i(f) = p$ $(0 < p < \infty)$. *If either*

$$\max\{i(A_j) \;\; (j = 0, 1, \cdots, k-1), i(F)\} < p$$

or

$$\max\{\rho_p(A_j) \;\; (j = 0, 1, \cdots, k-1), \rho_p(F)\} < \rho_p(f),$$

then

$$i_{\overline{\lambda}}(f) = i_{\lambda}(f) = i(f) = p$$

and

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f).$$

Using the same arguments as in the proof of Lemma 2.2 (see, the proof of Lemma 2.5 in [7]), we easily obtain the following lemma.

Lemma 2.3 *Let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be finite iterated $p-$order meromorphic functions in the unit disc Δ. If f is a meromorphic solution with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho < \infty$ of equation (2.1), then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.*

Lemma 2.4 [7] *Let $p \in \mathbb{N}$, and assume that the coefficients $A_0, \cdots, A_{k-1}$ and $F \not\equiv 0$ are analytic in Δ and $\rho_p(A_j) < \rho_p(A_0)$ for all $j = 1, \cdots, k-1$. Let $\alpha_M := \max\{\rho_{M,p}(A_j) : j = 0, \cdots, k-1\}$.*

(i) *If $\rho_{M,p+1}(F) > \alpha_M$, then all solutions f of (2.1) satisfy $\rho_{M,p+1}(f) = \rho_{M,p+1}(F)$.*

(ii) *If $\rho_{M,p+1}(F) < \alpha_M$, then all solutions f of (2.1) satisfy $\rho_p(A_0) \le \rho_{M,p+1}(f) \le \alpha_M$, with at most one exeption f_0 satisfying $\rho_{M,p+1}(f_0) < \rho_p(A_0)$.*

(iii) *If $\rho_{M,p+1}(F) < \rho_p(A_0)$, then all solutions f of (2.1) satisfy $\rho_p(A_0) \le \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{M,p+1}(f) \le \alpha_M$, with at most one exception f_0 satisfying $\rho_{M,p+1}(f_0) < \rho_p(A_0)$.*

3. Proof of Theorems

Proof of Theorem 1.1. For the proof, we use the principle of mathematical induction. Since $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$, then by using Lemma 2.3 we have

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho.$$

Dividing both sides of (1.2) by B, we obtain

$$\frac{1}{B}f'' + \frac{A}{B}f' + f = \frac{F}{B}. \tag{3.1}$$

Differentiating both sides of equation (3.1) , we have

$$\frac{1}{B}f^{(3)} + \left(\left(\frac{1}{B}\right)' + \frac{A}{B}\right)f'' + \left(\left(\frac{A}{B}\right)' + 1\right)f' = \left(\frac{F}{B}\right)'. \tag{3.2}$$

Multiplying now (3.2) by B, we get

$$f^{(3)} + A_1 f'' + B_1 f' = F_1, \tag{3.3}$$

where

$$A_1 = A - \frac{B'}{B},\ B_1 = A' - A\frac{B'}{B} + B$$

and

$$F_1 = F' - F\frac{B'}{B}.$$

Since $B_1 \not\equiv 0$ and $F_1 \not\equiv 0$ are meromorphic functions with finite iterated p-order, then by using Lemma 2.3 we obtain

$$\overline{\lambda}_p(f') = \lambda_p(f') = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(f') = \lambda_{p+1}(f') = \rho_{p+1}(f) = \rho.$$

Dividing now both sides of (3.3) by B_1, we obtain

$$\frac{1}{B_1}f^{(3)} + \frac{A_1}{B_1}f'' + f' = \frac{F_1}{B_1}. \tag{3.4}$$

Differentiating both sides of equation (3.4) and multplying by B_1, we get

$$f^{(4)} + A_2 f^{(3)} + B_2 f'' = F_2, \tag{3.5}$$

where $A_2, B_2 \not\equiv 0$ and $F_2 \not\equiv 0$ are meromorphic functions defined in $(1.3) - (1.5)$. By using Lemma 2.3 we obtain

$$\overline{\lambda}_p(f'') = \lambda_p(f'') = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(f'') = \lambda_{p+1}(f'') = \rho_{p+1}(f) = \rho.$$

