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A Study on Multi-Intuitionistic Fuzzy Sets and Their Application in Ordered Semigroups

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Abstract. In this paper, we introduce the notion of multi-intuitionistic fuzzy sets in ordered semigroups. The concepts of multi-intuitionistic fuzzy subsemigroups, multi-intuitionistic fuzzy left (right, two-sided, interior) ideals of an ordered semigroup are introduced and some algebraic properties of multi-intuitionistic fuzzy subsemigroups and such their multi-intuitionistic fuzzy ideals are studied. Moreover, the relationships among their multi-intuitionistic fuzzy ideals are investigated. We prove that in regular, intra-regular, and semisimple ordered semigroups, the concepts of multi-intuitionistic fuzzy interior ideals and multi-intuitionistic fuzzy ideals coincide. Finally, the new multi-intuitionistic fuzzy sets are considered.

1. Introduction

The theory of fuzzy sets, which is the most appropriate theory for dealing with uncertainty, was first introduced by Zadeh [1] in 1965. After Zadeh's introduction of fuzzy sets, several researchers explored generalizations of these notions, leading to significant applications in various fields such as computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machines, graph theory, logic, operations research, and many branches of pure and applied mathematics.

As a generalization of fuzzy set, Atanassov [2] created intuitionistic fuzzy set. Intuitionistic fuzzy set is widely used in all fields (See [3–6] for applications in algebraic structures). In [7], Kuroki initially applied fuzzy set theory to semigroup theory. After that, Kehayopulu and Tsingelis [8] first applied fuzzy set theory to ordered semigroup theory. In 2014, Chen et al. [9] introduced the idea of multi-polar fuzzy sets as an extension of bipolar fuzzy sets and showed that bipolar fuzzy

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sets and 2-polar fuzzy sets are cryptomorphic mathematical notions. Kazanci et al. [10] introduced the notion of multi-polar fuzzy sets in ordered semihypergroups and defined multi-polar fuzzy hyperideals (bi-hyperideals, quasi-hyperideals) in an ordered semihypergroup. Relations between multi-polar fuzzy hyperideals, multi-polar fuzzy bi-hyperideals and multi-polar fuzzy quasi-hyperideals are discussed.

We consider intuitionistic fuzzy sets, as a generalization of bipolar fuzzy sets. Based on the concept of [9,10]. In this paper, we introduce the notion of multi-intuitionistic fuzzy sets in ordered semigroups. The concepts of multi-intuitionistic fuzzy subsemigroups, multi-intuitionistic fuzzy left (right, two-sided, interior) ideals of an ordered semigroup are introduced and some algebraic properties of multi-intuitionistic fuzzy subsemigroups and such their multi-intuitionistic fuzzy ideals are studied. Moreover, the relationships among their multi-intuitionistic fuzzy ideals are investigated. In regular, intra-regular, and semisimple ordered semigroups, the concepts of multi-intuitionistic fuzzy interior ideals and multi-intuitionistic fuzzy ideals coincide are proved. Finally, the new multi-intuitionistic fuzzy sets are considered.

2. Preliminary

In this section, we will recall the basic terms and definitions from the ordered semigroup theory and the multi-intuitionistic fuzzy set theory that we will use in this paper.

Let S be a nonempty set and a binary operation "$\cdot$" on the set S. The structure $\langle S; \cdot\rangle$ is called *groupoid*. If the binary operation "$\cdot$" satisfied associative property, that is

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for all $x, y, z \in S$, the groupoid $\langle S; \cdot\rangle$ is called *semigroup*.

A binary relation $\leq$ on S is called *partial relation on S* if it satisfies the following three properties: (1) reflexive, i.e. $a \leq a$ for all $a \in S$. (2) antisymmetric, i.e., if $a \leq b$ and $b \leq a$, then $a = b$ for all $a, b \in S$. (3) transitive, i.e., if $a \leq b$ and $b \leq c$, then $a \leq c$ for all $a, b, c \in S$. The structure $\langle S; \leq\rangle$ is called a *partial ordered set* if the relation $\leq$ is a partial relation on S.

Definition 2.1. *[8] The structure $\langle S; \cdot, \leq\rangle$ is called an ordered semigroup if the following conditions are satisfied:*

(1) *$\langle S; \cdot\rangle$ is a semigroup.*
(2) *$\langle S; \leq\rangle$ is a partially ordered set.*
(3) *For every $a, b, c \in S$ if $a \leq b$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.*

For simplicity, we denoted an ordered semigroup $\langle S; \cdot, \leq\rangle$ by its carrier set as a bold letter $\mathbf{S}$ and if $a, b \in S$, we will instead of $a \cdot b$ by ab. Let A and B be two nonempty subsets of S. Then we define

$$AB := \{ab \ : \ a \in A \text{ and } b \in B\}.$$

Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S is called a *subsemigroup* of $\mathbf{S}$ [11] if $AA \subseteq A$.

Definition 2.2. *[8] Let* **S** *be an ordered semigroup. A nonempty subset A of S is called a left ideal of* **S** *if it satisfies*

(1) $SA \subseteq A$.
(2) *For $x, y \in S$, if $x \leq y$ and $y \in A$, then $x \in A$.*

Definition 2.3. *[8] Let* **S** *be an ordered semigroup. A nonempty subset A of S is called a right ideal of* **S** *if it satisfies*

(1) $AS \subseteq A$.
(2) *For $x, y \in S$, if $x \leq y$ and $y \in A$, then $x \in A$.*

A nonempty subset of S is called *two-sided ideal* or simply *ideal* if it is both a left and a right ideal of **S**.

Definition 2.4. *[12] Let* **S** *be an ordered semigroup. A subsemigroup A of* **S** *is called an interior ideal of* **S** *if it satisfies*

(1) $SAS \subseteq A$.
(2) *For $x, y \in S$, if $x \leq y$ and $y \in A$, then $x \in A$.*

A fuzzy subset (or fuzzy set) of a nonempty set X is a mapping $f : X \to [0, 1]$ from X to a unit closed interval (see [1]).

