International Journal of Analysis and Applications

Some Generalized Fuzzy Separation Axioms

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Abstract. This article's objective is to progress the field of generalized fuzzy topological spaces, particularly generalized fuzzy T_0 spaces. Various types of these spaces are introduced and examined. We investigate their hereditary, productive, and projective properties, and demonstrate that these properties are preserved under bijective generalized fuzzy continuous generalized fuzzy open mappings. Additionally, we explore these concepts in the context of initial and final generalized fuzzy topological spaces.

1. Introduction

In 1965, Zadeh [24] presented The idea of fuzzy sets, which have subsequently demonstrated valuable in addressing a variety of real-world physical problems (see, for example, [4-5], [13], [25]). The concept of fuzzy sets has provided a natural foundation for the development of a new branch of mathematics known as fuzzy topology. The field of fuzzy topological spaces has emerged as a vibrant field of mathematical research (see, for example, [7], [9], [12], [15-17], [21-23], [26]). The idea of a fuzzy topological space was initially proposed by Chang [9]. Subsequently, many mathematicians have contributed to the advancement of fuzzy topological spaces. For instance, A. Császár [6] presented the notion of generalized topological spaces, and Chetty [10] extended this to include generalized fuzzy topological spaces. Later, several studies (see, for example, [2-3], [8], [10-11], [14], [20]) further developed and explored the concept of generalized fuzzy topological spaces. Separation axioms are a key component of generalized fuzzy topological spaces, with the generalized fuzzy T_0 type being one such axiom that has already been introduced in the literature.

Received: Dec. 23, 2024.

2020 *Mathematics Subject Classification.* 54D10.

Key words and phrases. fuzzy topological space; quasi-coincidence; generalized fuzzy open; generalized fuzzy T_0-space; subspace; generalized lower semi continuous; initial and final generalized fuzzy topologies; sum of generalized fuzzy topological spaces .

https://doi.org/10.28924/2291-8639-23-2025-49
ISSN: 2291-8639

Developing the study of generalized fuzzy topological spaces is The aim of this work, with a particular focus on generalized fuzzy T_0 topological spaces. In this work, we introduce new concepts related to generalized fuzzy T_0 spaces and explore the relationships between them.
The format of the paper is as follows: A few preliminary results are shown in Section 2, and generalized fuzzy T_0 spaces are introduced and discussed in Section 3, along with the relationships between these concepts. In Section 4, we introduce the idea of a subspace in generalized fuzzy topological spaces and demonstrate that hereditary, projective, and additive properties apply to these new concepts of generalized fuzzy T_0 spaces. Section 5 explores how our concepts of generalized fuzzy T_0 spaces are maintained under bijective generalized fuzzy continuous generalized fuzzy open mappings. In Section 6, we discuss and examine a generalized lower semi-continuous function, along with initial and final generalized fuzzy topological spaces.

2. Preliminaries

This section provides essential concepts needed for the subsequent discussions. In this work, the closed unit interval $[0,1]$ is denoted by I, while non-empty sets are represented by X and Y.

Definition 2.1. [24] *A fuzzy set in X is a function from X to I. 0_X and 1_X represent the fuzzy sets defined by $0_X(x) = 0$, and $1_X(x) = 1$, $\forall x \in X$. U, V, W etc denotes the fuzzy sets on X. I^X represents the collection of all fuzzy sets on X.*

Definition 2.2. [16] *The complement of U, represented as U^c, is specified by $U^c(x) = 1_X(x) - U(x) = 1 - U(x)$, for every $x \in X$.*

Definition 2.3. [16] *Given an indexed set J and a group of fuzzy sets $\{H_k | k \in J\}$ in X, the union and intersection of these sets are defined, respectively, by:*
$(\bigcup_{k\in J} H_k)(x) = \bigvee\{H_k(x) : k \in J\}$, $\forall x \in X$
$(\bigcap_{k\in J} H_k)(x) = \bigwedge\{H_k(x) : k \in J\}$, $\forall x \in X$.

Definition 2.4. [16] *A fuzzy singleton in X is a fuzzy set that is 0 for every element except one, where it takes a value of α (with $0 < \alpha \leq 1$). It is represented by x_α, where x is its support. If $\alpha = 1$, it is called a crisp fuzzy singleton.*
The collection of all fuzzy singletons of X will be referred to as $FS(X)$. We say that two fuzzy singletons x_α, y_β are distinct if $x \neq y$ or $\alpha \neq \beta$.

Definition 2.5. [16] *The fuzzy singleton x_α is considered to be within U, referred to as $x_\alpha \in U$, if $\alpha \leq U(x)$.*

Definition 2.6. [9] *for a mapping $f : X \longrightarrow Y$ and $U \in I^X$, the image $f(U) \in I^Y$ defined as follows:*

$$f(U)(y) = \begin{cases} \bigvee U(x) & \text{if} \quad x \in f^{-1}(y) \neq \phi, x \in X \\ 0 & , \quad \text{otherwise} \end{cases}$$

Definition 2.7. [9] *For a mapping $f : X \longrightarrow Y$ and $V \subseteq Y$, the preimage $f^{-1}(V) \subseteq X$ defined by $f^{-1}(V)(x) = V(f(x))$, for every $x \in X$.*

Definition 2.8. [9] (X, τ) *is referred to as a fuzzy topological space if* $\tau \subseteq I^X$ *satisfying:*
1. $0_X, 1_X \in \tau$,
2. *if* $H, W \in \tau$, *then* $H \cap W \in \tau$,
3. *if* $H_k \in \tau$ *for* $k \in J$, *then* $\bigcup_{k \in J} H_K \in \tau$.

Theorem 2.1. [9] *Consider a mapping* $f : X \to Y$ *and let* $U \subseteq I^X$, $W \subseteq I^Y$. *Then:*
1. $(f(U))^c \subseteq f(U^c)$, $f^{-1}(W^c) = (f^{-1}(W))^c$;
2. $U \subseteq f^{-1}(f(U))$, $f(f^{-1}(W)) \subseteq W$;
3. *If* f *is injective, then* $f^{-1}(f(U)) = U$;
4. *If* f *is surjective, then* $f(f^{-1}(W)) = W$;
5. *If* f *is both injective and surjective, then* $(f(U))^c = f(U^c)$.

Definition 2.9. [16] *Two fuzzy sets* H *and* W *are considered quasi-coincident (written as* HqW*) if* $\exists\, x \in X$ *for which* $H(x) + W(x) > 1$. *They are not quasi-coincident (denoted* $H\, \bar{q}\, W$*) if* $H(x) + W(x) \leq 1$ *for every* $x \in X$. *Additionally, a fuzzy singleton* x_α *is quasi-coincident with* H *if* $\alpha + H(x) > 1$.

Theorem 2.2. [23] *Let* $f : X \to Y$ *be a function and let* x_α *represent a fuzzy singleton in* X.
(1) If $V \subseteq Y$ *and* $f(x_\alpha)qV$, *then* $x_\alpha q f^{-1}(V)$;
(2) If $U \subseteq X$ *and* $x_\alpha qU$, *then* $f(x_\alpha)qf(U)$.

Proposition 2.1. [16] *Suppose* $U, V \in I^X$ *and* x_α *be a fuzzy singleton. Then* $U \subseteq V$ *iff* $U\bar{q}V^c$; *particulary,* $x_\alpha \in U$ *if and only if* $x_\alpha \bar{q} U^c$.

Proposition 2.2. [7] *For any fuzzy sets* U, V, W *and fuzzy singletons* x_α, y_β, *the following hold:*
1. $U\bar{q}V \Leftrightarrow V\bar{q}U$;
2. $U \cap V = 0_X \Rightarrow U\bar{q}V$;
3. $U\bar{q}U^c$;
4. $U\bar{q}V, W \subseteq V \Rightarrow U\bar{q}W$;
5. $U \subseteq V \Leftrightarrow (x_\alpha qU \Rightarrow x_\alpha qV)$;
6. $x_\alpha q(\bigcup_{k \in J} U_k) \Leftrightarrow x_\alpha qU_k$, *for some* $k \in J$;
7. $x_\alpha q(U \cap V) \Leftrightarrow (x_\alpha qU$ *and* $x_\alpha qV)$;
8. $x_\alpha \bar{q} y_\beta \Leftrightarrow x \neq y$.

Definition 2.10. [10] *A subcollection* $\acute{g}$ *of* I^X *is referred to as a generalized fuzzy topology (abbreviated as GFT) if* $0_X \in g$ *and* g *is closed under arbitrary unions of its members.*
A nonempty set X *paired with a GFT* g, *denoted as* (X, g), *is referred to as a generalized fuzzy topological space (abbreviated as GFTS).*
The elements of g *are generalized fuzzy open sets (abbreviated as GFO(X)), while their complements are known as generalized fuzzy closed sets (abbreviated as GFC(X)).*

Definition 2.11. [19] *In a GFTS* (X, g), *a fuzzy set* U *is considered a* g*-neighborhood of a fuzzy singleton* x_α *if there is* $w \in g$ *s.t* $x_\alpha \in w \subseteq U$. *The collection of every such* g*- neighborhoods of* x_α *is represented by* $N_g(x_\alpha)$.

Definition 2.12. [14] *In a GFTS (X, g), a fuzzy set U is considered a g-Q-neighborhood of x_α if $\exists\, W \in g$ s.t $x_\alpha \, q \, W \subseteq U$. The set of all such g-Q-neighborhoods of x_α is represented by $N_g^Q(x_\alpha)$.*

Definition 2.13. [14] *A generalized fuzzy open set U is an open g-Q-neighborhood of x_α if $x_\alpha \, q \, U$. The set of all such open g-Q-neighborhoods of x_α is represented by $N_{og}^Q(x_\alpha)$.*

Definition 2.14. [14] *Let (X, g) be a GFTS. For any fuzzy set $H \in I^X$, the g-closure of H is the set $c_g(H)$ defined by $c_g(H) = \bigcap\{W : H \subseteq W, W \in GFC(X)\}$.*
Similarly, the g-interior of U is the set $i_g(U)$ defined by $i_g(U) = \bigcup\{W : W \subseteq U, W \in g\}$

Proposition 2.3. [14] *In a GFTS (X, g), the following properties hold:*
(1) $\forall\, H, W \in I^X$; $H \subseteq W \Rightarrow i_g(H) \subseteq i_g(W)$ and $c_g(H) \subseteq c_g(W)$;
(2) $\forall\, H \in I^X$, $i_g(H) \in GFO(X)$ with $i_g(H) \subseteq H$ and $c_g(H) \in GFC(X)$ with $H \subseteq c_g(H)$;
(3) $\forall\, U \in I^X$, $U \in GFO(X)$ iff $U = i_g(U)$ and $U \in GFC(X)$ iff $U = c_g(U)$;
(4) $\forall\, U \in I^X$, $i_g(i_g(U)) = i_g(U)$ and $c_g(c_g(U)) = c_g(U)$;
(5) $\forall\, U \in I^X$, $1 - c_g(U) = i_g(1 - U)$.

Proposition 2.4. [14] *Let (X, g) be a GFTS on X, $U \in I^X$, and x_α be a fuzzy singleton. Then $x_\alpha \in c_g(U)$ iff every open g-Q-neighborhood of x_α is quasi-coincident with U.*

Definition 2.15. [14] *Consider (X, g) and $(Y, \acute{g})$ as two GFTS's. A mapping $f : (X, g) \longrightarrow (Y, \acute{g})$ is defined as follows:*
(1) generalized fuzzy continous if $\forall\, U \in \acute{g}$, $f^{-1}(U) \in g$;
(2) generalized fuzzy open if $\forall\, U \in g$, $f(U) \in \acute{g}$.

Theorem 2.3. [1] *A bijective mapping between sets X and Y maintains the value of a fuzzy singleton. Also, the preimage of any fuzzy singleton under such a mapping maintains its original value.*

3. Generalized fuzzy T_0 spaces

This section presents several concepts related to generalized fuzzy T_0 spaces and examines the connections between them.

Proposition 3.1. *Given a GFTS (X, g) and two fuzzy sets U and V. Subsequently UqV iff $U q\, c_g(V)$, when $U \in g$ in X.*

Proof. Necessity. Let $U \in g$ such that $U\bar{q}V$. Then $V \subseteq U^c$. Since $U^c \in GFC(X)$, we have $c_g(V) \subseteq c_g(U^c) = U^c$. Hence $U\bar{q}\, c_g(V)$.
Sufficiency. Suppose that $U \,\bar{q}\, c_g(V)$. Then $c_g(V) \subseteq U^c$. Since $V \subseteq c_g(V)$, we have $V \subseteq U^c$. Hence $U\bar{q}V$. □

Corollary 3.1. *Given a GFTS (X, g) and two fuzzy sets U and V. Subsequently $U\bar{q}V$ iff $U\bar{q}\, c_g(V)$, when $U \in g$ in X.*

Corollary 3.2. *Given a GFTS (X, g) and two fuzzy sets U and V and let y_β be fuzzy singleton. Then $U\bar{q}V$ if and only if $y_\beta\bar{q}\, c_g(U)$, when $y_\beta \in V \in g$.*

Now, we introduce our definitions of a generalized fuzzy T_0 topological space.

Definition 3.1. *A GFTS (X, g) is referred to as a generalized fuzzy T_0 space (gFT_0 space, for short) provided that the following conditions are met for each pair of different fuzzy singletons x_α and y_β:*
1. When $x \neq y$, either $\exists H \in N_g(x_\alpha)$ s.t $y_\beta \bar{q} H$ or $\exists W \in N_g(y_\beta)$ s.t $x_\alpha \bar{q} W$;
2. When $x = y$ and $\alpha < \beta$(say), $\exists\, V \in N_g^Q(y_\beta)$ such that $x_\alpha \bar{q} V$.