Suppose now that

(3.6)
$$\overline{\lambda}_p\left(f^{(k)}\right)=\lambda_p\left(f^{(k)}\right)=\rho_p\left(f\right)=\infty,\ \overline{\lambda}_{p+1}\left(f^{(k)}\right)=\lambda_{p+1}\left(f^{(k)}\right)=\rho_{p+1}\left(f\right)=\rho$$

for all $k=0,1,2,\cdots,j-1$, and we prove that (3.6) is true for $k=j$. By the same procedure as before, we can obtain

$$f^{(j+2)}+A_jf^{(j+1)}+B_jf^{(j)}=F_j,$$

where $A_j, B_j\not\equiv 0$ and $F_j\not\equiv 0$ are meromorphic functions defined in $(1.3)-(1.5)$. By using Lemma 2.3 we obtain

$$\overline{\lambda}_p\left(f^{(j)}\right)=\lambda_p\left(f^{(j)}\right)=\rho_p\left(f\right)=\infty$$

and

$$\overline{\lambda}_{p+1}\left(f^{(j)}\right)=\lambda_{p+1}\left(f^{(j)}\right)=\rho_{p+1}\left(f\right)=\rho.$$

The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. By a similar reasoning as Theorem 1.1 and by using Lemma 2.2, we obtain

$$\overline{\lambda}_p\left(f^{(j)}\right)=\lambda_p\left(f^{(j)}\right)=\rho_p\left(f\right)\quad\left(j=0,1,2,\cdots\right).$$

Proof of Theorem 1.3. By Lemma 2.4 (iii), all nontrivial solutions of (1.2) satisfy

$$\rho_p\left(B\right)\leq\overline{\lambda}_{p+1}\left(f\right)=\lambda_{p+1}\left(f\right)=\rho_{p+1}\left(f\right)\leq\rho_{M,p}\left(B\right)$$

with at most one exceptional solution f_0 such that $\rho_p\left(B\right)>\rho_{p+1}\left(f_0\right)$. By using (1.3) and Lemma 2.1 we have

$$m\left(r,A_j\right)\leq m\left(r,A_{j-1}\right)+O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right)\quad\left(\beta=\rho_p\left(B_{j-1}\right)\right)$$

outside a set $E_1\subset[0,1)$ with $\int_{E_1}\frac{dr}{1-r}<\infty$, for all $j=1,2,3,\cdots$, which we can write as

$$m\left(r,A_j\right)\leq m\left(r,A\right)+O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right)\quad\left(j=1,2,3,\cdots\right).\tag{3.7}$$

On the other hand, we have from (1.4)

$$B_j = A_{j-1}\left(\frac{A'_{j-1}}{A_{j-1}} - \frac{B'_{j-1}}{B_{j-1}}\right) + B_{j-1}$$

$$= A_{j-1}\left(\frac{A'_{j-1}}{A_{j-1}} - \frac{B'_{j-1}}{B_{j-1}}\right) + A_{j-2}\left(\frac{A'_{j-2}}{A_{j-2}} - \frac{B'_{j-2}}{B_{j-2}}\right) + B_{j-2}$$

$$= \sum_{k=0}^{j-1} A_k\left(\frac{A'_k}{A_k} - \frac{B'_k}{B_k}\right) + B. \tag{3.8}$$

Now we prove that $B_j \not\equiv 0$ for all $j = 1, 2, 3, \cdots$. For that we suppose there exists $j \in \mathbb{N}$ such that $B_j = 0$. By (3.7) and (3.8) we have

$$T(r, B) = m(r, B) \leq \sum_{k=0}^{j-1} m(r, A_k) + O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right)$$

$$\leq jm(r, A) + O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right)$$

$$= jT(r, A) + O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right) \tag{3.9}$$

which implies the contradiction $\rho_p(B) \leq \rho_p(A)$. Hence $B_j \not\equiv 0$ for all $j = 1, 2, 3, \cdots$. Suppose now there exists $j \in \mathbb{N}$ such that $F_j = 0$. Then, from (1.5)

$$F'_{j-1}(z) - F_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} = 0$$

which implies

$$F_{j-1}(z) = cB_{j-1}(z), \tag{3.10}$$

where $c \in \mathbb{C}^*$. By (3.8) and (3.10) we have

$$\frac{1}{c}F_{j-1} = \sum_{k=0}^{j-2} A_k\left(\frac{A'_k}{A_k} - \frac{B'_k}{B_k}\right) + B. \tag{3.11}$$

On the other hand, we have from (1.5)

$$m(r, F_j) \leq m(r, F) + O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right) \quad (j = 1, 2, 3, \cdots). \tag{3.12}$$