Definition 2.5. *Let X be a nonempty set. An multi-intuitionistic fuzzy set f of X is an object having the form*

$$f := \{(x, f^+(x), f^*(x)) : x \in X,\ \mathbf{0} \leq f^+(x) \oplus f^*(x) \leq \mathbf{1}\},$$

where $f^+ : X \to [0, 1]^m$, $f^* : X \to [0, 1]^m$ and $\mathbf{0} \in \{0\}^m$, $\mathbf{1} \in \{1\}^m$. The value of f^+ and f^* are denoted by

$$f^+(x) := \big((\pi_1 \circ f^+)(x), \ldots, (\pi_m \circ f^+)(x)\big),$$

and

$$f^*(x) := ((\pi_1 \circ f^*)(x), \ldots, (\pi_m \circ f^*)(x))$$

such that $\pi_i : [0, 1]^m \to [0, 1]$ is a projection i-th mapping and $\mathbf{0} \leq f^+(x) \oplus f^*(x) \leq \mathbf{1}$, it means that $0 \leq (\pi_i \circ f^+)(x) + (\pi_i \circ f^*)(x) \leq 1$ for all $i \in \{1, \ldots, m\}$.

For the sake of simplicity, we shall use the symbol $f = (f^+, f^*)$ for $f = \{(x, f^+(x), f^*(x)) : x \in X,\ \mathbf{0} \leq f^+(x) \oplus f^*(x) \leq \mathbf{1}\}$.

Remark 2.1. *It is easy to see that if $m = 1$, then by Definition 2.5 the multi-intuitionistic fuzzy set f become intutionistic fuzzy set. This means that the multi-intuitionistic fuzzy set is a generalization of intutionistic fuzzy set.*

Example 2.1. *Let $H = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of five employees in a company. We shall characterize them according to four qualities in the form of 4-intuitionistic fuzzy set, given in the follows:*

	Honesty	*Punctual*	*Communication*	*Hardworking*
a_1	0.6	0.5	0.8	1
a_2	1	0.8	0.5	0.4
a_3	0.5	1	1	0.8
a_4	0.8	0.5	1	0.7
a_5	1	0.5	0	0.6

We define 4-intuitionistic fuzzy set $f = (f^+, f^*)$ as follows.

H	$f^+(x)$	$f^*(x)$
a_1	$(0.6, 0.5, 0.8, 1)$	$(0.3, 0.5, 0.1, 0)$
a_2	$(1, 0.8, 0.5, 0.4)$	$(0, 0.1, 0.2, 0.4)$
a_3	$(0.5, 1, 1, 0.8)$	$(0.2, 0, 0, 0.1)$
a_4	$(0.8, 0.5, 1, 0.7)$	$(0.1, 0.5, 0, 0.2)$
a_5	$(1, 0.5, 0, 0.6)$	$(0, 0.5, 0.5, 0.3)$

We denoted by $mIF(X)$ the set of all multi-intuitionistic fuzzy sets of X. Now we will give the relation and the operation on $mIF(X)$ as follows.

Definition 2.6. *Let $f_1 = (f_1^+, f_1^*)$ and $f_2 = (f_2^+, f_2^*)$ be elements of $mIF(X)$. Then for each $x \in X$ and $i \in \{1, \ldots, m\}$,*

(1) *$f_1 \sqsubseteq f_2$ if $(\pi_i \circ f_1^+)(x) \leq (\pi_i \circ f_2^+)(x)$ and $(\pi_i \circ f_1^*)(x) \geq (\pi_i \circ f_2^*)(x)$.*
(2) *$f_1 \sqcap f_2 = (f_1^+ \cap f_2^+, f_1^* \cup f_2^*)$ is an element of $mIF(X)$ and is defined by*

$$((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(x) := \min\{(\pi_i \circ f_1^+)(x), (\pi_i \circ f_2^+)(x)\},$$

and

$$((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(x) := \max\{(\pi_i \circ f_1^*)(x), (\pi_i \circ f_2^*)(x)\}.$$

It is easy to see that the structure $\langle mIF(X); \sqsubseteq \rangle$ is a partially ordered set. Let $\mathbf{S}$ be an ordered semigroup and $a \in S$. We set

$$\mathbf{S}_a := \{(x, y) \in S \times S : a \leq xy \text{ for some } x, y \in S\}.$$

We now define more operation on $mIF(S)$ as follows.

Definition 2.7. *Let $f_1 = (f_1^+, f_1^*)$ and $f_2 = (f_2^+, f_2^*)$ be elements of $mIF(S)$. The product of f_1 and f_2 is an element of $mIF(S)$, denoted by $f_1 \diamond f_2 := (f_1^+ \diamond f_2^+, f_1^* \diamond f_2^*)$ and is defined as follows. For each $a \in S$ and $i \in \{1, \ldots, m\}$,*

$$\Big((\pi_i \circ f_1^+) \diamond (\pi_i \circ f_2^+)\Big)(a) := \begin{cases} \bigvee\limits_{(x,y) \in \mathbf{S}_a} \{\min\{(\pi_i \circ f_1^+)(x), (\pi_i \circ f_2^+)(y)\}\} & \text{if } \mathbf{S}_a \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\left((\pi_i \circ f_1^*) \diamond (\pi_i \circ f_2^*)\right)(a) := \begin{cases} \bigwedge\limits_{(x,y)\in \mathbf{S}_a} \{\max\{(\pi_i \circ f_1^*)(x), (\pi_i \circ f_2^*)(y)\}\} & \text{if } \mathbf{S}_a \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Let f, g and h be elements of $mIF(S)$. If $f \sqsubseteq g$, then by the definitions of the relation $\sqsubseteq$ and the operation $\diamond$, it is easy to see that

$$f \diamond h \sqsubseteq g \diamond h \text{ and } h \diamond f \sqsubseteq h \diamond g.$$

The structure $\langle mIF(S); \diamond, \sqsubseteq\rangle$ is an ordered semigroup and its called an *multi-intuitionistic fuzzy ordered semigroup*.

3. Main Results

In this main section, we introduce the concepts of multi-intuitionistic fuzzy subsemigroups, multi-intuitionistic fuzzy left (right, two-sided, interior) ideals and study algebraic properties of multi-intuitionistic fuzzy subsemigroups and such their multi-intuitionistic fuzzy ideals. The relationships among their multi-intuitionistic fuzzy ideals are investigated. In regular, intra-regular, and semisimple ordered semigroups, multi-intuitionistic fuzzy interior ideals and multi-intuitionistic fuzzy ideals coincide are proved. Finally of this main section, the new multi-intuitionistic fuzzy sets are considered.