Example 3.1. *Consider $X = \{x, y, z\}$ and let $g = \{0_X, H\}$ where $H = \{(x, 1)\}$. We have $\alpha \leq H(x)$ for any $\alpha \in (0, 1]$, implies $x_\alpha \in H$ and $\beta + H(y) \leq 1$ for any $\beta \in (0, 1]$ implies $y_\beta \bar{q} H$. Hence (X, g) is gFT_0 space.*

The following theorem gives some equvalents properties of gFT_0 space.

Theorem 3.1. *Consider (X, g) as a GFTS. The statements listed below are equivalent:*
1. (X, g) is a gFT_0 space;
2. For every pair of distinct crisp fuzzy singletons $x_\alpha, y_\beta \in I^X$, either $x_\alpha \notin c_g(y_\beta)$ or $y_\beta \notin c_g(x_\alpha)$;
3. For every pair of distinct fuzzy singletons $x_\alpha, y_\beta \in I^X$ such that $x_\alpha \bar{q} y_\beta$, either $x_\alpha \bar{q} c_g(y_\beta)$ or $y_\beta \bar{q} c_g(x_\alpha)$;
4. For every pair of distinct fuzzy singletons $x_\alpha, y_\beta \in I^X$, $\exists\, H, W \in g$ s.t $x_\alpha \in H \subseteq (y_\beta)^c$ or $y_\beta \in W \subseteq (x_\alpha)^c$.

Proof. $1 \Rightarrow 2$ Assume that (X, g) is gFT_0 and x_1, y_1 be two distinct crisp fuzzy singletons in X. When $x \neq y$, either $\exists H \in N_g(x_1)$ s.t $y_\beta \bar{q} H$ or $\exists W \in N_g(y_1)$ s.t $x_\alpha \bar{q} W$. Suppose , without loss of generality, $\exists\, H \in N_g(x_1)$ s.t $y_\beta \bar{q} H$. Therefore, $H \in N_g^Q(x_\alpha)$ and $y_\beta \bar{q} H$. Hence $x_\alpha \notin c_g(y_\beta)$.
$2 \Rightarrow 3$ Let x_α, y_β be two distinct crisp fuzzy singletons in X. Then either $x_\alpha \notin c_g(y_\beta)$ or $y_\beta \notin c_g(x_\alpha)$. Suppose, without loss of generality, $x_\alpha \notin c_g(y_\beta)$. Then from Proposition 2.1, $x_\alpha q (c_g(y_\beta))^c$ and $(c_g(y_\beta))^c \in g$. Say $(c_g(y_\beta))^c = U$. Hence $x_\alpha q U \subseteq (y_\beta)^c$. Thus $x_\alpha \in U$ and $U \in g$. Also, we have $y_\beta \bar{q} U$ and $x_\alpha \in U \in g$. Hence from Corollary 3.2, $x_\alpha \bar{q} c_g(y_\beta)$.
$3 \Rightarrow 4$ Let x_α, y_β be two distinct fuzzy singletons such that $x_\alpha \bar{q} y_\beta$. Suppose $x_\alpha \bar{q} c_g(y_\beta)$. Then from Proposition 2.1, $x_\alpha \in (c_g(y_\beta))^c$ and $(c_g(y_\beta))^c \in g$. Also, we have $(c_g(y_\beta))^c \bar{q} y_\beta$. Say $(c_g(y_\beta))^c = U$, then $U \bar{q} y_\beta$ implies $U \subseteq (y_\beta)^c$. Hence $x_\alpha \in U \subseteq (y_\beta)^c$.
$4 \Rightarrow 1$ Let x_α, y_β be two distinct fuzzy singletons in X, then $\exists\, H, W \in g$ s.t $x_\alpha \in H \subseteq (y_\beta)^c$ or $y_\beta \in W \subseteq (x_\alpha)^c$. Suppose $\exists\, H \in g$ s.t $x_\alpha \in H \subseteq (y_\beta)^c$. From Proposition 2.1, $H \subseteq (y_\beta)^c$ implies $H \bar{q} y_\beta$. Hence $\exists\, H \in N_g(x_\alpha)$ s.t $y_\beta \bar{q} H$. Therefore (X, g) is a gFT_0 space. □

Consider the following property P. We state that a GFTS (X, g) has property P if $\forall x \in X$ and $\rho \in [0, 1]$, $\exists\, U \in g$ with $U(x) = \rho$.

Definition 3.2. *A GFTS (X, g) is called*
1. $gFT_0^{(i)}$ if for any pair $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$, $\exists\, H \in g$ s.t either $x_\alpha \in H$ and $y_\beta \notin H$ or $y_\beta \in H$ and $x_\alpha \notin H$;
2. $gFT_0^{(ii)}$ if (X, g) has property P and for any $r, \delta \in [0, 1)$ and $x, y \in X$, $x \neq y$, there is $W \in g$ s.t either $W(x) = r$ and $W(y) > \delta$, or $W(x) > r$ and $W(y) = \delta$;
3. $gFT_0^{(iii)}$ if $\forall\, x_\alpha, y_\beta \in FS(X)$ with $x \neq y$, $\exists\, H \in g$ s.t $x_\alpha q H$ and $y_\beta \cap H = 0_X$ or $\exists\, W \in g$ s.t $y_\beta q W$ and $x_\alpha \cap W = 0_X$;

4. $gFT_0^{(iv)}$ *if* $\forall\, x_\alpha, y_\beta \in FS(X)$ *with* $x \neq y$, $\exists\, H \in g$ *s.t either* $x_\alpha q H \subseteq (y_\beta)^c$ *or* $y_\beta q H \subseteq (x_\alpha)^c$.

The relations between the concepts given in Definition 3.2 are given in the subsequent theorem.

Theorem 3.2. *As for a GFTS* (X, g) *the subsequent implications are true:*
1. $gFT_0^{(i)} \Rightarrow gFT_0^{(ii)}$*;*
2. $gFT_0^{(i)} \Rightarrow gFT_0^{(iii)}$*;*
3. $gFT_0^{(i)} \Rightarrow gFT_0$*;*
4. $gFT_0^{(iii)} \Rightarrow gFT_0^{(iv)}$*;*
5. $gFT_0 \Rightarrow gFT_0^{(iv)}$*.*
But, in general, the converses are not true.

Proof. 1. Let (X, g) be $gFT_0^{(i)}$ and x_α, $y_\beta \in FS(X)$ s.t $x \neq y$ and $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{n}$, $n \in N$. Then $\exists$ $H_n \in g$ s.t, either $x_\alpha \in H_n$ and $y_\beta \notin H_n$ or $y_\beta \in H_n$ and $x_\alpha \notin H_n$. Assume, without loss of generality, $x_\alpha \in H_n$ and $y_\beta \notin H_n$ then $H_n(x) > 1 - \frac{1}{n}$. Define $H = \bigcup_n H_n$ then $H \in g$ and $H(x) = 1$, $H(y) = 0$. So, $H(x) = \rho$ and $H(y) = \rho$, $\rho \in [0, 1]$. Hence (X, g) satisfy property P. Further choose any $r, \delta \in [0, 1)$ and $x, y \in X$ where $x \neq y$. Since $\exists\, H \in g$ s.t either $H(x) = 1$ and $H(y) = 0$ or $H(y) = 1$ and $H(x) = 0$, we consider $W(x) = \max\{H(x), r\}$. Then $W(x) = 1$, $W(y) = r$ or W(y) = 1, W(x) = r, implying the existance of $W \in GFO(X)$ satisfying $W(x) > \delta$, $W(y) = r$ or $W(y) > \delta$, $W(x) = r$. Hence (X, g) is a $gFT_0^{(ii)}$ space.
2. Let (X, g) be $gFT_0^{(i)}$ and x_α, $y_\beta \in FS(X)$ s.t $x \neq y$ and $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{n}$, $n \in N$. Afterward $\exists\, G_n \in g$ s.t, either $x_\alpha \in G_n$ and $y_\beta \notin G_n$ or $y_\beta \in G_n$ and $x_\alpha \notin G_n$. Assume ,without loss of generality, $x_\alpha \in G_n$ and $y_\beta \notin G_n$ then $G_n(x) > 1 - \frac{1}{n}$. Define $G = \bigcup_n G_n$ then $G \in g$ and $G(x) = 1$, $G(y) = 0$. $G(x) = 1$ implies $G(x) + \alpha > 1$ for any $\alpha \in (0, 1]$. Therefore $x_\alpha q G$ and $H(y) = 0$ indicates $y_\beta \cap G = 0_X$. So $\exists\, G \in g$ s.t $x_\alpha q G$ and $y_\beta \cap G = 0_X$. Therefore, (X, g) is $gFT_0^{(iii)}$.
3. Suppose (X, g) be a $gFT_0^{(i)}$ space and $x_\alpha, y_\beta \in FS(X)$ s.t $x \neq y$. Therefore, $\exists\, G \in g$ s.t either $G(x) = 1$ and $G(y) = 0$ or $G(y) = 1$ and $G(x) = 0$. Let takes $G(x) = 1$ and $G(y) = 0$. Now $G(x) = 1$ implies $\alpha \leq G(x)$, for any $\alpha \in (0, 1]$. Therefore $x_\alpha \in G$ for every $G \in g$. $G(y) = 0$ implies $\beta + G(y) \leq 1$ for any $\beta \in (0, 1]$. Therefore $y_\beta \bar{q} G$. So, from Corollary 3.2, $x_\alpha \bar{q} c_g(y_\beta)$. Hence from Theorem 3.1, (X, g) is a gFT_0 space.
4. Assume that (X, g) is $gFT_0^{(iii)}$ and x_α, y_β be any two distinct fuzzy singletons, then $\exists\, H \in g$ s.t $x_\alpha q H$ and $y_\beta \cap H = 0_X$. $y_\beta \cap H = 0_X$ implies $H(y) = 0$ and so $H(y) + \beta \leq 1$ for any $\beta \in (0, 1]$. Therefore $H \subseteq (y_\beta)^c$. Hence $x_\alpha q H \subseteq (y_\beta)^c$. Therefore (X, g) is $gFT_0^{(iv)}$.
5. Suppose (X, g) be a gFT_0 space, $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$ and $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{n}$, $n \in N$. Then $\exists\, G_n \in g$ s.t, either $x_\alpha \in G_n$ and $y_\beta \bar{q} G_n$ or $y_\beta \in G_n$ and $x_\alpha \bar{q} G_n$. Assume $x_\alpha \in G_n$ and $y_\beta \bar{q} G_n$. Then $\alpha \leq G_n(x)$ implies $1 - \frac{1}{n} < G_n(x)$. Define $G = \bigcup_n G_n$ then $G \in g$ and $G(x) = 1$. Afterward $G(x) + \alpha > 1$ for any $\alpha \in (0, 1]$ and Therefore $x_\alpha q G$ and $G \subseteq y_\beta^c$. Hence (X, g) is $gFT_0^{(iv)}$. □

The opposite direction of the above implications does not hold, as demonstrated through the subsequent examples.

Example 3.2. . *Consider* $X = \{x, y\}$ *and let* $g = \{0_X, U_1, U_2\}$ *where* $U_1(x) = 1 - \frac{\epsilon}{2}$, $U_1(y) = 1$ *and* $U_2(x) = 1$, $U_2(y) = 1 - \frac{\epsilon}{2}$, *such that* $\epsilon \in (0, 1]$. *Then for any* $r, \delta \in [0, 1)$, *we have* (X, g) *is* $gFT_0^{(ii)}$ *space. But for any* $\alpha, \beta \in (0, 1]$ *there exists* $U_1 \in g$ *such that* $x_\alpha \in U_1$ *and* $y_\beta \in U_1$. *Hence* (X, g) *is not* $gFT_0^{(i)}$.

Example 3.3. *Consider* $X = \{x, y\}$ *and* $g = \{0_X, G\}$ *where* $G(x) = 1 - \epsilon$ *and* $G(y) = 0$, *such that* $\epsilon = \frac{\alpha}{2}$ *for* $\alpha \in (0, 1]$. *Then* $G(x) = 1 - \frac{\alpha}{2} \Rightarrow H(x) + \frac{\alpha}{2} = 1 \Rightarrow G(x) + \alpha > 1 \Rightarrow x_\alpha q G$ *and* $G(y) = 0 \Rightarrow y_\beta \cap G = 0_X$. *Therefore,* (X, g) *is* $gFT_0^{(iii)}$ *space. But for* $\alpha = 1$ *we get* $x_\alpha \notin G$ *and* $G(y) = 0$ *implies* $y_\beta \notin G$ *for any* $\beta \in (0, 1]$. *Hence* (X, g) *is not* $gFT_0^{(i)}$ *space.*

Example 3.4. *Consider* $X = \{x, y, z\}$ *and* $g = \{0_X, W_1, W_2, W_3, W_4\}$ *where* $W_1 = \{(x, 1)\}, W_2 = \{(y, \frac{1}{3})\}, W_3 = \{(x, 1), (y, \frac{1}{3})\}$ *and* $W_4 = \{(x, 1), (y, \frac{1}{3}), (z, 1)\}$. *For any* $\alpha, \beta \in (0, 1]$, *we get* $x_\alpha \bar{q} c_g(y_\beta)$. *So* (X, g) *is a* gFT_0 *space but not* $gFT_0^{(i)}$ *space as there exists* $W_2 \in g$ *such that* $x_\alpha \notin W_2$ *and* $y_\beta \notin W_2$.