By $(3.11)\,,(3.12)$ and Lemma 2.1, we have

$$\begin{aligned}
T(r,B) &= m(r,B) \le \sum_{k=0}^{j-2} m(r,A_k) + m(r,F_{j-1}) + O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right) \\
&\le (j-1)\,m(r,A) + m(r,F) + O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right) \\
&= (j-1)\,T(r,A) + T(r,F) + O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right)
\end{aligned}$$

which implies the contradiction $\rho_p(B) \le \max\{\rho_p(A), \rho_p(F)\}$. Since $B_j \not\equiv 0$, $F_j \not\equiv 0$ $(j = 1, 2, 3, \cdots)$, then by applying Theorem 1.1 and Lemma 2.4 (iii) we have

$$\rho_p(B) \le \overline{\lambda}_{p+1}\left(f^{(j)}\right) = \lambda_{p+1}\left(f^{(j)}\right) = \rho_{p+1}(f) \le \rho_{M,p}(B) \quad (j = 0, 1, 2, \cdots)$$

with at most one exceptional solution f_0 such that $\rho_p(B) > \rho_{p+1}(f_0)$.

4. Proof of Theorem 1.4

Using the same reasoning as Theorem 1.1, we obtain Theorem 1.4.

References

[1] S. Bank and I. Laine, *On the oscillation theory of $f'' + A(z)f = 0$. where A is entire*, Trans. Amer. Math. Soc. 273 (1982), no. 1, 351–363.

[2] B. Belaïdi, *Oscillation of fast growing solutions of linear differential equations in the unit disc*, Acta Univ. Sapientiae Math. 2 (2010), no. 1, 25–38.

[3] B. Belaïdi, A. El Farissi, *Fixed points and iterated order of differential polynomial generated by solutions of linear differential equations in the unit disc*, J. Adv. Res. Pure Math. 3 (2011), no. 1, 161–172.

[4] L. G. Bernal, *On growth k-order of solutions of a complex homogeneous linear differential equation*, Proc. Amer. Math. Soc. 101 (1987), no. 2, 317–322.

[5] T. B. Cao and H. X. Yi, *The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc*, J. Math. Anal. Appl. 319 (2006), no. 1, 278–294.

[6] T. B. Cao, *The growth, oscillation and fixed points of solutions of complex linear differential equations in the unit disc*, J. Math. Anal. Appl. 352 (2009), no. 2, 739-748.

[7] T. B. Cao and Z. S. Deng, *Solutions of non-homogeneous linear differential equations in the unit disc*, Ann. Polo. Math. 97(2010), no. 1, 51-61.

[8] T. B. Cao, C. X. Zhu, K. Liu, *On the complex oscillation of meromorphic solutions of second order linear differential equations in the unit disc*, J. Math. Anal. Appl. 374 (2011), no. 1, 272–281.

[9] Z. X. Chen and K. H. Shon, *The growth of solutions of differential equations with coefficients of small growth in the disc*, J. Math. Anal. Appl. 297 (2004), no. 1, 285–304.

[10] I. E. Chyzhykov, G. G. Gundersen and J. Heittokangas, *Linear differential equations and logarithmic derivative estimates*, Proc. London Math. Soc. (3) 86 (2003), no. 3, 735–754.

[11] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[12] J. Heittokangas, *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss. 122 (2000), 1-54.

[13] J. Heittokangas, R. Korhonen and J. Rättyä, *Fast growing solutions of linear differential equations in the unit disc*, Results Math. 49 (2006), no. 3-4, 265–278.

[14] L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math. 22 (1998), no. 4, 385-405.

[15] I. Laine, *Nevanlinna theory and complex differential equations*, de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin-New York, 1993.

[16] I. Laine, *Complex differential equations*, Handbook of differential equations: ordinary differential equations. Vol. IV, 269–363, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008.

[17] Z. Latreuch and B. Belaïdi, *On the zeros of solutions and their derivatives of second order non-homogenous linear differential equations*, submitted.

[18] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea, New York, (1975), reprint of the 1959 edition.

[19] J. Tu, H. Y. Xu and C. Y. Zhang, *On the zeros of solutions of any order of derivative of second order linear differential equations taking small functions*, Electron. J. Qual. Theory Differ. Equ. 2011, No. 23, 1-17.

[20] G. Zhang, A. Chen, *Fixed points of the derivative and k-th power of solutions of complex linear differential equations in the unit disc*, Electron J. Qual. Theory Differ. Equ., 2009, No. 48, 1-9.

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