Definition 3.1. *Let* $\mathbf{S}$ *be an ordered semigroup. An multi-intuitionistic fuzzy set* $f = (f^+, f^*)$ *of* S *is called an multi-intuitionistic fuzzy subsemigroup of* $\mathbf{S}$ *if the following conditions are satisfied. For each* $x, y \in S$ *and* $i \in \{1, \ldots, m\}$,

(1) $(\pi_i \circ f^+)(xy) \geq \min\{(\pi_i \circ f^+)(x), (\pi_i \circ f^+)(y)\}$,
(2) $(\pi_i \circ f^*)(xy) \leq \max\{(\pi_i \circ f^*)(x), (\pi_i \circ f^*)(y)\}$.

Example 3.1. *Let* $S = \{a, b, c, d, e, k\}$ *with the following operation "$*$" and the order "$\leq$" on* S *be defined as follows:*

$*$	a	b	c	d	e	k
a	a	a	a	d	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
k	a	b	c	d	e	k

and

$$\leq := \{(k, e)\} \cup \Delta_S,$$

where Δ_S is the identity relation on S. It is easy to verify that $\langle S; *, \preceq\rangle$ is an ordered semigroup. We define 3-intuitionistic fuzzy set $f = (f^+, f^*)$ as follows.

S	$f^+(x)$	$f^*(x)$
a	$(0.4, 0.5, 0.6)$	$(0.3, 0.4, 0.2)$
b	$(0.4, 0.5, 0.6)$	$(0.3, 0.4, 0.2)$
c	$(0.4, 0.5, 0.6)$	$(0.3, 0.4, 0.2)$
d	$(0.6, 0.3, 0.1)$	$(0.1, 0.6, 0.6)$
e	$(0.6, 0.3, 0.1)$	$(0.1, 0.6, 0.6)$
k	$(0.6, 0.3, 0.1)$	$(0.1, 0.6, 0.6)$

It is easy to see that f is a 3-intuitionistic fuzzy subsemigroup of $\langle S; *, \preceq\rangle$.

Definition 3.2. *Let* **S** *be an ordered semigroup. An multi-intuitionistic fuzzy set $f = (f^+, f^*)$ of S is called an multi-intuitionistic fuzzy left ideal of* **S** *if the following conditions are satisfied. For each $x, y \in S$ and $i \in \{1, \ldots, m\}$,*

(1) $(\pi_i \circ f^+)(xy) \geq (\pi_i \circ f^+)(y)$ *and* $(\pi_i \circ f^*)(xy) \leq (\pi_i \circ f^*)(y)$.
(2) *If $x \preceq y$, then* $(\pi_i \circ f^+)(x) \geq (\pi_i \circ f^+)(y)$ *and* $(\pi_i \circ f^*)(x) \leq (\pi_i \circ f^*)(y)$.

Example 3.2. *Let $S = \{k, l, m, n\}$ with the following operation "$*$" and the order "$\preceq$" on S be defined as follows:*

$*$	k	l	m	n
k	k	k	k	k
l	k	k	k	k
m	k	k	l	l
n	k	k	l	k

and

$$\preceq := \{(k, n), (l, n), (m, n)\} \cup \Delta_S,$$

where Δ_S is the identity relation on S. It is easy to verify that $\langle S; *, \preceq\rangle$ is an ordered semigroup. We define 3-intuitionistic fuzzy set $f = (f^+, f^*)$ as follows.

S	$f^+(x)$	$f^*(x)$
k	$(0.4, 0.5, 0.7)$	$(0.2, 0.4, 0.2)$
l	$(0.3, 0.4, 0.6)$	$(0.3, 0.4, 0.4)$
m	$(0.2, 0.3, 0.1)$	$(0.5, 0.5, 0.6)$
n	$(0.2, 0.3, 0.1)$	$(0.5, 0.6, 0.7)$

It is easy to see that f is a 3-intuitionistic fuzzy left ideal of $\langle S; *, \preceq\rangle$.

Definition 3.3. *Let* **S** *be an ordered semigroup. An multi-intuitionistic fuzzy set $f = (f^+, f^*)$ of S is called an multi-intuitionistic fuzzy right ideal of* **S** *if the following conditions are satisfied. For each $x, y \in S$ and $i \in \{1, \ldots, m\}$,*

(1) $(\pi_i \circ f^+)(xy) \geq (\pi_i \circ f^+)(x)$ *and* $(\pi_i \circ f^*)(xy) \leq (\pi_i \circ f^*)(x)$.

(2) *If* $x \leq y$, *then* $(\pi_i \circ f^+)(x) \geq (\pi_i \circ f^+)(y)$ *and* $(\pi_i \circ f^*)(x) \leq (\pi_i \circ f^*)(y)$.

Example 3.3. *Consider the ordered semigroup* $\langle S; *, \preceq \rangle$ *given in Example 3.2. We define the* 3*-intuitionistic fuzzy set* $f = (f^+, f^*)$ *as follows.*

S	$f^+(x)$	$f^*(x)$
k	$(0.40, 0.55, 0.80)$	$(0.09, 0.18, 0.15)$
l	$(0.29, 0.45, 0.75)$	$(0.29, 0.28, 0.25)$
m	$(0.19, 0.42, 0.78)$	$(0.39, 0.35, 0.22)$
n	$(0.19, 0.42, 0.78)$	$(0.45, 0.45, 0.22)$

It is easy to see that f *is a* 3*-intutionistic fuzzy right ideal of* $\langle S; *, \preceq \rangle$. *But it is not a* 3*-intuitionistion fuzzy left ideal of* $\langle S; *, \preceq \rangle$. *Since* $(\pi_3 \circ f^+)(m * n) = (\pi_3 \circ f^+)(l) = 0.75 \ngeq 0.78 = (\pi_3 \circ f^+)(n)$.

An multi-intuitionistic fuzzy set f of S is called an *multi-intuitionistic fuzzy two-sided ideal of* **S** (or *multi-intuitionistic fuzzy ideal of* **S**) if f is both an multi-intuitionistic fuzzy left and an multi-intuitionistic fuzzy right ideal of **S**.