Example 3.5. *Consider* $X = \{x, y\}$ *and* $g = \{0_X, W\}$ *where* $W(x) = 1$ *and* $W(y) = 0.1$. *For* $0 < \alpha \leq 1$, $0 < \beta < 0.9$, *we obtain* $W(x) + \alpha > 1 \Rightarrow x_\alpha q W$ *and* $W(y) + \beta \leq 1 \Rightarrow y_\beta \bar{q} W$. *Therefore* $\exists$ *open g-Q-neighborhood* W *of* x_α *that is not not quasi-coincident with* y_β. *This indicates that* (X, g) *is* $gFT_0^{(iv)}$ *space. However, since* $W(y) \neq 0 \Rightarrow y_\beta \cap W \neq 0_X$. *it is evident that* (X, g) *is not a* $gFT_0^{(iii)}$ *space.*

Example 3.6. *Take the GFTS* (X, g) *to be described in Example 3.3,* (X, g) *is* $gFT_0^{(iv)}$ *but not* gFT_0 *as for* $\alpha = 1, \beta \in (0, 1]$ $\exists H \in g$ *s.t* $y_\beta \bar{q} H$ *but* $H \notin N_{go}(x_\alpha)$.

Theorem 3.3. *For a GFTS* (X, g), *The statements listed below are equivalent:*
1. (X, g) *is a* $gFT_0^{(i)}$ *space;*
2. $\forall x, y \in X$ *where* $x \neq y$, $\exists W \in g$ *s.t either* $W(x) = 1$ *and* $W(y) = 0$ *or* $W(y) = 1$ *and* $W(x) = 0$;
3. for any pair $x_\alpha, y_\beta \in FS(X)$ *where* $x \neq y$, $\exists W \in g$ *s.t* $x_\alpha \in W$ *and* $y_\beta \cap W = 0_X$ *or* $y_\beta \in W$ *and* $x_\alpha \cap W = 0_X$.

Proof. 1 ⇔ 2 Necessity. Consider (X, g) as a $gFT_0^{(i)}$ space and let x_α, $y_\beta \in FS(X)$ s.t, $x \neq y$ and $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{n}$, $n \in N$. Then $\exists W_n \in g$ s.t, either $x_\alpha \in W_n$ and $y_\beta \notin W_n$ or $y_\beta \in W_n$ and $x_\alpha \notin W_n$. Assume ,without loss of generality, $x_\alpha \in W_n$ and $y_\beta \notin W_n$. Then $W_n(x) > 1 - \frac{1}{n}$. Define $W = \bigcup_n W_n$ Subsequently $W \in g$ and $W(x) = 1$, $W(y) = 0$.
Sufficiency. Consider x_α, $y_\beta \in FS(X)$ s.t, $x \neq y$ and values $\alpha, \beta \in (0, 1]$, Subsequently there is $W \in g$ such that, $W(x) = 1$, $W(y) = 0$. Since $W(x) = 1$, then $\alpha \leq W(x)$ for any $\alpha \in (0, 1]$, so $x_\alpha \in W$, also when $W(y) = 0$ then $\beta \nleq W(y)$ for any $\beta \in (0, 1]$, so $y_\beta \notin W$. Hence (X, g) is $gFT_0^{(i)}$.
1 ⇔ 3 Necessity. Consider (X, g) as a $gFT_0^{(i)}$ space and x_α, $y_\beta \in FS(X)$ s.t $x \neq y$. Then $\exists W \in g$ s.t, $W(x) = 1$ and $W(y) = 0$ or $W(y) = 1$ and $W(x) = 0$. Now $W(x) = 1$ implies $\alpha \leq W(x)$ for any $\alpha \in (0, 1]$. Hence $x_\alpha \in W$. $W(y) = 0$ implies $y_\beta \cap W = 0_X$.
Sufficiency. Let x_α, $y_\beta \in FS(X)$ s.t, $x_\alpha(x) = y_\beta(y) = 1 - \frac{1}{2n}$, where n is a natural number. $\exists W_n \in g$ s.t, either $x_\alpha \in W_n$ and $y_\beta \cap W_n = 0_X$ or $y_\beta \in W_n$ and $x_\alpha \cap W_n = 0_X$. Assume that there is an infinite subsets of N such that $x_\alpha \in W_n$ and $y_\beta \cap W_n = 0_X$ for all $n \in K$. Now if $x_\alpha \in W_n$, then $W_n(x) > 1 - \frac{1}{2n}$. Additionally, if $y_\beta \cap W_n = 0_X$, then $W_n(y) = 0$ for every $n \in K$. Define $W = \bigcup_{n \in K} W_n$. Therefore, $W \in g$, $W(x) = 1$ and $W(y) = \bigcup_{n \in K} W_n(y) = 0$. Hence, (X, g) is a $gFT_0^{(i)}$ space. □

Theorem 3.4. *A GFTS (X, g) is $gFT_0^{(iv)}$ iff $\forall\, x_\alpha, y_\beta \in FS(X)$ with $x \neq y$, $c_g(x_\alpha) \neq c_g(y_\beta)$.*

Proof. Necessity. Suppose (X, g) is a $gFT_0^{(iv)}$ space. Then $\forall\, x_\alpha, y_\beta \in FS(X)$ where $x \neq y$, $\exists\, U \in g$ s.t $x_\alpha q U \subseteq (y_\beta)^c$ or $y_\beta q U \subseteq (x_\alpha)^c$. If $x_\alpha q U \subseteq (y_\beta)^c$, then $x_\alpha \nsubseteq U^c$ and $U \subseteq (y_\beta)^c$, that is, $x_\alpha \nsubseteq U^c$ and $y_\beta \subseteq U^c$. Since $U^c \in GFC(X)$ and $c_g(y_\beta) \in GFC(X)$ (the smallest one) containing y_β, then $c_g(y_\beta) \subseteq U^c$. Since $x_\alpha \notin U^c$ and $x_\alpha \in c_g(x_\alpha)$, then $c_g(x_\alpha) \neq c_g(y_\beta)$.
Sufficiency. suppose $x_\alpha, y_\beta \in FS(X)(x \neq y)$ and $c_g(x_\alpha) \neq c_g(y_\beta)$. Let $z_\lambda \in FS(X)$ such that $z_\lambda \in c_g(x_\alpha)$ and $z_\lambda \notin c_g(y_\beta)$. We claim $x_\alpha \notin c_g(y_\beta)$ (Indeed, if $x_\alpha \in c_g(y_\beta)$, $c_g(x_\alpha) \subseteq c_g(y_\beta)$. This contradicts the fact that $z_\lambda \notin c_g(y_\beta)$). Hence $x_\alpha \notin c_g(y_\beta)$, that is, $x_\alpha q (c_g(y_\beta))^c$ and $U = (c_g(y_\beta))^c \in GFO(X)$, then $x_\alpha q U \subseteq (y_\beta)^c$. □

4. Generalized fuzzy subspace, the product and the sum generalized fuzzy topological spaces

In this part, we examine the hereditary property and provide the idea of a subspace in generalized fuzzy topology.Additionally, we examine the additive, productive, and projective characteristics of generalized fuzzy T_0 spaces.

Lemma 4.1. *Consider (X, g) as a GFTS and let $B \subseteq X$. Subsequently $g_B = \{H \cap B : H \in g\}$ is a GFT on B.*

Proof. Since g is GFT , $\phi \in g$. Hence $\phi \cap B = \phi \in g_B$.
Now let $\{H_k : k \in J\}$ be a subcollection of g_B. By definition of g_B, for each $k \in J$, $\exists\, D_k \in g$ s.t $H_k = D_k \cap B$. Then $\bigcup_{k \in J} H_k = \bigcup_{k\in J}(D_k \cap B) = (\bigcup_{k\in J} D_k) \cap B$. But $\bigcup_{k\in J} D_k \in g$. Hence $\bigcup_{k\in J} H_k \in g_B$. So, g_B is GFT on B. □

Definition 4.1. *Consider (X, g) as a GFTS and $B \subseteq X$. The collection $g_B = \{U \cap B : U \in g\}$ is referred to as the relative generalized fuzzy topology on B. The space (B, g_B) is known as a generalized fuzzy subspace of (X, g).*
Members of g_B are known as generalized fuzzy open sets on B (GFO(B), for short) and their complements are referred to as generalized fuzzy closed sets on B (GFC(B), for short).

Definition 4.2. *A property P of a GFTS is considered hereditary if every subspace of a GFTS that possesses property P, also retains property P.*

Now, we shall show that our notions of a gFT_0 spaces satisfies the hereditary property.

Theorem 4.1. *Consider (X, g) is a GFTS and $B \subseteq X$, then*
1. (X, g) is $gFT_0 \Rightarrow (B, g_B)$ is gFT_0;
2. (X, g) is $gFT_0^{(i)} \Rightarrow (B, g_B)$ is $gFT_0^{(i)}$;
3. (X, g) is $gFT_0^{(ii)} \Rightarrow (B, g_B)$ is $gFT_0^{(ii)}$;
4. (X, g) is $gFT_0^{(iii)} \Rightarrow (B, g_B)$ is $gFT_0^{(iii)}$;
5. (X, g) is $gFT_0^{(iv)} \Rightarrow (B, g_B)$ is $gFT_0^{(iv)}$.

Proof. 1. Suppose (X, g) is gFT_0 and $x_\alpha, y_\beta \in FS(B)$. Since $B \subseteq X$, then $x_\alpha, y_\beta \in FS(X)$. Furthermore, since (X, g) is a gFT_0 space, it follows that

When $x \neq y$, either $\exists\ G \in N_g(x_\alpha)$ s.t, $y_\beta \bar{q} G$ or $\exists\ D \in N_g(y_\beta)$ s.t, $x_\alpha \bar{q} D$. For a subset B of X, both $G \cap B, D \cap B \in g_B$. $x_\alpha \in H \Rightarrow \alpha \leq G(x), x \in X \Rightarrow \alpha \leq (G \cap B)(x), x \in B \subseteq X \Rightarrow x_\alpha \in G \cap B$. Also, $y_\beta \bar{q} G \Rightarrow \beta + G(y) \leq 1, y \in X \Rightarrow \beta + (G \cap B)(y) \leq 1,\ y \in B \subseteq X \Rightarrow y_\beta \bar{q}(G \cap B)$. Consequently, $G \cap B \in N_{g_B}(x_\alpha)$ and $y_\beta \bar{q}(G \cap B)$.

When $x = y$ and $\alpha < \beta$, $\exists\ D \in N_g^Q(y_\beta)$ s.t, $x_\alpha \bar{q} D$. $D \in N_g^Q(y_\beta)$ implies $\exists\ G \in g$ s.t, $y_\beta q G \subseteq D$. For $B \subseteq X, G \cap B \in g_B$. $y_\beta q G \Rightarrow \beta + G(y) > 1, y \in X \Rightarrow \beta + (G \cap B)(y) > 1, y \in B \subseteq X \Rightarrow y_\beta q(G \cap B)$ and $G \cap B \subseteq D$. Then we have $D \in N_{g_B}^Q(y_\beta)$. $x_\alpha \bar{q} D \Rightarrow \alpha + D(x) \leq 1, x \in X \Rightarrow \alpha + (D \cap B)(x) \leq 1, x \in B \subseteq X$. So $x_\alpha \bar{q}(D \cap B)$. Hence (B, g_B) is also gFT_0. Proof of (2) is similar to proof of (1).

3. Suppose (X, g) is $gFT_0^{(ii)}$ and $x, y \in B$ with $x \neq y$. Since $B \subseteq X$, then $x, y \in X$. Also, (X, g) is $gFT_0^{(ii)}$, then for any $r, \delta \in [0, 1)$, $\exists$ an $G \in g$ s.t, $G(x) = r$ and $G(y) > \delta$ or $G(x) > r$ and $G(y) = \delta$. For $B \subseteq X$, it follows that $G \cap B \in g_B$. $G(x) = r \Rightarrow (G \cap B)(x) = r, x \in B \subseteq X$ and $G(y) > \delta \Rightarrow (G \cap B)(y) > \delta, y \in B \subseteq X$. Hence (B, g_B) is also $gFT_0^{(ii)}$.