Example 3.4. *Consider the ordered semigroup* $\langle S; *, \preceq \rangle$ *given in Example 3.2. We define the* 4*-intuitionistic fuzzy set* $f = (f^+, f^*)$ *as follows.*

S	$f^+(x)$	$f^*(x)$
k	$(0.65, 0.55, 0.85, 0.95)$	$(0.35, 0.45, 0.15, 0.05)$
l	$(0.35, 0.45, 0.75, 0.80)$	$(0.35, 0.45, 0.25, 0.05)$
m	$(0.35, 0.45, 0.75, 0.80)$	$(0.35, 0.45, 0.25, 0.05)$
n	$(0.35, 0.45, 0.75, 0.80)$	$(0.65, 0.55, 0.25, 0.05)$

It is easy to see that f *is a* 4*-intuitionistic fuzzy ideal of* $\langle S; *, \preceq \rangle$.

Definition 3.4. *Let* **S** *be an ordered semigroup. An multi-intuitionistic fuzzy subsemigroup* $f = (f^+, f^*)$ *of* **S** *is called an multi-intuitionistic interior ideal of* **S** *if the following conditions are satisfied. For each* $x, y, z \in S$ *and* $i \in \{1, \ldots, m\}$,

(1) $(\pi_i \circ f^+)(xyz) \geq (\pi_i \circ f^+)(y)$ *and* $(\pi_i \circ f^*)(xyz) \leq (\pi_i \circ f^*)(y)$.
(2) *If* $x \leq y$, *then* $(\pi_i \circ f^+)(x) \geq (\pi_i \circ f^+)(y)$ *and* $(\pi_i \circ f^*)(x) \leq (\pi_i \circ f^*)(y)$.

Example 3.5. *Let* $S = \{a, b, c, d\}$ *with the following operation "*$*$*" and the order "*$\preceq$*" on* S *be defined as follows:*

$*$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and

$$\preceq := \{(a, b), (a, d)\} \cup \Delta_S,$$

where Δ_S is the identity relation on S. It is easy to verify that $\langle S; *, \leq\rangle$ is an ordered semigroup. We define 3-intuitionistic fuzzy set $f = (f^+, f^*)$ as follows.

S	$f^+(x)$	$f^*(x)$
a	$(0.8, 0.5, 0.7)$	$(0.2, 0.4, 0.2)$
b	$(0.7, 0.4, 0.6)$	$(0.3, 0.4, 0.4)$
c	$(0.5, 0.3, 0.1)$	$(0.5, 0.5, 0.6)$
d	$(0.3, 0.3, 0.1)$	$(0.5, 0.5, 0.6)$

It is easy to verify that f is a 3-intutionistic fuzzy interior ideal of $\langle S; *, \leq\rangle$.

Next propositions, we study the algebraic properties of multi-intuitionistic fuzzy subsemigroups and their multi-intuitionistic fuzzy ideals as follows.

Proposition 3.1. *Let $\mathbf{S}$ be an ordered semigroup and let $f_1,\ f_2$ be multi-intuitionistic fuzzy subsemigroups of $\mathbf{S}$. Then $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy subsemigroup of $\mathbf{S}$.*

Proof. Let $f_1 = (f_1^+, f_1^*), f_2 = (f_2^+, f_2^*)$ be multi-intuitionistic fuzzy subsemigroups of $\mathbf{S}$ and $x, y \in S$. Let us consider as follows. For each $i \in \{1, \ldots, m\}$,

$$\begin{aligned}
\Big((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+)\Big)(xy) &= \min\{(\pi_i \circ f_1^+)(xy), (\pi_i \circ f_2^+)(xy)\} \\
&\geq \min\{\min\{(\pi_i \circ f_1^+)(x), (\pi_i \circ f_1^+)(y)\}, \min\{(\pi_i \circ f_2^+)(x), (\pi_i \circ f_2^+)(y)\}\} \\
&= \min\{\min\{(\pi_i \circ f_1^+)(x), (\pi_i \circ f_2^+)(x)\}, \min\{(\pi_i \circ f_1^+)(y), (\pi_i \circ f_2^+)(y)\}\} \\
&= \min\{((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(x), ((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\Big((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*)\Big)(xy) &= \max\{(\pi_i \circ f_1^*)(xy), (\pi_i \circ f_2^*)(xy)\} \\
&\leq \max\{\max\{(\pi_i \circ f_1^*)(x), (\pi_i \circ f_1^*)(y)\}, \max\{(\pi_i \circ f_2^*)(x), (\pi_i \circ f_2^*)(y)\}\} \\
&= \max\{\max\{(\pi_i \circ f_1^*)(x), (\pi_i \circ f_2^*)(x)\}, \max\{(\pi_i \circ f_1^*)(y), (\pi_i \circ f_2^*)(y)\}\} \\
&= \max\{((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(x), ((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(y)\}.
\end{aligned}$$

Therefore $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy subsemigroup of $\mathbf{S}$. □

Proposition 3.2. *Let $\mathbf{S}$ be an ordered semigroup and let $f_1,\ f_2$ be multi-intuitionistic fuzzy left ideals of $\mathbf{S}$. Then $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy left ideal of $\mathbf{S}$.*

Proof. Let $f_1 = (f_1^+, f_1^*), f_2 = (f_2^+, f_2^*)$ be multi-intuitionistic fuzzy left ideals of $\mathbf{S}$ and $x, y \in S$. Let us consider as follows. For each $i \in \{1, \ldots, m\}$,

$$\begin{aligned}
\Big((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+)\Big)(xy) &= \min\{(\pi_i \circ f_1^+)(xy), (\pi_i \circ f_2^+)(xy)\} \\
&\geq \min\{(\pi_i \circ f_1^+)(y), (\pi_i \circ f_2^+)(y)\} \\
&= ((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(y),
\end{aligned}$$

and

$$\begin{aligned}\big((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*)\big)(xy) &= \max\{(\pi_i \circ f_1^*)(xy), (\pi_i \circ f_2^*)(xy)\} \\ &\leq \max\{(\pi_i \circ f_1^*)(y), (\pi_i \circ f_2^*)(y)\} \\ &= ((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(y).\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then, we obtain

$$\begin{aligned}\big((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+)\big)(x) &= \min\{(\pi_i \circ f_1^+)(x), (\pi_i \circ f_2^+)(x)\} \\ &\geq \min\{(\pi_i \circ f_1^+)(y), (\pi_i \circ f_2^+)(y)\} \\ &= ((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(y),\end{aligned}$$

and

$$\begin{aligned}\big((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*)\big)(x) &= \max\{(\pi_i \circ f_1^*)(x), (\pi_i \circ f_2^*)(x)\} \\ &\leq \max\{(\pi_i \circ f_1^*)(y), (\pi_i \circ f_2^*)(y)\} \\ &= ((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(y).\end{aligned}$$

Therefore $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy left ideal of $\mathbf{S}$. □

Similar to Proposition 3.2, we have the following proposition.

Proposition 3.3. *Let $\mathbf{S}$ be an ordered semigroup and let f_1, f_2 be multi-intuitionistic fuzzy right ideals of $\mathbf{S}$. Then $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy right ideal of $\mathbf{S}$.*

Combining Proposition 3.2 and Proposition 3.3, we obtain the following corollary.

Corollary 3.1. *Let $\mathbf{S}$ be an ordered semigroup and let f_1, f_2 be multi-intuitionistic fuzzy ideals of $\mathbf{S}$. Then $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy ideal of $\mathbf{S}$.*

Proposition 3.4. *Let $\mathbf{S}$ be an ordered semigroup and let f_1, f_2 be multi-intuitionistic fuzzy interior ideals of $\mathbf{S}$. Then $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy interior ideal of $\mathbf{S}$.*

Proof. Let $f_1 = (f_1^+, f_1^*)$ and $f_2 = (f_2^+, f_2^*)$ be multi-intuitionistic fuzzy interior ideals of $\mathbf{S}$. First, we shows that $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy subsemigroup of $\mathbf{S}$. Let $x, y \in S$ and $i \in \{1, \ldots, m\}$. Then, let us consider as follows.

$$\begin{aligned}&((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(xy) = \min\{(\pi_i \circ f_1^+)(xy), (\pi_i \circ f_2^+)(xy)\} \\ \geq\ & \min\{\min\{(\pi_i \circ f_1^+)(x), (\pi_i \circ f_1^+)(y)\}, \min\{(\pi_i \circ f_2^+)(x), (\pi_i \circ f_2^+)(y)\}\} \\ =\ & \min\{\min\{(\pi_i \circ f_1^+)(x), (\pi_i \circ f_2^+)(x)\}, \min\{(\pi_i \circ f_1^+)(y), (\pi_i \circ f_2^+)(y)\}\} \\ =\ & \min\{((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(x), ((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(y)\},\end{aligned}$$

and

$$\begin{aligned}((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(xy) &= \max\{(\pi_i \circ f_1^*)(xy), (\pi_i \circ f_2^*)(xy)\} \\ &\leq \max\{\max\{(\pi_i \circ f_1^*)(x), (\pi_i \circ f_1^*)(y)\}, \max\{(\pi_i \circ f_2^*)(x), (\pi_i \circ f_2^*)(y)\}\} \\ &= \max\{\max\{(\pi_i \circ f_1^*)(x), (\pi_i \circ f_2^*)(x)\}, \max\{(\pi_i \circ f_1^*)(y), (\pi_i \circ f_2^*)(y)\}\} \\ &= \max\{((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(x), ((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(y)\}.\end{aligned}$$

This complete to prove that $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy subsemigroup of $\mathbf{S}$.
Let $x, y, z \in S$ and $i \in \{1, \ldots, m\}$. Then, let us consider as follows.

$$\begin{aligned}((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(xyz) &= \min\{(\pi_i \circ f_1^+)(xyz), (\pi_i \circ f_2^+)(xyz)\} \\ &\geq \min\{(\pi_i \circ f_1^+)(y), (\pi_i \circ f_2^+)(y)\} \\ &= ((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(y),\end{aligned}$$

and

$$\begin{aligned}((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(xyz) &= \max\{(\pi_i \circ f_1^*)(xyz), (\pi_i \circ f_2^*)(xyz)\} \\ &\leq \max\{(\pi_i \circ f_1^*)(y), (\pi_i \circ f_2^*)(y)\} \\ &= ((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(y).\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then, we obtain

$$\begin{aligned}((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(x) &= \min\{(\pi_i \circ f_1^+)(x), (\pi_i \circ f_2^+)(x)\} \\ &\geq \min\{(\pi_i \circ f_1^+)(y), (\pi_i \circ f_2^+)(y)\} \\ &= ((\pi_i \circ f_1^+) \cap (\pi_i \circ f_2^+))(y),\end{aligned}$$

and

$$\begin{aligned}((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(x) &= \max\{(\pi_i \circ f_1^*)(x), (\pi_i \circ f_2^*)(x)\} \\ &\leq \max\{(\pi_i \circ f_1^*)(y), (\pi_i \circ f_2^*)(y)\} \\ &= ((\pi_i \circ f_1^*) \cup (\pi_i \circ f_2^*))(y).\end{aligned}$$

Therefore $f_1 \sqcap f_2$ is an multi-intuitionistic fuzzy interior ideal of $\mathbf{S}$. □

Remark 3.1. *Every multi-intuitionistic fuzzy ideal of* $\mathbf{S}$ *is an multi-intuitionistic fuzzy subsemigroup of* $\mathbf{S}$.

We, now study the relationships between multi-intuitionistic fuzzy ideals and multi-intuitionistic fuzzy interior ideals.

Lemma 3.1. *Let* $\mathbf{S}$ *be an ordered semigroup and let* f *be an multi-intuitionistic fuzzy ideal of* $\mathbf{S}$. *Then* f *is an multi-intuitionistic fuzzy interior ideal of* $\mathbf{S}$.

Proof. Let $f = (f^+, f^*)$ be an multi-intuitionistic fuzzy ideal of $\mathbf{S}$. By Remark 3.1, we obtain f is an multi-intuitionistic fuzzy subsemigroup of $\mathbf{S}$. Let $x, y, z \in S$ and for each $i \in \{1, \dots, m\}$. Then we obtain

$$\begin{aligned}(\pi_i \circ f^+)(xyz) &= (\pi_i \circ f^+)(x(yz)) \\ &\geq (\pi_i \circ f^+)(yz) \\ &\geq (\pi_i \circ f^+)(y),\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ f^*)(xyz) &= (\pi_i \circ f^*)(x(yz)) \\ &\leq (\pi_i \circ f^*)(yz) \\ &\leq (\pi_i \circ f^*)(y).\end{aligned}$$

Therefore f is an multi-intuitionistic fuzzy interior ideal of $\mathbf{S}$. □

The converse of Lemma 3.1 may not be true in general, as can be seen by the following example.