4. Suppose (X, g) is $gFT_0^{(iii)}$ and $x_\alpha, y_\beta \in FS(B)$ with $x \neq y$. Since $B \subseteq X$, then $x_\alpha, y_\beta \in FS(X)$. Given that (X, g) is a $gFT_0^{(iii)}$, then either $\exists\ H \in g$ s.t, $x_\alpha q H$ and $y_\beta \cap H = 0_X$ or $\exists\ W \in g$ s.t, $y_\beta q W$ and $x_\alpha \cap W = 0_X$. For $B \subseteq X$, it can be inferred that $H \cap B, W \cap B \in g_B$. $x_\alpha q H \Rightarrow H(x) + \alpha > 1$, $x \in X \Rightarrow (H \cap B)(x) + \alpha > 1, x \in B \subseteq X \Rightarrow x_\alpha q(H \cap B)$ and $y_\beta \cap H = 0_X \Rightarrow H(y) = 0 \Rightarrow (H \cap B)(y) = 0, y \in B \subseteq X \Rightarrow y_\beta \cap (H \cap B) = 0_X$. In a similar manner, it can be demonstrated that. $y_\beta q(W \cap B), x_\alpha \cap (W \cap B) = 0_X$. Hence (B, g_B) is also $gFT_0^{(iii)}$. Proof of (5) is similar to proof of (4). □

Definition 4.3. *Consider $\{ X_k, k \in J\}$ be a collection of non empty sets. Define $X = \prod_{k \in J} X_k$ as the product of the X_k and let π_k denote the projection map from X to X_k. Additionally, assume that X_k is a GFTS with GFT g_k. The GFT on X is then generated by using $\{\pi_k^{-1}(b_k) : b_k \in g_k, k \in J\}$ as a subbasis, and this is referred to as the product GFTS on X.*

Definition 4.4. *A property P of a GFTS is said to be productive, if, given a collection $\{(X_k, g_k) : k \in J\}$, where each space has the property p, $(\prod X_k, \prod g_k)$ also possesses the property P.*

Definition 4.5. *A property P of a GFTS is said to be projective if $(\prod X_k, \prod g_k)$ has the property P implies that each individual coordinate space (X_i, g_i) also possesses the property P.*

Definition 4.6. *Consider (X, g) and $(Y, \acute{g})$ as two GFTS's. A mapping $f : (X, g) \longrightarrow (Y, \acute{g})$ is referred to as a generalized fuzzy homeomorphism if f is bijective and both f and its inverse f^{-1} are generalized fuzzy continuous.*

Now, we shall show that our notions of a gFT_0 spaces satisfies the projective and the productive properties

Theorem 4.2. *Let $\{(X_k, g_k), k \in J\}$, is a collection of GFTS's and $X = \prod_{k \in J} X_k$ and g be the product generalized topology on X. Then, $\forall\ k \in J$,*

1. (X_k, g_k) *is* gFT_0 *iff* (X, g) *is* gFT_0*;*
2. (X_k, g_k) *is* $gFT_0^{(i)}$ *iff* (X, g) *is* $gFT_0^{(i)}$*;*
3. (X_k, g_k) *is* $gFT_0^{(ii)}$ *iff* (X, g) *is* $gFT_0^{(ii)}$*;*
4. (X_k, g_k) *is* $gFT_0^{(iii)}$ *iff* (X, g) *is* $gFT_0^{(iii)}$*;*
5. (X_k, g_k) *is* $gFT_0^{(iv)}$ *iff* (X, g) *is* $gFT_0^{(iv)}$*.*

Proof. 1. Necessity. Assume $\forall\, k \in J$, (X_k, g_k) is gFT_0. We need to prove that (X, g) is gFT_0. Suppose $x_\alpha, y_\beta \in FS(X)$, there exist two possible scenarios (i) $x \neq y$, (ii) $x = y$ and $\alpha < \beta$, for instance. Whenever x is not equal to y, then $(x_k)_\alpha, (y_k)_\beta \in FS(X_k)$ satisfy x_k is not equal to y_k for at least one $k \in J$. Since (X_k, g_k) is a gFT_0 space, then either $\exists\, H_k \in N_{g_k}((x_k)_\alpha)$ s.t, $(y_k)_\beta \bar{q} H_k$ or $\exists\, V_k \in N_{g_k}((y_k)_\beta)$ such that $(x_k)_\alpha \bar{q} V_k$. Additionally, $\pi_k(x) = x_k$ and $\pi_k(y) = y_k$. Assume, without losing generality, that $\exists H_k \in N_{g_k}((x_k)_\alpha)$ s.t $(y_k)_\beta \bar{q} H_k$.
Now, $H_k \in N_{g_k}((x_k)_\alpha) \Rightarrow (x_k)_\alpha \in H_k \Rightarrow \alpha \leq H_k(x_k) \Rightarrow \alpha \leq H_k(\pi_k(x)) \Rightarrow \alpha \leq (H_k \circ \pi_k)(x) \Rightarrow x_\alpha \in (H_k \circ \pi_k) \Rightarrow (H_k \circ \pi_k) \in N_g(x_\alpha)$ and $(y_k)_\beta \bar{q} H_k \Rightarrow H_k(y_k) + \beta \leq 1 \Rightarrow H_k(\pi_k(y)) + \beta \leq 1 \Rightarrow (H_k \circ \pi_k)(y) + \beta \leq 1 \Rightarrow y_\beta \bar{q} (H_k \circ \pi_k)$.
When $x = y$ and $\alpha < \beta$,for instance, $\exists V_k \in N^Q_{g_k}((y_k)_\beta)$ such that $(x_k)_\alpha \bar{q} V_k$.
Now, $V_k \in N^Q_{g_k}((y_k)_\beta)$ implies that there exists $G_k \in g_k$ such that $(y_k)_\beta q G_k \subseteq V_k$. $(y_k)_\beta q G_k \Rightarrow G_k(y_k) + \beta > 1 \Rightarrow G_k(\pi_k(y)) + \beta > 1 \Rightarrow (G_k \circ \pi_k)(y) + \beta > 1 \Rightarrow y_\beta q (G_k \circ \pi_k)$ and $G_k \subseteq V_k \Rightarrow (G_k \circ \pi_k) \subseteq (V_k \circ \pi_k)$. Therefore, $y_\beta q (G_k \circ \pi_k) \subseteq (V_k \circ \pi_k)$. Hence, $(V_k \circ \pi_k) \in N^Q_g(y_\beta)$. $(x_k)_\alpha \bar{q} V_k \Rightarrow V_k(x_k) + \alpha \leq 1 \Rightarrow V_k(\pi_k(x)) + \alpha \leq 1 \Rightarrow (V_k \circ \pi_k)(x) + \alpha \leq 1 \Rightarrow x_\alpha \bar{q} (V_k \circ \pi_k)$. Hence, (X, g) is gFT_0.
Sufficiency. Let (X, g) be gFT_0. We need to prove that $(X_k, g_k) fork \in J$ is also gFT_0. Choose a constant element b_k in X_k. Define $B_k = \{x \in X = \prod_{k \in J} X_k : x_j = b_j\, forsome\, k \neq j\}$. Then $B_k \subseteq X$, so (B_k, g_{B_k}) is a subspace of (X, g). Given that (X, g) is gFT_0, it follows that (B_k, g_{B_k}) is also gFT_0. Furthermore, B_k is homeomorphic to X_k. Therefore, (X_k, g_k) is a gFT_0 space $\forall\, k \in J$. Proof of (2) is similar to (1).
3. Necessity. Assume $\forall\, k \in J$, (X_k, g_k) is $gFT_0^{(ii)}$. We need to demonstrate that (X, g) meets the criteria for a $gFT_0^{(ii)}$ space. Consider x and y in X with x is not equal to y. Then $x_k, y_k \in X_k$ with $x_k \neq y_k$ for some $k \in J$. Since each (X_k, g_k) is a $gFT_0^{(ii)}$ space, then for any $r, \delta \in [0, 1)$, $\exists\, H_k \in g_k$ s.t, $H_k(x_k) = r$ and $H_k(y_k) > \delta$ or $H_k(x_k) > r$ and $H_k(y_k) = \delta$. Note that $\pi_k(x) = x_k$ and $\pi_k(y) = y_k$. Suppose, without loss of generality, that $\exists H_k \in g_k$ such that $H_k(x_k) = r$ and $H_k(y_k) > \delta$.
Now, $H_k(x_k) = r \Rightarrow H_k(\pi_k(x)) = r \Rightarrow (H_k \circ \pi_k)(x) = r$ and $H_k(y_k) > \delta \Rightarrow H_k(\pi_k(y)) > \delta \Rightarrow (H_k \circ \pi_k)(y) > \delta$. This means that $(H_k \circ \pi_k) \in g$ s.t, $(H_k \circ \pi_k)(x) = r$ and $(H_k \circ \pi_k)(y) > \delta$. Similarly, we can show that $(U_i \circ \pi_i)(x) > r$ and $(U_i \circ \pi_i)(y) = \delta$. Hence (X, g) is $gFT_0^{(ii)}$ space.
Sufficiency. Let (X, g) be $gFT_0^{(ii)}$. We need to show that (X_k, g_k) is $gFT_0^{(ii)}$ $\forall k \in J$. Consider a constant element b_k in X_k. Define $B_k = \{x \in X = \prod_{k \in J} X_k : x_j = b_j\, forsome\, k \neq j\}$. Hence $B_k \subseteq X$, and thus (B_k, g_{B_k}) is a subspace of (X, g). Since (X, g) is $gFT_0^{(ii)}$, it follows that (B_k, g_{B_k}) is $gFT_0^{(ii)}$. At this point, we find B_k is homeomorphic to X_i. Hence (X_i, g_i) is a $gFT_0^{(ii)}$ space, $\forall k \in J$.
4. Necessity. Assume $\forall\, k \in J$, (X_k, g_k) is $gFT_0^{(iii)}$. We need to show that (X, g) is $gFT_0^{(iii)}$. Suppose

$x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Then, for some $k \in J$, $(x_k)_\alpha, (y_k)_\beta \in FS(X_k)$ with $x_k \neq y_k$. Since (X_k, g_k) is $agFT_0^{(iii)}$ space, then either $\exists\, H_k \in g_k$ s.t, $(x_k)_\alpha q H_k$ and $(y_k)_\beta \cap H_k = 0_X$ or $\exists\, W_k \in g_k$ s.t, $(y_k)_\beta q W_k$ and $(x_k)_\alpha \cap W_k = 0_X$. Note that $\pi_k(x) = x_k$ and $\pi_k(y) = y_k$. Suppose, without loss of generality, that $\exists U_i \in g_i$ such that $(x_i)_\alpha q U_i$ and $(y_i)_\beta \cap U_i = 0_X$.
Now, $(x_k)_\alpha q H_k \Rightarrow H_k(x_k) + \alpha > 1 \Rightarrow H_k(\pi_k(x)) + \alpha > 1 \Rightarrow (H_k \circ \pi_k)(x) + \alpha > 1 \Rightarrow x_\alpha q (H_k \circ \pi_k)$ and $(y_k)_\beta \cap H_k = 0_X \Rightarrow H_k(y_k) = 0 \Rightarrow H_k(\pi_k(y)) = 0 \Rightarrow (H_k \circ \pi_k)(y) = 0 \Rightarrow y_\beta \cap (H_k \circ \pi_k) = 0_X$. Therefore, $(H_k \circ \pi_k) \in g$ satisfies $x_\alpha q (H_k \circ \pi_k)$ and $y_\beta \cap (H_k \circ \pi_k) = 0_X$. Similarly, one can demonstrate that $y_\beta q (W_k \circ \pi_k)$ and $x_\alpha \cap (W_k \circ \pi_k) = 0_X$. So, (X, g) is $gFT_0^{(iii)}$.
Sufficiency. Let (X, g) be $gFT_0^{(iii)}$. We need to show that (X_k, g_k) is $gFT_0^{(iii)}$, $\forall k \in J$. Consider a constant element b_k in X_k. Define $B_k = \{x \in X = \prod_{k \in J} X_k : x_j = b_j \, for some \, k \neq j\}$. Then $B_k \subseteq X$, and thus (B_k, g_{B_k}) is a subspace of (X, g). Since (X, g) is $gFT_0^{(iii)}$, so (B_k, g_{B_k}) is also $gFT_0^{(iii)}$. Furthermore, B_k is homeomorphic to X_k. Therefore (X_i, g_i) is a $gFT_0^{(iii)}$ space, $\forall k \in J$. Proof of (5) is similar to (4). □

Proposition 4.1. *Let $\{(X_k, g_k) : k \in J\}$ be a collection of disjoint GFTS's and let $X = \bigcup_{k \in J} X_k$. The class $\oplus_{k \in J} g_k = \{H | H \in \wp(X) \wedge (H \cap X_k) \in g_k \, \forall\, k \in J\}$ defines a GFT on X. Where $\wp(X)$ is the fuzzy power class of the universe.*

Proof. it is evident that $0_X \in \oplus_{k \in J} g_k$. Consider an arbitrary collection $\{H_i : i \in \Delta\}$ of sets from $\oplus_{k \in J} g_k$. For each $i \in \Delta$ and $k \in J$, $H_i \cap X_k \in g_k$. Thus $\bigcup_{i \in \Delta} H_i \cap X_k \in g_k \forall k \in J$. Therefore, $\bigcup_{i \in \Delta} H_i \in \oplus_{k \in J} g_k \forall k \in J$. Hence, $\oplus_{k \in J} g_k$ is GFT on X. □

Definition 4.7. *The GFT $\oplus_{i \in I} g_i$ described in the above propostion is said to be the sum GFT on X. The corresponding pair $(X, \oplus_{i \in I} g_i)$ is known as the sum GFTS for the family $\{(X_i, g_i) : i \in I\}$.*

Definition 4.8. *A property P of a GFTS is said to be additive if, for any family of GFTS $\{(X_i, g_i), i \in \Lambda\}$ with the property P, the sum of this family $(X, \oplus_{i \in I} g_i)$ also has property P.*

Now, we shall show that our notions of gFT_0 spaces satisfies the additive property.

Theorem 4.3. *The property of being a gFT_0 space is an additive property.*

Proof. Suppose (X_k, g_k) is a gFT_0 space, $\forall k \in J$. We have to prove that $\oplus_{k \in J} X_k$ is gFT_0 space. To do so, we consider two fuzzy singletons x_α, y_β in $X = \bigcup_{k \in J} X_k$ with x is not equal to y. If x and y are a member of different sets X_k and X_j one easily obtain, $x_\alpha \in X_k \subseteq X_j - \{y_\beta\}$ or $y_\beta \in X_j \subseteq X_k - \{x_\alpha\}$. $X_k \subseteq X_j^c$, both X_k and X_j are generalized fuzzy open sets in X under $\oplus_{k \in J}$. If x and y belong to the same gFT_0 space (X_{k_0}, g_{k_0}), then there exists $U_0, V_0 \in GFO(X_{k_0})$ such that $x_\alpha \in U_0 \subseteq X_{k_0} - \{y_\beta\}$ or $y_\beta \in V_0 \subseteq X_{k_0} - \{x_\alpha\}$. Since $X_{k_0} \in GFO(X), X = \oplus_{k \in J} X_k$, one finds $U_0, V_0 \in GFO(X)$ and hence the result. □

Theorem 4.4. *The property of being a $gFT_0^{(i)}$ space is an additive property.*

Proof. Is similar to proof of Theorem (4.3). □

Theorem 4.5. *The property of being a $gFT_0^{(ii)}$ space is an additive property.*

Proof. Suppose (X_k, g_k) is a $gFT_0^{(ii)}$ space, $\forall k \in J$. We aim to show that $\oplus_{k\in J} X_k$ is also $gFT_0^{(ii)}$ space. To do so, we consider two fuzzy singletons x_α, y_β in $X = \bigcup_{k\in J} X_k$ with x is not equal to y. If x and y is a member of different sets X_k and X_j one easily obtain, $X_k(x) = r$ and $X_k(y) > \delta$ or $X_k(x) > r$ and $X_k(y) = \delta$. $X_k \subseteq X_j^c$, both X_k and X_j are generalized fuzzy open sets in X under $\oplus_{k\in J}$. If x and y belong to the same $gFT_0^{(ii)}$ space (X_{k_0}, g_{k_0}), then there exists $U_0 \in GFO(X_{k_0})$ s.t, $U_0(x) = r$ and $U_0(y) > \delta$ or $U_0(x) > r$ and $U_0(y) = \delta$. Since $X_{k_0} \in GFO(X), X = \oplus_{k\in J} X_k$, one finds $U_0 \in GFO(X)$ and hence the result. □

Theorem 4.6. *The property of being a $gFT_0^{(iii)}$ space is an additive property.*

Proof. Suppose (X_k, g_k) is a $gFT_0^{(iii)}$ space, $\forall k \in J$. We aim to show that $\oplus_{k\in J} X_k$ is $gFT_0^{(iii)}$ space. To do so, we consider two fuzzy singletons x_α, y_β in $X = \bigcup_{k\in J} X_k$ with different supports x and y. If x and y belongs to different sets X_k and X_j one easily obtain, $x_\alpha(x) q X_k$ and $y_\beta \cap X_k = 0_X$ or $y_\beta q X_k$ and $x_\alpha \cap X_k = 0_X$. $X_k \subseteq X_j^c$, both X_k and X_j are generalized fuzzy open sets in X under $\oplus_{k\in J}$. If x and y belong to the same $gFT_0^{(iii)}$ space (X_{k_0}, g_{k_0}), then there exists $U_0 \in GFO(X_{k_0})$ such that $x_\alpha q U_0$ and $y_\beta \cap U_0 = 0_X$ or $y_\beta q U_0$ and $x_\alpha \cap U_0 = 0_X$. Since $X_{k_0} \in GFO(X), X = \oplus_{k\in J} X_k$, one finds $U_0 \in GFO(X)$ and hence the result. □

Theorem 4.7. *The property of being a $gFT_0^{(iv)}$ space is an additive property.*

Proof. Is similar to proof of Theorem (4.6). □

5. Mappings in gFT_0 spaces

In this part, we demonstrate the preservation of our concepts of generalized fuzzy T_0 spaces under bijective generalized fuzzy continuous generalized fuzzy open mappings.

Theorem 5.1. *Assume (X, g) and $(Y, \acute{g})$ are two GFTS's and let $f : X \longrightarrow Y$ be a bijective generalized fuzzy open map. Then*

1. (X, g) is $gFT_0 \Rightarrow (Y, \acute{g})$ is gFT_0;
2. (X, g) is $gFT_0^{(i)} \Rightarrow (Y, \acute{g})$ is $gFT_0^{(i)}$;
3. (X, g) is $gFT_0^{(ii)} \Rightarrow (Y, \acute{g})$ is $gFT_0^{(ii)}$;
4. (X, g) is $gFT_0^{(iii)} \Rightarrow (Y, \acute{g})$ is $gFT_0^{(iii)}$;
5. (X, g) is $gFT_0^{(iv)} \Rightarrow (Y, \acute{g})$ is $gFT_0^{(iv)}$.

Proof. 1. Consider (X, g) as a gFT_0 space and let $\acute{x}_\alpha, \acute{y}_\beta \in FS(Y)$ where $\acute{x}$ is not equal to $\acute{y}$. Given that f is surjective, there is $x, y \in X$ s.t $f(x) = \acute{x}$, $f(y) = \acute{y}$. Here, $x_\alpha, y_\beta \in FS(X)$ where x is not equal to y because f is injective. Given that (X, g) is gFT_0, then either $\exists\ H \in N_g(x_\alpha)$ s.t, $y_\beta \bar{q} H$ or $\exists\ W \in N_g(y_\beta)$ s.t, $x_\alpha \bar{q} W$. Suppose, for simplicity, that there is $G \in N_g(x_\alpha)$ and $y_\beta \bar{q} G$. Now, $G \in N_g(x_\alpha)$ implies that $\exists\ W \in g$ s.t, $x_\alpha \in W \subseteq G$ and $y_\beta \bar{q} G \Rightarrow G(y) + \beta \leq 1$. Given that $f(G)(\acute{x}) = \sup\{G(x) : f(x) = \acute{x}\} = G(x)$, for some x. Also $f(G)(\acute{y}) = G(y)$, for a particular y. Since $G, W \in GFO(X)$ and f is a generalized fuzzy open map, it can be inferred that $f(G), f(W) \in GFO(Y)$.

Again, $x_\alpha \in W \Rightarrow \alpha \leq W(x) \Rightarrow \alpha \leq f(W)(\acute{x}) \Rightarrow \acute{x}_\alpha \in f(W)$. Since $W \subseteq G \Rightarrow f(W) \subseteq f(G)$, so $\acute{x}_\alpha \in f(W) \subseteq f(G)$ and $G(y) + \beta \leq 1 \Rightarrow f(G)(\acute{y}) + \beta \leq 1 \Rightarrow \acute{y}_\beta \bar{q} f(G)$. Thus, $\exists\ f(G) \in N_{\acute{g}}(\acute{x}_\alpha)$ and $\acute{y}_\beta \bar{q} f(G)$. Similarly, we can demonstrate that $f(W) \in N_{\acute{g}}(\acute{y}_\beta)$ and $\acute{x}_\alpha \bar{q} f(W)$. Alternatively, consider $\acute{x}_\alpha, \acute{y}_\beta \in FS(Y)$ with $\acute{x} = \acute{y}$ and $\alpha < \beta$(say). Since f is surjective, $\exists\ x, y \in X$ s.t $f(x) = \acute{x}$, $f(y) = \acute{y}$ and $x_\alpha, y_\beta \in FS(X)$ where $x = y$ due to the injective of f. Given that (X, g) is a gFT_0 space, then $\exists\ G \in N_g^Q(y_\beta)$ s.t, $x_\alpha \bar{q} G$. Since $G \in N_g^Q(y_\beta)$, $\exists\ W \in g$ .t, $y_\beta q W \subseteq G$. $y_\beta q W \Rightarrow \beta + W(y) > 1 \Rightarrow f(W)(\acute{y}) + \beta > 1 \Rightarrow \acute{y}_\beta q f(W)$ and $W \subseteq G \Rightarrow f(W) \subseteq f(G)$. Then $\acute{y}_\beta q f(W) \subseteq f(G)$. Also we observe, $x_\alpha \bar{q} G \Rightarrow G(x) + \alpha \leq 1 \Rightarrow f(G)(\acute{x}) + \alpha \leq 1 \Rightarrow \acute{x}_\alpha \bar{q} f(G)$. Since $W \in GFO(X)$ and f is a generalized fuzzy open mapping, then $f(W) \in GFO(Y)$. Consequently, $\exists\ f(G) \in N_{\acute{g}}^Q(\acute{y}_\beta)$ s.t, $\acute{x}_\alpha q f(G)$. Hence (Y, μ) is gFT_0.

2. Is similar to (1).

3. Consider (X, g) is $gFT_0^{(ii)}$ and let $\acute{x}, \acute{y} \in Y$ where $\acute{x}$ is not equal to $\acute{y}$. Since f is surjective, $\exists\ x, y \in X$ s.t $f(x) = \acute{x}$, $f(y) = \acute{y}$. Given that $f(x) \neq f(y)$ and f is injective, it can be inferred that x is not equal to y. Considering that (X, g) is a $gFT_0^{(ii)}$ space, $\exists\ G \in g$ s.t $G(x) = r$ and $G(y) > \delta$ or $G(x) > r$ and $G(y) = \delta$. For simplicity, assume there exists $G \in g$ s.t, $G(x) = r$ and $G(y) > \delta$. Given that, $f(G)(\acute{x}) = \sup\{G(x) : f(x) = \acute{x}\} = G(x)$, for some x. Also $f(G)(\acute{y}) = G(y)$, for some y. Since $G \in GFO(X)$ and f is a generalized fuzzy open mapping, $f(G) \in GFO(Y)$.

Again, $G(x) = r \Rightarrow f(G)(\acute{x}) = r$ and $G(y) > \delta \Rightarrow f(G)(\acute{y}) > \delta$. Similarly, we can show that $f(G)(\acute{y}) = \delta$ and $f(G)(\acute{x}) > r$. Therefore, (Y, μ) is $gFT_0^{(ii)}$.

4. Consider (X, g) is $gFT_0^{(iii)}$ and let $\acute{x}_\alpha, \acute{y}_\beta \in FS(Y)$ with $\acute{x}$ is not equal to $\acute{y}$. Since f is surjective, $\exists$ x and y are elements of X s.t $f(x) = \acute{x}$, $f(y) = \acute{y}$. $x_\alpha, y_\beta \in FS(X)$ with $x \neq y$ because f is injective. Given (X, g) is $gFT_0^{(iii)}$, then either $\exists\ H \in g$ s.t, $x_\alpha q H$ and $y_\beta \cap H = 0_X$, or $\exists\ W \in g$ s.t $y_\beta q W$ and $x_\alpha \cap W = 0_X$. Assuming, for simplicity, there exists an $G \in g$ s.t $x_\alpha q G$ and $y_\beta \cap G = 0_X$. This means $x_\alpha q G \Rightarrow G(x) + \alpha > 1$ and $y_\beta \cap G = 0_X \Rightarrow G(y) = 0$. Since, $f(G)(\acute{x}) = \sup\{G(x) : f(x) = \acute{x}\} = G(x)$, for some x. Also $f(G)(\acute{y}) = G(y)$, for a certain y. Since $G \in GFO(X)$ and f is a generalized fuzzy open mapping, then $f(G) \in GFO(Y)$.

Moreover, $G(x) + \alpha > 1 \Rightarrow f(G)(\acute{x}) + \alpha > 1 \Rightarrow \acute{x}_\alpha q f(G)$ and $G(y) = 0 \Rightarrow f(G)(\acute{y}) = 0 \Rightarrow \acute{y}_\beta \cap f(G) = 0_X$. Likewise, it can be shown that $\acute{y}_\beta q f(W)$ and $\acute{x}_\alpha \cap f(W) = 0_X$. Therefore, $(Y, \acute{g})$ is $gFT_0^{(iii)}$.

5. Is similar to (4). □

Theorem 5.2. *Consider (X, g) and $(Y, \acute{g})$ as two GFTS's and let $f : X \longrightarrow Y$ is an injective and generalized fuzzy continous map. Then*

1. $(Y, \acute{g})$ is $gFT_0 \Rightarrow (X, g)$ is gFT_0;

2. $(Y, \acute{g})$ is $gFT_0^{(i)} \Rightarrow (X, g)$ is $gFT_0^{(i)}$;

3. $(Y, \acute{g})$ is $gF_0^{(ii)} \Rightarrow (X, g)$ is $gFT_0^{(ii)}$;

4. $(Y, \acute{g})$ is $gFT_0^{(iii)} \Rightarrow (X, g)$ is $gFT_0^{(iii)}$;
5. $(Y, \acute{g})$ is $gFT_0^{(iv)} \Rightarrow (X, g)$ is $gFT_0^{(iv)}$.