Example 3.6. *Let $S = \{a, b, c, d\}$ with the following operation "$*$" and the order "$\leq$" on S be defined as follows:*

$*$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	b
d	a	a	b	b

and

$$\leq := \{(a,b), (a,c), (b,d), (c,d)\} \cup \Delta_S$$

where Δ_S is the identity relation on S. It is easy to verify that $\langle S; *, \leq \rangle$ is an ordered semigroup. We define 3-intuitionistic fuzzy set $f = (f^+, f^*)$ as follows.

S	$f^+(x)$	$f^*(x)$
a	$(1, 0.8, 0.90)$	$(0, 0.1, 0.05)$
b	$(0, 0.5, 0.48)$	$(1, 0.95, 0.52)$
c	$(0.8, 0.65, 0.57)$	$(0.1, 0.25, 0.35)$
d	$(0.5, 0.62, 0.45)$	$(0.4, 0.30, 0.48)$

Then f is a 3-intuitionistic fuzzy interior ideal of $\langle S; *, \leq \rangle$ but f is not 3-intuitionistic fuzzy ideal of $\langle S; *, \leq \rangle$ because of $(\pi_1 \circ f^+)(c * d) = (\pi_1 \circ f^+)(b) = 0 \ngeq 0.5 = (\pi_1 \circ f^+)(d)$.

Corollary 3.2, Corollary 3.3, and Corollary 3.4 show that the concepts of multi-intuitionistic fuzzy ideals and multi-intuitionistic fuzzy interior ideals are the same thing in certain classes of ordered semigroups: regular, intra-regular, and semisimple.

An ordered semigroup $\mathbf{S}$ is *regular* if for each element $a \in S$ there exists element $x \in S$ such that $a \leq axa$. Firstly, we illustrate the converse of Lemma 3.1 for regular ordered semigroups.

Theorem 3.1. *Let* $\mathbf{S}$ *be a regular ordered semigroup. Then every multi-intuitionistic fuzzy interior ideal of* $\mathbf{S}$ *is an multi-intuitionistic fuzzy ideal of* $\mathbf{S}$*.*

Proof. Let $f = (f^+, f^*)$ be an multi-intuitionistic fuzzy interior ideal of $\mathbf{S}$ and $a, b \in S$. Since $\mathbf{S}$ is regular, there exists $x \in S$ such that $a \leq axa$. Then $ab \leq (axa)b = (ax)ab$ and, since f is multi-intuitionistic fuzzy interior ideal, for each $i \in \{1, \ldots, m\}$, we obtain

$$(\pi_i \circ f^+)(ab) \geq (\pi_i \circ f^+)(axab) = (\pi_i \circ f^+)((ax)ab) \geq (\pi_i \circ f^+)(a),$$

and

$$(\pi_i \circ f^*)(ab) \leq (\pi_i \circ f^*)(axab) = (\pi_i \circ f^*)((ax)ab) \leq (\pi_i \circ f^*)(a).$$

Thus, f is an multi-intuitionistic fuzzy right ideal of $\mathbf{S}$. Similarly, we can show that f is an multi-intuitionistic fuzzy left ideal of $\mathbf{S}$. It is complete to prove that f is an multi-intuitionistic fuzzy ideal of $\mathbf{S}$. □

Combining Lemma 3.1 and Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let* $\mathbf{S}$ *be a regular ordered semigroup. Then the following conditions are equivalent.*

(1) f *is an multi-intuitionistic fuzzy ideal of* $\mathbf{S}$*.*
(2) f *is an multi-intuitionistic fuzzy interior ideal of* $\mathbf{S}$*.*

An ordered semigroup $\mathbf{S}$ is called *intra-regular* if for each element $a \in S$ there exist elements $x, y \in S$ such that $a \leq xa^2y$. For intra-regular ordered semigroups, the other direction of Lemma 3.1 is provided below.

Theorem 3.2. *Let* $\mathbf{S}$ *be an intra-regular ordered semigroup. Then every an multi-intuitionistic fuzzy interior ideal of* $\mathbf{S}$ *is an multi-intuitionistic fuzzy ideal of* $\mathbf{S}$*.*

Proof. Let $f = (f^+, f^*)$ be an multi-intuitionistic fuzzy interior ideal of $\mathbf{S}$ and $a, b \in S$. Since $\mathbf{S}$ is an intra-regular ordered semigroup, there exist $x, y \in S$ such that $a \leq xa^2y$. Then $ab \leq xa^2yb = (xa)a(yb)$ and, since f is multi-intuitionistic fuzzy interior ideal, for each $i \in \{1, \ldots, m\}$, we have

$$(\pi_i \circ f^+)(ab) \geq (\pi_i \circ f^+)(xa^2yb) = (\pi_i \circ f^+)((xa)a(yb)) \geq (\pi_i \circ f^+)(a),$$

and

$$(\pi_i \circ f^*)(ab) \leq (\pi_i \circ f^*)(xa^2yb) = (\pi_i \circ f^*)((xa)a(yb)) \leq (\pi_i \circ f^*)(a).$$

Thus, f is an multi-intuitionistic fuzzy right ideal of $\mathbf{S}$. Similarly, we can prove f is also an multi-intuitionistic fuzzy left ideal of $\mathbf{S}$. Therefore, f is an multi-intuitionistic fuzzy ideal of $\mathbf{S}$. □

We obtain the coincidence of multi-intuitionistic fuzzy ideals and multi-intuitionistic fuzzy interior ideals by combining Lemma 3.1 and Theorem 3.2 as the following corollary.

Corollary 3.3. *Let* $\mathbf{S}$ *be an intra-regular ordered semigroup. Then the following conditions are equivalent.*

(1) *f is an multi-intuitionistic fuzzy ideal of* **S**.
(2) *f is an multi-intuitionistic fuzzy interior ideal of* **S**.

An ordered semigroup **S** is called *semisimple* if for each element $a \in S$ there exist elements $x, y, z \in S$ such that $a \preceq xayaz$. If an ordered semigroup is semisimple, the following theorem demonstrates the converse of Lemma 3.1.

Theorem 3.3. *Let* **S** *be a semisimple ordered semigroup. Then every multi-intuitionistic fuzzy interior ideal of* **S** *is an multi-intuitionistic fuzzy ideal of* **S**.