Proof. 1. Suppose $(Y, \acute{g})$ is gFT_0 and let $x_\alpha, y_\beta \in FS(X)$ with x is not equal to y. Consequently, $(f(x))_\alpha$, $(f(y))_\beta \in FS(Y)$ with $f(x)$ is not equal to $f(y)$ due to f being injective. Given that $(Y, \acute{g})$ is a gFT_0 space, $\exists\, \acute{H} \in N_{\acute{g}}((f(x))_\alpha)$ and $(f(y))_\beta\, \bar{q}\, \acute{H}$. $\acute{H} \in N_{\acute{g}}((f(x))_\alpha)$ implies $\exists\, \acute{W} \in \acute{g}$ s.t, $(f(x))_\alpha \in \acute{w} \subseteq \acute{H}$ Which indicates that $f^{-1}((f(x))_\alpha) \in f^{-1}(\acute{W}) \subseteq f^{-1}(\acute{H})$. Thus $x_\alpha \in f^{-1}(\acute{W}) \subseteq f^{-1}(\acute{H})$ and $(f(y))_\beta\, \bar{q}\, \acute{H} \Rightarrow \acute{H}(f(y)) + \beta \le 1 \Rightarrow f^{-1}(\acute{H})(y) + \beta \le 1$. Therefore $y_\beta\, \bar{q}\, f^{-1}(\acute{H})$. Since f is a generalized fuzzy continous map and $\acute{W}, \acute{H} \in GFO(Y)$, it can be inferred that $f^{-1}(\acute{W}), f^{-1}(\acute{H}) \in GFO(X)$. Consequently, $\exists\, f^{-1}(\acute{H}) \in N_g(x_\alpha)$ s.t, $f^{-1}(\acute{H})\, \bar{q}\, y_\beta$.
Alternatively, let $x_\alpha, y_\beta \in FS(X)$ with x is not equal to y. Afterward $(f(x))_\alpha$, $(f(y))_\beta \in FS(Y)$ with $f(x)$ is equal to $f(y)$ since f is injective. Given that $(Y, \acute{g})$ is gFT_0, $\exists\, \acute{H} \in N^Q_{\acute{g}}((f(y))_\beta)$ s.t, $(f(x))_\alpha \bar{q} \acute{H}$. Since, $\acute{U} \in N^Q_{\acute{g}}((f(y))_\beta)$, $\exists\, \acute{W} \in \acute{g}$ s.t, $(f(y))_\beta\, q\, \acute{W} \subseteq \acute{H}$. This indicates $(f(y))_\beta q \acute{W}$ $\Rightarrow \acute{W}(f(y)) + \beta > 1 \Rightarrow f^{-1}(\acute{W})(y) + \beta > 1 \Rightarrow y_\beta\, q\, f^{-1}(\acute{W})$ and $\acute{W} \subseteq \acute{H} \Rightarrow f^{-1}(\acute{W}) \subseteq f^{-1}(\acute{H})$. So $y_\beta\, q\, f^{-1}(\acute{W}) \subseteq f^{-1}(\acute{H})$. Also we obtain, $(f(x))_\alpha \bar{q} \acute{H} \Rightarrow \acute{H}(f(x)) + \alpha \le 1 \Rightarrow f^{-1}(\acute{H})(x) + \alpha \le 1 \Rightarrow x_\alpha\, \bar{q}\, f^{-1}(\acute{H})$. Given that f is a generalized fuzzy continous map and $\acute{W} \in \acute{g}$, it implies $f^{-1}(\acute{W}) \in g$. Therefore, $\exists\, f^{-1}(\acute{H}) \in N^Q_g(y_\beta)$ s.t, $x_\alpha \bar{q} f^{-1}(\acute{H})$. Hence (X, g) is gFT_0.
2. Is similar to (1).
3. Assume $(Y, \acute{g})$ is $gFT_0^{(ii)}$ and consider $x, y \in X$ with x is not equal to y. Because f is injective, the images $f(x)$ and $f(y)$ in Y are also distinct. Given that $(Y, \acute{g})$ is a $gFT_0^{(ii)}$ space, for any $r, \delta \in [0, 1)$ and distinct $f(x)$ and $f(y)$, there is an $\acute{G} \in \acute{g}$ s.t $\acute{G}(f(x)) = r$ and $\acute{G}(f(y)) > \delta$ or $\acute{G}(f(y)) = \delta$ and $\acute{G}(f(x)) > r$. Without limiting generality, assume $\exists\, \acute{G} \in \acute{g}$ s.t $\acute{G}(f(x)) = r$ and $\acute{G}(f(y)) > \delta$.
Now, $\acute{G}(f(x)) = r \Rightarrow f^{-1}(\acute{G})(x) = r$ and $\acute{G}(f(y)) > \delta \Rightarrow f^{-1}(\acute{G})(y) > \delta$. Given that f is a generalized fuzzy continous map and $\acute{G} \in \acute{g}$, it can be inferred that $f^{-1}(\acute{G}) \in g$. Similarly, if $\acute{G}(f(x)) > r$ and $\acute{G}(f(y)) = \delta$, afterward $f^{-1}(\acute{G})(x) > r$ and $f^{-1}(\acute{G})(y) = \delta$. Thus, (X, g) is $gFT_0^{(ii)}$.
4. Consider $(Y, \acute{g})$ as a $gFT_0^{(iii)}$ space and let $x_\alpha, y_\beta \in FS(X)$ with different supports. Therefore $(f(x))_\alpha$, $(f(y))_\beta \in FS(Y)$ with $f(x)$ is not equal to $f(y)$ since f is injective.Because $(Y, \acute{g})$ is $gFT_0^{(iii)}$, $\exists \acute{H} \in \acute{g}$ s.t $(f(x))_\alpha q \acute{H}$ and $(f(y))_\beta \cap \acute{H} = 0_X$ or $\exists\, \acute{W} \in \acute{g}$ s.t $(f(y))_\beta q \acute{W}$ and $(f(x))_\alpha \cap \acute{W} = 0_X$. Assume, for simplicity, that $\exists \acute{H} \in \acute{g}$ s.t $(f(x))_\alpha q \acute{H}$ and $(f(y))_\beta \cap \acute{H} = 0_X$.
Now, $(f(x))_\alpha q \acute{H} \Rightarrow \acute{H}(f(x)) + \alpha > 1 \Rightarrow f^{-1}(\acute{H}(x)) + \alpha > 1 \Rightarrow (f^{-1}(\acute{H}))(x) + \alpha > 1 \Rightarrow x_\alpha q f^{-1}(\acute{H})$ and $(f(y))_\beta \cap \acute{H} = 0_X \Rightarrow \acute{H}(f(y)) = 0 \Rightarrow f^{-1}(\acute{H}(y)) = 0 \Rightarrow (f^{-1}(\acute{H}))(y) = 0 \Rightarrow y_\beta \cap f^{-1}(\acute{H}) = 0_X$. Given that f is a generalized fuzzy continous map and $\acute{H} \in \acute{g}$, it can be inferred that $f^{-1}(\acute{H}) \in g$. Similarly, we can show that $y_\beta q f^{-1}(\acute{W})$ and $x_\alpha \cap f^{-1}(\acute{W}) = 0_X$. Therefore, (X, g) is a $gFT_0^{(iii)}$ space.
5. Is similar to (4). □

Theorem 5.3. *A GFTS (X, g) is considered a gFT_0 space iff $\forall\, x_\alpha, y_\beta \in FS(X)$ with x is not equal to y, $\exists$ a generalized fuzzy continuous mapping f from X to a gFT_0 space $(Y, \acute{g})$ s.t $f(x)$ is not equal to $f(y)$.*

Proof. Necessity. Assume (X, g) is gFT_0. Assume $(Y, \acute{g}) = (X, g)$ and let f be the identity mapping id_X. Clearly, $(Y, \acute{g})$ and f meet the required criteria.

Sufficiency. Consider fuzzy singletons x_α and, y_β in $FS(X)$. We examine two scenarios: (i) $x \neq y$ and (ii) $x = y$ with $\alpha < \beta$

For the situation where $x \neq y$, according to the hypothesis, $\exists$ a generalized fuzzy continous mapping f from (X, g) to gFT_0 $(Y, \acute{g})$ with $f(x)$ is not equal to $f(y)$. Given that $(Y, \acute{g})$ is a gFT_0 space and $(f(x))_\alpha, (f(y))_\beta \in FS(Y)$ s.t $f(x)$ is not equal to $f(y)$, either $\exists\, H \in N^Y_{\acute{g}}((f(x))_\alpha)$ s.t $(f(y))_\beta \bar{q} H$ or $\exists\, W \in N^Y_{\acute{g}}((f(y))_\beta)$ s.t $(f(x))_\alpha \bar{q} W$. It follows from generalized fuzzy continouity of f that either $f^{-1}(H) \in N^X_g(x_\alpha)$ s.t $y_\beta \bar{q} f^{-1}(H)$ or $f^{-1}(W) \in N^X_g(y_\beta)$ s.t $x_\alpha \bar{q} f^{-1}(W)$.

If $x = y$ and $(\alpha < \beta)$, then $(f(x))_\alpha, (f(y))_\beta \in FS(Y)$ with $f(x) = f(y)$. Given that $(Y, \acute{g})$ is a gFT_0 space, there exists $W \in N^{Q^Y}_{\acute{g}}((f(y))_\beta)$ s.t $(f(x))_\alpha \bar{q} W$. Therefore, $f^{-1}(V) \in N^{Q^X}_g(y_\beta)$ s.t $x_\alpha \bar{q} f^{-1}(V)$.Thus (X, g) is also a gFT_0 space. □

Theorem 5.4. *A GFTS (X, g) is a $gFT_0^{(i)}$ space iff $\forall\, x_\alpha, y_\beta \in FS(X)$ with $x \neq y$, $\exists$ a generalized fuzzy continuous mapping f from X to a $gFT_0^{(i)}$ space $(Y, \acute{g})$ s.t $f(x)$ is not equal to $f(y)$.*

Proof. Is analogous to the proof of Theorem 5.3. □

Theorem 5.5. *A GFTS (X, g) is a $gFT_0^{(ii)}$ space iff $\forall\, x, y \in X$ where x is not equal to y, $\exists$ a generalized fuzzy continuous mapping f from X to a $gFT_0^{(ii)}$ space $(Y, \acute{g})$ s.t $f(x) \neq f(y)$.*

Proof. Necessity. Suppose (X, g) is $gFT_0^{(ii)}$. Consider $(Y, \acute{g})$ be defined as (X, g) and let f be the identity mapping id_X. Clearly, $(Y, \acute{g})$ and f meet the required criteria.

Sufficiency. Assume $x, y \in X$ with $x \neq y$. Based on the hypothesis, $\exists$ a generalized fuzzy continous mapping f from (X, g) to $gFT_0^{(ii)}$ $(Y, \acute{g})$ s.t, $f(x)$ is not equal to $f(y)$. Since $(Y, \acute{g})$ is $gFT_0^{(ii)}$ and $f(x), f(y) \in Y$ with $f(x)$ is not equal to $f(y)$, then for any $r, \delta \in [0, 1)$ and $f(x), f(y) \in Y$ with $f(x)$ is not equal to $f(y)$. $\exists\, G \in \acute{g}$ s.t either $G(f(x)) = r$ and $G(f(y)) > \delta$ or $G(f(x)) > r$ and $G(f(y)) = \delta$. It follows from generalized fuzzy continouity of f that $f^{-1}(G) \in g$ s.t either $f^{-1}(G)(x) = r$ and $f^{-1}(G)(y) > \delta$ or $f^{-1}(G)(x) > r$ and $f^{-1}(G)(y) = \delta$. Hence (X, g) is a $gFT_0^{(ii)}$ space. □

Theorem 5.6. *A GFTS (X, g) is $gFT_0^{(iii)}$ iff $\forall\, x_\alpha, y_\beta \in FS(X)$ with x is not equal to y, $\exists$ a generalized fuzzy continuous mapping f from X to a $gFT_0^{(iii)}$ space $(Y, \acute{g})$ s.t $f(x)$ is not equal to $f(y)$.*

Proof. Necessity. assuming (X, g) is $gFT_0^{(iii)}$. Consider $(Y, \acute{g}) = (X, g)$ and let f be the identity mapping id_X. Clearly, $(Y, \acute{g})$ and f meet the required criteria.

Sufficiency. Consider fuzzy singletons x_α and, y_β in $FS(X)$ with $x \neq y$. According to the assumption, $\exists$ a generalized fuzzy continous mapping f from (X, g) to $gFT_0^{(iii)}$ $(Y, \acute{g})$ s.t $f(x)$ is not equal to $f(y)$. Given that $(Y, \acute{g})$ is $gFT_0^{(iii)}$ and $(f(x))_\alpha, (f(y))_\beta \in FS(Y)$ s.t $f(x)$ is not equal to $f(y)$, $\exists\, H \in \acute{g}$ s.t $(f(x))_\alpha q H$ and $(f(y))_\beta \cap H = 0_X$ or $\exists\, W \in \acute{g}$ s.t $(f(y))_\beta q W$ and $(f(x))_\alpha \cap W = 0_X$. It follows from generalized fuzzy continouity of f that $f^{-1}(H) \in g$ s.t $x_\alpha q f^{-1}(H)$ and $y_\beta \cap f^{-1}(H) = 0_X$ or $f^{-1}(W) \in g$ s.t $y_\beta q f^{-1}(W)$ and $x_\alpha \cap f^{-1}(W) = 0_X$. Thuse, (X, g) is a $gFT_0^{(iii)}$ space. □

Theorem 5.7. *A GFTS (X, g) is a $gFT_0^{(iv)}$ space iff $\forall\, x_\alpha, y_\beta \in FS(X)$ with $x \neq y$, $\exists$ a generalized fuzzy continuous mapping f from X to a $gFT_0^{(iv)}$ space $(Y, \acute{g})$ s.t $f(x)$ is not equal to $f(y)$.*

Proof. Is analogous to the proof of Theorem 5.6. □

6. A generalized lower semi-continuous function, initial and final generalized fuzzy topological spaces

This section covers the initial and final generalized fuzzy topologies as well as the introduction and examination of a generalized lower semi-continuous function.

Definition 6.1. *A real-valued function f on a GTS is called a generalized lower semi-continuous function if the collection $\{x : f(x) > \beta\}$ is generalized open $\forall$ real β.*

Definition 6.2. *Consider a nonempty set X having a generalized topology g. Let $\omega(g)$ represent the collection of all generalized lower semi-continuous functions from (X, g) to I. Hence, $\omega(g) = \{H \in I^X : H^{-1}(\beta, 1] \in g\}$, $\forall\, \beta \in [0, 1)$. It can be demonstrated that $\omega(g)$ forms a GFT on X.*

Theorem 6.1. *Consider (X, g) be a GTS . The subsequent statements are equivalent:*
1. (X, g) is a gT_0 space;
2. $(X, \omega(g))$ is a gFT_0 space;
3. $(X, \omega(g))$ is a $gFT_0^{(i)}$ space;
4. $(X, \omega(g))$ is a $gFT_0^{(ii)}$ space;
5. $(X, \omega(g))$ is a $gFT_0^{(iii)}$ space;
6. $(X, \omega(g))$ is a $gFT_0^{(iv)}$ space.