Proof. Let $f = (f^+, f^*)$ be an multi-intuitionistic fuzzy interior ideal of **S** and $a, b \in S$. Since **S** is a semisimple ordered semigroup, there exist $x, y, z \in S$ such that $a \preceq xayaz$. Then $ab \preceq xayazb = (xay)a(zb)$. Since f is an multi-intuitionistic fuzzy interior ideal of **S**, we have

$$(\pi_i \circ f^+)(ab) \geq (\pi_i \circ f^+)(xayazb) = (\pi_i \circ f^+)((xay)a(zb)) \geq (\pi_i \circ f^+)(a),$$

and

$$(\pi_i \circ f^*)(ab) \leq (\pi_i \circ f^*)(xayazb) = (\pi_i \circ f^*)((xay)a(zb)) \leq (\pi_i \circ f^*)(a),$$

for all $i \in \{1, \ldots, m\}$. Thus, f is an multi-intuitionistic fuzzy right ideal of **S**. Similarly, we can prove that f is an multi-intuitionistic fuzzy left ideal of **S**. Therefore, f is an multi-intuitionistic fuzzy ideal of **S**. □

Lemma 3.1 and Theorem 3.3 lead in the following corollary.

Corollary 3.4. *Let* **S** *be a semisimple ordered semigroup. Then the following conditions are equivalent.*

(1) *f is an multi-intuitionistic fuzzy ideal of* **S**.
(2) *f is an multi-intuitionistic fuzzy interior ideal of* **S**.

Let $\mathbf{L} := (L; \circ_L, \preceq_L)$ and $\mathbf{M} := (M; \circ_M, \preceq_M)$ be ordered semigroups. Let $\theta : L \to M$ be a mapping from a set L to a set M. For an multi-intuitionistic fuzzy set $g = (g^+, g^*)$ of M, consider an multi-intuitionistic fuzzy set $\theta^{-1}(g) = (\theta^{-1}(g^+), \theta^{-1}(g^*))$ of L and it define as follows.

$$(\pi_i \circ \theta^{-1}(g^+))(x) := (\pi_i \circ g^+)(\theta(x)) \text{ and } (\pi_i \circ \theta^{-1}(g^*))(x) := (\pi_i \circ g^*)(\theta(x))$$

for every $x \in L$ and $i \in \{1, \ldots, m\}$. We say that $\theta^{-1}(g)$ is the preimage multi-intuitionistic fuzzy set of g under θ. The mapping $\theta : L \to M$ is called a *homomorphism* of ordered semigroup, if its satisfied the following conditions: For $a, b \in L$,

(1) $\theta(a \circ_L b) = \theta(a) \circ_M \theta(b)$,
(2) if $a \preceq_L b$, then $\theta(a) \preceq_M \theta(b)$.

Theorem 3.4. *Let* **L**, **M** *be ordered semigroups and let $\theta : L \to M$ be a homomorphism of ordered semigroup. If g is an multi-intuitionistic fuzzy left ideal of* **M**, *then $\theta^{-1}(g)$ is an multi-intuitionistic fuzzy left ideal of* **L**.

Proof. Let $g = (g^+, g^*)$ be an multi-intuitionistic fuzzy left ideal of $\mathbf{M}$ and $a, b \in L$. Then, let us consider as follows. For each $i \in \{1, \ldots, m\}$,

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^+))(a \circ_L b) &= (\pi_i \circ g^+)(\theta(a \circ_L b)) \\ &= (\pi_i \circ g^+)(\theta(a) \circ_M \theta(b)) \\ &\geq (\pi_i \circ g^+)(\theta(b)) \\ &= (\pi_i \circ \theta^{-1}(g^+))(b),\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^*))(a \circ_L b) &= (\pi_i \circ g^*)(\theta(a \circ_L b)) \\ &= (\pi_i \circ g^*)(\theta(a) \circ_M \theta(b)) \\ &\leq (\pi_i \circ g^*)(\theta(b)) \\ &= (\pi_i \circ \theta^{-1}(g^*))(b).\end{aligned}$$

Let $a, b \in L$ be such that $a \preceq_L b$. Then $\theta(a) \preceq_M \theta(b)$ and for each $i \in \{1, \ldots, m\}$, we obtain

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^+))(a) &= (\pi_i \circ g^+)(\theta(a)) \\ &\geq (\pi_i \circ g^+)(\theta(b)) \\ &= (\pi_i \circ \theta^{-1}(g^+))(b),\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^*))(a) &= (\pi_i \circ g^*)(\theta(a)) \\ &\leq (\pi_i \circ g^*)(\theta(b)) \\ &= (\pi_i \circ \theta^{-1}(g^*))(b).\end{aligned}$$

This shows that $\theta^{-1}(g)$ is an multi-intuitionistic fuzzy left ideal of $\mathbf{L}$. □

Similar to Theorem 3.4, we obtain the following theorem.

Theorem 3.5. *Let* $\mathbf{L}$, $\mathbf{M}$ *be ordered semigroups and let* $\theta : L \to M$ *be a homomorphism of ordered semigroup. If* g *is an multi-intuitionistic fuzzy right ideal of* $\mathbf{M}$*, then* $\theta^{-1}(g)$ *is an multi-intuitionistic fuzzy right ideal of* $\mathbf{L}$.

Combining Theorem 3.4 and Theorem 3.5, we have the following result.

Corollary 3.5. *Let* $\mathbf{L}$, $\mathbf{M}$ *be ordered semigroups and let* $\theta : L \to M$ *be a homomorphism of ordered semigroup. If* g *is an multi-intuitionistic fuzzy ideal of* $\mathbf{M}$*, then* $\theta^{-1}(g)$ *is an multi-intuitionistic fuzzy ideal of* $\mathbf{L}$.

Theorem 3.6. *Let* $\mathbf{L}$, $\mathbf{M}$ *be ordered semigroups and let* $\theta : L \to M$ *be a homomorphism of ordered semigroup. If* g *is an multi-intuitionistic fuzzy interior ideal of* $\mathbf{M}$*, then* $\theta^{-1}(g)$ *is an multi-intuitionistic fuzzy interior ideal of* $\mathbf{L}$.