Proof. 1 $\Leftrightarrow$ 2. Necessity: Suppose (X, g) is gT_0. We shall demonstrate that $(X, \omega(g))$ is gFT_0. Assume $x_\alpha, y_\beta \in FS(X)$ where x is not equal to y. Given that (X, g) is gT_0, $\exists\, G \in g$ s.t $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$. According to the concept of the generalized lower semi-continuous function, $1_G \in \omega(g)$ and satisfies $1_G(x) = 1$ and $1_G(y) = 0$. Therefore:
• Since $1_G(x) = 1$, it can be concluded that $\alpha \leq 1_G(x)$, so $x_\alpha \in 1_G$
• Since $1_G(y) = 0$, it can be concluded that $1_G(y) + \beta \leq 1$, so $y_\beta \bar{q} 1_G$. Consequently, $1_G \bar{q} y_\beta$, meaning $1_G \subseteq (y_\beta)^c$. Thus, $1_G \in \omega(g)$ and $x_\alpha \in 1_U \subseteq (y_\beta)^c$. Similarly, we can prove that $y_\beta \in 1_G \subseteq (x_\alpha)^c$. Hence $(X, \omega(g))$ is a gFT_0 space.
Sufficiency: Considering $(X, \omega(g))$ is gFT_0. We must demonstrate that (X, g) is gT_0. Assume $x, y \in X$ where $x \neq y$. Because $(X, \omega(g))$ is a gFT_0 space,so $\forall\, x_\alpha, y_\alpha \in FS(X)$, $\exists\, G \in \omega(g)$ s.t $x_\alpha \in G \subseteq (y_\alpha)^c$ or $y_\alpha \in G \subseteq (x_\alpha)^c$. Without loss of generality, $x_\alpha \in G \subseteq (y_\alpha)^c$.
Now, $x_\alpha \in G \Rightarrow \alpha < G(x) \Rightarrow 1 - \alpha = m < G(x) \Rightarrow x \in G^{-1}(m, 1]$ and $G \subseteq (y_\alpha)^c \Rightarrow G \bar{q} y_\alpha \Rightarrow G(y) + \alpha \leq 1 \Rightarrow G(y) \leq 1 - \alpha = m \Rightarrow y \notin G^{-1}(m, 1]$. Likewise, it can be demonstrated that if $y \in G^{-1}(m, 1]$, then $x \notin G^{-1}(m, 1]$. Additionally, $G^{-1}(m, 1]$ is generalized open set. Hence (X, g) is a gT_0 space.
1 $\Leftrightarrow$ 3. Necessity: Suppose (X, g) is gT_0. We want to demonstrate that $(X, \omega(g))$ is $gFT_0^{(i)}$. Consider $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Because (X, g) is gT_0,so $\exists\, H \in g$ s.t $x \in H$ and $y \notin H$ or $y \in H$ and $x \notin H$. By the concept of the generalized lower semi continuous function, we know $1_H \in \omega(g)$. Then $1_H(x) = 1$ and $1_H(y) = 0$ or $1_H(y) = 1$ and $1_H(x) = 0$. Hence $(X, \omega(g))$ is a $gFT_0^{(i)}$ space.
Sufficiency: Considering $(X, \omega(g))$ is a $gFT_0^{(i)}$ space. We need to demonstrate that (X, g) is gT_0.

Assume $x, y \in X$ where $x \neq y$. Given that $(X, \omega(g))$ is $gFT_0^{(i)}$ space, $\forall\, x_\alpha, y_\alpha \in FS(X)$, $\exists\, G \in \omega(g)$ s.t $G(x) = 1$ and $G(y) = 0$ or $G(y) = 1$ and $G(x) = 0$. Assume, without affecting generality, $G(x) = 1$ and $G(y) = 0$.

Now, $G(x) = 1 \Rightarrow G(x) + \alpha > 1 \Rightarrow G(x) > 1 - \alpha = m \Rightarrow x \in G^{-1}(m, 1]$ and $G(y) = 0 \Rightarrow G(y) + \alpha \leq 1 \Rightarrow G(y) \leq 1 - \alpha = m \Rightarrow y \notin G^{-1}(m, 1]$. This can be similarly shown for the reverse case. Additionally, $G^{-1}(m, 1]$ is generalized open set. Hence (X, g) is a gT_0 space.

$1 \Leftrightarrow 4$. Necessity: Assuming (X, g) is gT_0. We aim to demonstrate that $(X, \omega(g))$ is $gFT_0^{(ii)}$. Suppose $x, y \in X$ where $x \neq y$ and assume that $r, \delta \in [0, 1)$. Since (X, g) is gT_0, $\exists\, G \in g$ s.t $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$. By the concept of the generalized lower semi continuous function, $1_G \in \omega(g)$ with $1_G(x) = 1$ and $1_G(y) = 0$. Thus:

- Since $1_G(x) = 1$, we have $1_G(x) > r$.
- Since $1_G(y) = 0$, we have $1_G(y) = \delta$.

Hence $(X, \omega(g))$ is a $gFT_0^{(ii)}$ space.

Sufficiency: Suppose $(X, \omega(g))$ is a $gFT_0^{(ii)}$ space. We need to demonstrate that (X, g) is gT_0. Assume $x, y \in X$ where x is not equal to y. Given that $(X, \omega(g))$ is a $gFT_0^{(ii)}$ space, $\forall\, r, \delta \in [0, 1)$, $\exists\, G \in \omega(g)$ s.t $G(x) = r$ and $G(y) > \delta$ or $G(x) > r$ and $G(y) = \delta$. Suppose, without loss of generality, $G(x) = r$ and $G(y) > \delta$.

Now, $G(x) = r \Rightarrow x \notin G^{-1}(r, 1]$ and $G(y) > \delta \Rightarrow y \in G^{-1}(\delta, 1]$. Hence (X, g) is a gT_0 space.

$1 \Leftrightarrow 5$. Necessity: Suppose (X, g) is gT_0. We shall demonstrate that $(X, \omega(g))$ is $gFT_0^{(iii)}$. Assume $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Given that (X, g) is gT_0, $\exists\, H \in g$ s.t $x \in H$ and $y \notin H$ or $\exists\, W \in g$ s.t $y \in W$ and $x \notin W$. According to the concept of the generalized lower semi continuous function, we know $1_H, 1_W \in \omega(g)$. Thus:

- Since $1_H(x) = 1$, it can be concluded that $1_H(x) + \alpha > 1$, so $x_\alpha q 1_H$.
- Since $1_H(y) = 0$, it can be concluded that $y_\beta \cap 1_H(y) = 0_X$. This can be similarly shown for the reverse case. Therefore, $(X, \omega(g))$ is a $gFT_0^{(iii)}$ space.

Sufficiency: Suppose $(X, \omega(g))$ is a $gFT_0^{(iii)}$ space. We need to demonstrate that (X, g) is gT_0. Assume $x, y \in X$where x is not equal to y. Given that $(X, \omega(g))$ is $gFT_0^{(iii)}$, $\forall\, x_\alpha, y_\alpha \in FS(X)$, $\exists\, H \in \omega(g)$ s.t $x_\alpha q H$ and $y_\alpha \cap H = 0_X$ or $\exists\, W \in \omega(g)$ s.t $y_\alpha q W$ and $x_\alpha \cap W = 0_X$. Suppose, without loss of generality, $\exists\, H \in \omega(g)$ s.t $x_\alpha q H$ and $y_\alpha \cap H = 0_X$.

Now, $x_\alpha q H \Rightarrow H(x) + \alpha > 1 \Rightarrow H(x) > 1 - \alpha = m \Rightarrow x \in H^{-1}(m, 1]$ and $y_\alpha \cap H = 0_X \Rightarrow H(y) = 0 \Rightarrow H(y) + \alpha \leq 1 \Rightarrow H(y) < 1 - \alpha = m \Rightarrow y \notin H^{-1}(m, 1]$. This can be similarly shown for the reverse case. Additionally, $H^{-1}(m, 1]$ and $W^{-1}(r, 1]$ are generalized open sets. Hence (X, g) is a gT_0 space.

$1 \Leftrightarrow 6$. Necessity. Suppose (X, g) is gT_0. We want to demonstrate that $(X, \omega(g))$ is $gFT_0^{(iv)}$. Assume $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Given that (X, g) is gT_0, $\exists\, G \in g$ s.t $x \in G$ and $y \notin G$ or $\exists\, W \in g$ s.t $y \in W$ and $x \notin W$. According to the concept of the generalized lower semi continous function, we know $1_G, 1_W \in \omega(g)$. Thus

$1_G(x) = 1$ implies $1_G(x) + \alpha > 1 \Rightarrow x_\alpha q 1_G$. $1_G(y) = 0$ implies $1_G(y) + \beta \leq 1 \Rightarrow y_\beta \bar{q} 1_G \Rightarrow 1_G \bar{q} y_\beta \Rightarrow$

$1_G \subseteq (y_\beta)^c$. Hence, $x_\alpha q 1_G \subseteq (y_\beta)^c$. Similarly, we can show that $y_\beta q 1_G \subseteq (x_\alpha)^c$. Hence, $(X, \omega(g))$ is a $gFT_0^{(iv)}$ space.

Sufficiency: Assume $(X, \omega(g))$ is $gFT_0^{(iv)}$. We need to demonstrate that (X, g) is gT_0. Consider $x, y \in X$ where x is not equal to y. Given that $(X, \omega(g))$ is $gFT_0^{(iii)}$, $\forall\, x_\alpha, y_\alpha \in FS(X)$, $\exists\, G \in \omega(g)$ s.t $x_\alpha q G \subseteq (y_\beta)^c$ or $y_\beta q G \subseteq (x_\alpha)^c$. Suppose, without loss of generality, $\exists\, G \in \omega(g)$ s.t $x_\alpha q G \subseteq (y_\beta)^c$.
Then:

- $x_\alpha q G$ implies $G(x) + \alpha > 1$, so $G(x) > 1 - \alpha = m$, meaning $x \in G^{-1}(m, 1]$.
- $G \subseteq (y_\beta)^c$ implies $G \bar{q} y_\beta$, so $\beta + G(y) \leq 1$, hence $G(y) < 1 - \beta = m$ meaning $y \notin G^{-1}(m, 1]$. Since $G^{-1}(m, 1]$ is generalized open sets. Hence (X, g) is a gT_0 space.

□

Definition 6.3. *Given the family of GFTS $\{(X_k, g_k)\}_{k \in J}$ and the the collection of functions $\{f_k : X \longrightarrow (X_k, g_k)\}_{k \in J}$, The initial GFTS on a set X is specified as the smallest GFT that makes each f_k generalized fuzzy continuous. This GFT is generated by the family $\{f_k^{-1}(H_k) : H_k \in g_k\}_{k \in J}\}$.*

Theorem 6.2. *If $\{(X_k, g_k)\}$ represents a collection of gFT_0 spaces and $\{f_k : X \to (X_k, g_k)\}$ denotes a collection of injective and generalized fuzzy continuous functions; thus, the initial GFT induced by the collection $\{f_k\}_{k \in J}$ is also a gFT_0 space.*

Proof. Assume that g denote the initial GFT on X for the family $\{f_k\}_{k \in J}$. Consider $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Since f_k is a one-to-one function, $f_k(x)$ and $f_k(y)$ are distinct elements in X_k. Given that (X_k, g_k) is a gFT_0 space, for every different fuzzy singletons $(f_k(x))_\alpha$ and $(f_k(y))_\beta$ either $\exists\, H_k \in g_k$ s.t $(f_k(x))_\alpha \in H_k \subseteq ((f_k(y))_\beta)^c$ or $\exists\, W_k \in g_k$ s.t $(f_k(y))_\beta \in H_k \subseteq ((f_k(x))_\alpha)^c$. For simplicity, assume $\exists\, H_k \in g_k$ s.t $(f_k(x))_\alpha \in H_k \subseteq ((f_k(y))_\beta)^c$.
Given that, $(f_k(x))_\alpha \in H_k \Rightarrow \alpha \leq H_k(f_k(x)) \Rightarrow \alpha \leq f_k^{-1}(H_k)(x)$. This condition holds $\forall\, k \in J$, so $\alpha \leq \bigvee_{k \in J} f_k^{-1}(H_k)(x)$. Also $H_k \subseteq ((f_k(y))_\beta)^c \Rightarrow H_k \bar{q} (f_k(y))_\beta \Rightarrow H_k(f_k(y)) + \beta \leq 1 \Rightarrow f_k^{-1}(H_k)(y) + \beta \leq 1$. This is true for every $k \in J$. $\bigvee_{k \in J} f_k^{-1}(H_k)(y) + \beta \leq 1$. Let $H = \bigvee_{k \in J} f_k^{-1}(H_k)$. Then $H \in g$ as f_k is a generalized fuzzy continous. so, $\alpha \leq H(x)$ and $H(y) + \beta \leq 1$. Therefore, $x_\alpha \in H$ and $H \bar{q} y_\beta \Rightarrow H \subseteq (y_\beta)^c$. Hence $x_\alpha \in H \subseteq (y_\beta)^c$. Similarly, $y_\beta \in H \subseteq (x_\alpha)^c$. Hence (X, g) is a gFT_0 space. □

Theorem 6.3. *If (X_k, g_k) represents a collection of $gFT_0^{(i)}$ spaces and $\{f_k : X \to (X_k, g_k)\}$ denotes a collection of injective and generalized fuzzy continuous functions; thus, the initial GFT induced by the collection $\{f_k\}_{k \in J}$ is also a $gFT_0^{(i)}$ space.*

Proof. Is analogous to the proof of Theorem 6.2. □

Theorem 6.4. *If (X_k, g_k) represents a collection of $gFT_0^{(ii)}$ spaces and $\{f_k : X \to (X_k, g_k)\}$ denotes a collection of injective and generalized fuzzy continuous functions; thus, the initial GFT on X induced by the collection $\{f_k\}_{k \in J}$ is also a $gFT_0^{(ii)}$ space.*

Proof. Assume that g denote the initial GFT for the collection $\{f_k\}_{k \in J}$. Suppose x and y are elements of X where x is not equal to y and any $r, \delta \in [0, 1)$. Since f_k is a one-to-one function, $f_k(x)$ and

$f_k(y)$ are distinct elements in X_k. Given that (X_k, g_k) is a $gFT_0^{(ii)}$ space, for any $r, \delta \in [0, 1)$ and distinct $f_k(x)$ and $f_k(y)$ in X_k, $\exists H_k \in g_k$ s.t $H_k(f_k(x)) = r$ and $H_k(f_k(y)) > \delta$, or $H_k(f_k(x)) > r$ and $H_k(f_k(y)) = \delta$. For simplicity, assume $\exists H_k \in g_k$ s.t $H_k(f_k(x)) = r$ and $H_k(f_k(y)) > \delta$.
Now, $H_k(f_k(x)) = r \Rightarrow f_k^{-1}(H_k)(x) = r$. This condition holds $\forall k \in J$. Thus, $\bigvee_{k \in J} f_k^{-1}(H_k)(x) = r$. Also, $H_k(f_k(y)) > \delta \Rightarrow f_k^{-1}(H_k)(y) > \delta \Rightarrow \bigvee_{k \in J} f_k^{-1}(H_k)(y) > \delta$. Suppose $H = \bigvee_{k \in J} f_k^{-1}(H_k)$. Since f_k is a generalized fuzzy continous, $H \in g$. Therefore, $H(x) = r$ and $H(y) > \delta$. This can be similarly shown for the reverse case. Thus, (X, g) is a $gFT_0^{(ii)}$ space. □

Theorem 6.5. *If (X_k, g_k) represents a collection of $gFT_0^{(iii)}$ spaces and $\{f_k : X \to (X_k, g_k)\}$ denotes a collection of injective and generalized fuzzy continuous functions; thus, the initial GFT on X induced by the collection $\{f_k\}_{k \in J}$ is also a $gFT_0^{(iii)}$ space.*

Proof. Assume that g denote the initial GFT on X for the family $\{f_k\}_{k \in J}$. Consider $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. Since f_k is a one-to-one function, $f_k(x)$ and $f_k(y)$ are distinct elements in X_k. Given that (X_k, g_k) is a $gFT_0^{(iii)}$ space, for every different fuzzy singletons $(f_k(x))_\alpha$ and $(f_k(y))_\beta$ either $\exists H_k \in g_k$ s.t $(f_k(x))_\alpha q H_k$ and $(f_k(y))_\beta \cap H_k = 0_X$, or $\exists W_k \in g_k$ s.t $(f_k(y))_\beta q W_k$ and $(f_k(x))_\alpha \cap W_k = 0_X$. For simplicity, assume $\exists H_k \in g_k$ s.t $(f_k(x))_\alpha q H_k$ and $(f_k(y))_\beta \cap H_k = 0_X$.
Given that, $(f_k(x))_\alpha q H_k \Rightarrow H_k(f_k(x)) + \alpha > 1 \Rightarrow f_k^{-1}(H_k)(x) + \alpha > 1$. This condition holds $\forall k \in J$. Hence $\bigvee_{k \in J} f_k^{-1}(H_k)(x) + \alpha > 1$. Also, $(f_k(y))_\beta \cap H_k = 0_X \Rightarrow H_k(f_k(y)) = 0_X \Rightarrow f_k^{-1}(H_k)(y) = 0_X \Rightarrow \bigvee_{k \in J} f_k^{-1}(H_k)(y) = 0_X$. Suppose $H = \bigvee_{k \in J} f_k^{-1}(H_k)$. Since f_k is a generalized fuzzy continous, $H \in g$. Therefore, $H(x) + \alpha > 1$ and $H(y) = 0_X$. Hence, $x_\alpha q H$ and $y_\beta \cap H = 0_X$. Similarly, $y_\beta q W$ and $x_\alpha \cap W = 0_X$. Hence (X, g) is a $gFT_0^{(iii)}$ space. □

Theorem 6.6. *If (X_k, g_k) represents a collection of $gFT_0^{(iv)}$ spaces and $\{f_k : X \to (X_k, g_k)\}$ denotes a collection of injective and generalized fuzzy continuous functions; thus, the initial GFT on X induced by the collection $\{f_k\}_{k \in J}$ is also a $gFT_0^{(iv)}$ space.*

Proof. Is analogous to the proof of Theorem 6.5. □

Definition 6.4. *Given the family of GFTS $\{(X_k, g_k)\}_{k \in J}$ and the family of functions $\{f_k : X \longrightarrow (X_k, g_k)\}_{k \in J}$, The final GFTS on a set X is defined as the finest GFT on X that ensures every f_k generalized fuzzy continuous.*

Theorem 6.7. *If (X_k, g_k) represents a collection of gFT_0 spaces and $\{f_k : (X_k, g_k) \to X\}$ denotes a set of bijective and generalized fuzzy open functions; thus, the final GFT corresponding to the collection $\{f_k\}_{k \in J}$ will be a gFT_0 space.*

Proof. Suppose g represent the final GFT for the collection $\{f_k\}_{k \in J}$. Assume $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. For each k, $f_k^{-1}(x)$ and $f_k^{-1}(y)$ are elements of X_k, and since f_i is bijective, $f_k^{-1}(x) \neq f_k^{-1}(y)$. Given that (X_k, g_k) is a gFT_0 space, then $\forall$ $(f_k^{-1}(x))_\alpha, (f_k^{-1}(y))_\beta \in FS(X_k)$ with $f_k^{-1}(x) \neq f_k^{-1}(y)$, $\exists H_k, W_k \in g_k$ s.t $(f_k^{-1}(x))_\alpha \in H_k \subseteq ((f_k^{-1}(y))_\beta)^c$ or $(f_k^{-1}(y))_\beta \in W_k \subseteq ((f_k^{-1}(x))_\alpha)^c$. Suppose, without loss of generality, $\exists H_k \in g_k$ s.t $(f_k^{-1}(x))_\alpha \in H_k \subseteq ((f_k^{-1}(y))_\beta)^c$.
Consider, $(f_k^{-1}(x))_\alpha \in H_k \Rightarrow \alpha \leq H_k(f_k^{-1}(x)) \Rightarrow \alpha \leq f_k(H_k)(x)$ and $H_k \subseteq ((f_k^{-1}(y))_\beta)^c \Rightarrow$

$H_k\bar{q}(f_k^{-1}(y))_\beta \Rightarrow \beta + H_k(f_k^{-1}(y)) \leq 1 \Rightarrow \beta + f_k(H_k)(y) \leq 1$. This is holds $\forall\, k \in J$, so it follows that $\alpha \leq \bigvee_{k\in J} f_k(H_k)(x)$ and $\beta + \bigvee_{k\in J} f_k(H_k)(y) \leq 1$. Define $H = \bigvee_{k\in J} f_k(H_k)$. Since f_k is a generalized fuzzy open function, $H \in g$. Thus, $\alpha \leq H(x)$ and $\beta + H(y) \leq 1$. Consequently, $x_\alpha \in H$ and $H\bar{q}y_\beta \Rightarrow H \subseteq (y_\beta)^c$. Therefore, $x_\alpha \in H \subseteq (y_\beta)^c$. Hence (X, g) is a gFT_0 space. □

Theorem 6.8. *If (X_k, g_k) represents a collection of $gFT_0^{(i)}$ spaces and $\{f_k : (X_k, g_k) \to X\}$ denotes a set of bijective and generalized fuzzy open functions; thus, the final GFT corresponding to the collection $\{f_k\}_{k\in J}$ will be a $gFT_0^{(i)}$ space.*

Proof. Resembles the proof of Theorem 6.7. □

Theorem 6.9. *If (X_k, g_k) represents a collection of $gFT_0^{(ii)}$ spaces and $\{f_k : (X_k, g_k) \to X\}$ denotes a set of bijective and generalized fuzzy open functions; thus, the final GFT corresponding to the collection $\{f_k\}_{k\in J}$ will be a $gFT_0^{(ii)}$ space.*

Proof. Suppose g represent the final GFT for the collection $\{f_k\}_{k\in J}$. Assume $x, y \in X$ wih $x \neq y$ and any $r, \delta \in [0, 1)$. For each k, $f_k^{-1}(x)$ and $f_k^{-1}(y)$ are elements of X_k, and since f_k is bijective, $f_k^{-1}(x) \neq f_k^{-1}(y)$. Given that (X_k, g_k) is $gFT_0^{(ii)}$ space, for each pair of fuzzy singletons $f_k^{-1}(x)$ and $f_k^{-1}(y)$ in X_k and for any $r, \delta \in [0, 1)$, $\exists\, H_k \in g_k$ s.t $H_k(f_k^{-1}(x)) = r$ and $H_k(f_k^{-1}(y)) > \delta$ or $H_k(f_k^{-1}(x)) > r$ and $H_k(f_k^{-1}(y)) = \delta$. Suppose, without loss of generality, $\exists\, H_k \in g_k$ s.t $H_k(f_k^{-1}(x)) = r$ and $H_k(f_k^{-1}(y)) > \delta$.

Now, $H_k(f_k^{-1}(x)) = r \Rightarrow f_k(H_k)(x) = r$ and $H_k(f_k^{-1}(y)) > \delta \Rightarrow f_k(H_k)(y) > \delta$. This is holds $\forall\, k \in J$. Therefore, $\bigvee_{k\in J} f_k(H_k)(x) = r$ and $\bigvee_{k\in J} f_k(H_k)(y) > \delta$. Define $H = \bigvee_{k\in J} f_k(H_k)$. Since f_k is a generalized fuzzy open function, $H \in g$. Consequently, $H(x) = r$ and $H(y) > \delta$. Similarly, $H(x) > r$ and $H(y) = \delta$. Then (X, g) is a $gFT_0^{(ii)}$ space. □

Theorem 6.10. *If (X_k, g_k) represents a collection of $gFT_0^{(iii)}$ spaces and $\{f_k : (X_k, g_k) \to X\}$ denotes a set of bijective and generalized fuzzy open functions; thus, the final GFT corresponding to the collection $\{f_k\}_{k\in J}$ will be a $gFT_0^{(iii)}$ space.*

Proof. Suppose g represent the final GFT for the collection $\{f_k\}_{k\in J}$. Assume $x_\alpha, y_\beta \in FS(X)$ where $x \neq y$. For each k, $f_k^{-1}(x)$ and $f_k^{-1}(y)$ are elements of X_k, and since f_i is bijective, $f_k^{-1}(x) \neq f_k^{-1}(y)$. Given that (X_k, g_k) is a $gFT_0^{(iii)}$ space, then $\forall\, (f_k^{-1}(x))_\alpha, (f_k^{-1}(y))_\beta \in FS(X_k)$ with $f_k^{-1}(x) \neq f_k^{-1}(y)$, $\exists\, H_k \in g_k$ s.t $(f_k^{-1}(x))_\alpha q H_k$ and $(f_k^{-1}(y))_\beta \cap H_k = 0_X$ or $\exists\, W_k \in g_k$ s.t $(f_k^{-1}(y))_\beta q W_k$ and $(f_k^{-1}(x))_\alpha \cap W_k = 0_X$. Suppose, without loss of generality, $\exists H_k \in g_k$ s.t $(f_k^{-1}(x))_\alpha q H_k$ and $(f_k^{-1}(y))_\beta \cap H_k = 0_X$.

Now, $(f_k^{-1}(x))_\alpha q H_k \Rightarrow H_k(f_k^{-1}(x)) + \alpha > 1 \Rightarrow f_k(H_k)(x) + \alpha > 1 \Rightarrow \bigvee_{k\in J}(f_k(H_k))(x) + \alpha > 1$. Also, $(f_k^{-1}(y))_\beta \cap H_k = 0_X \Rightarrow H_k(f_k^{-1}(y)) = 0_X \Rightarrow f_k(H_k)(y) = 0_X \Rightarrow \bigvee_{k\in J}(f_k(H_k))(y) = 0_X$. Define $H = \bigvee_{k\in J} f_k(H_k)$. Since f_k is a generalized fuzzy open function, $H \in g$. Consequently, $x_\alpha q H$ and $y_\beta \cap H = 0_X$. Similarly, $y_\beta q W$ and $x_\alpha \cap W = 0_X$. Therefore, (X, g) is a $gFT_0^{(iii)}$ space. □

Theorem 6.11. *If (X_k, g_k) represents a collection of $gFT_0^{(iv)}$ spaces and $\{f_k : (X_k, g_k) \to X\}$ denotes a set of bijective and generalized fuzzy open functions; thus, the final GFT corresponding to the collection $\{f_k\}_{k\in J}$ will be a $gFT_0^{(iv)}$ space.*

Proof. Resembles the proof of Theorem 6.10. □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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