Proof. Let $g = (g^+, g^*)$ be an multi-intuitionistic fuzzy interior ideal of **M** and $a, b \in L$. Then, let us consider as follows. For each $i \in \{1, \ldots, m\}$,

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^+))(a \circ_L b) &= (\pi_i \circ g^+)(\theta(a \circ_L b)) \\ &= (\pi_i \circ g^+)(\theta(a) \circ_M \theta(b)) \\ &\geq \min\{(\pi_i \circ g^+)(\theta(a)), (\pi_i \circ g^+)(\theta(b))\} \\ &= \min\{(\pi_i \circ \theta^{-1}(g^+))(a), (\pi_i \circ \theta^{-1}(g^+))(b)\},\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^*))(a \circ_L b) &= (\pi_i \circ g^*)(\theta(a \circ_L b)) \\ &= (\pi_i \circ g^*)(\theta(a) \circ_M \theta(b)) \\ &\leq \max\{(\pi_i \circ g^*)(\theta(a)), (\pi_i \circ g^*)(\theta(b))\} \\ &= \max\{(\pi_i \circ \theta^{-1}(g^*))(a), (\pi_i \circ \theta^{-1}(g^*))(b)\}.\end{aligned}$$

It is complete to prove that $\theta^{-1}(g)$ is an multi-intuitionistic fuzzy subsemigroup of **L**.
Let $a, b, c \in L$. Then, let us consider as follows. For each $i \in \{1, \ldots, m\}$,

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^+))(a \circ_L b \circ_L c) &= (\pi_i \circ g^+)(\theta(a \circ_L b \circ_L c)) \\ &= (\pi_i \circ g^+)(\theta(a) \circ_M \theta(b) \circ_M \theta(c)) \\ &\geq (\pi_i \circ g^+)(\theta(b)) \\ &= (\pi_i \circ \theta^{-1}(g^+))(b),\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^*))(a \circ_L b \circ_L c) &= (\pi_i \circ g^*)(\theta(a \circ_L b \circ_L c)) \\ &= (\pi_i \circ g^*)(\theta(a) \circ_M \theta(b) \circ_M \theta(c)) \\ &\leq (\pi_i \circ g^*)(\theta(b)) \\ &= (\pi_i \circ \theta^{-1}(g^*))(b).\end{aligned}$$

Let $a, b \in L$ be such that $a \leq_L b$. Then $\theta(a) \leq_M \theta(b)$ and for each $i \in \{1, \ldots, m\}$, we obtain

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^+))(a) &= (\pi_i \circ g^+)(\theta(a)) \\ &\geq (\pi_i \circ g^+)(\theta(b)) \\ &= (\pi_i \circ \theta^{-1}(g^+))(b),\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \theta^{-1}(g^*))(a) &= (\pi_i \circ g^*)(\theta(a)) \\ &\leq (\pi_i \circ g^*)(\theta(b)) \\ &= (\pi_i \circ \theta^{-1}(g^*))(b).\end{aligned}$$

This shows that $\theta^{-1}(g)$ is an multi-intuitionistic fuzzy interior ideal of $\mathbf{L}$. $\square$

Let $\mathbf{L} := (L; \circ_L, \preceq_L)$ and $\mathbf{M} := (M; \circ_M, \preceq_M)$ be ordered semigroups. A mapping $\theta : L \to M$ is called *epimorphism* of ordered semigroup, if its satisfied the following conditions,

(1) θ is an onto mapping,
(2) θ is a homomorphism of ordered semigroup.

An epimorphism of ordered semigroup $\theta : L \to M$ is called *reverse isotone* if for any $x, y \in L$, we have $x \preceq_L y$ whenever $\theta(x) \preceq_M \theta(y)$.

Theorem 3.7. *Let* $\mathbf{L}$, $\mathbf{M}$ *be ordered semigroups and let* $\theta : L \to M$ *be a reverse isotone epimorphism of ordered semigroup. If* $\theta^{-1}(f)$ *is an multi-intuitionistic fuzzy left ideal of* $\mathbf{L}$*, then* f *is an multi-intuitionistic fuzzy left ideal of* $\mathbf{M}$.

Proof. Let $f = (f^+, f^*)$ be an multi-intuitionistic fuzzy set of M and let $\theta^{-1}(f)$ be an multi-intuitionistic fuzzy left ideal of $\mathbf{L}$ and $a, b \in M$. Since θ is onto, there exist $x, y \in L$ such that $\theta(x) = a$ and $\theta(y) = b$ and then, for each $i \in \{1, \ldots, m\}$, we obtain

$$\begin{aligned}
(\pi_i \circ f^+)(a \circ_M b) &= (\pi_i \circ f^+)(\theta(x) \circ_M \theta(y)) \\
&= (\pi_i \circ f^+)(\theta(x \circ_L y)) \\
&= (\pi_i \circ \theta^{-1}(f^+))(x \circ_L y) \\
&\geq (\pi_i \circ \theta^{-1}(f^+))(y) \\
&= (\pi_i \circ f^+)(\theta(y)) \\
&= (\pi_i \circ f^+)(b),
\end{aligned}$$

and

$$\begin{aligned}
(\pi_i \circ f^*)(a \circ_M b) &= (\pi_i \circ f^*)(\theta(x) \circ_M \theta(y)) \\
&= (\pi_i \circ f^*)(\theta(x \circ_L y)) \\
&= (\pi_i \circ \theta^{-1}(f^*))(x \circ_L y) \\
&\leq (\pi_i \circ \theta^{-1}(f^*))(y) \\
&= (\pi_i \circ f^*)(\theta(y)) \\
&= (\pi_i \circ f^*)(b).
\end{aligned}$$

Let $a, b \in M$ be such that $a \preceq_M b$. Since θ is onto, there exist $x, y \in L$ such that $\theta(x) = a$ and $\theta(y) = b$. Then, since θ is reverse isotone, so $x \preceq_L y$ and for each $i \in \{1, \ldots, m\}$, we obtain

$$\begin{aligned}
(\pi_i \circ f^+)(a) &= (\pi_i \circ f^+)(\theta(x)) \\
&= (\pi_i \circ \theta^{-1}(f^+))(x) \\
&\geq (\pi_i \circ \theta^{-1}(f^+))(y) \\
&= (\pi_i \circ f^+)(\theta(y))
\end{aligned}$$

$$= (\pi_i \circ f^+)(b),$$

and

$$\begin{aligned}
(\pi_i \circ f^*)(a) &= (\pi_i \circ f^*)(\theta(x)) \\
&= (\pi_i \circ \theta^{-1}(f^*))(x) \\
&\leq (\pi_i \circ \theta^{-1}(f^*))(y) \\
&= (\pi_i \circ f^*)(\theta(y)) \\
&= (\pi_i \circ f^*)(b).
\end{aligned}$$

It is complete to prove that f is an multi-intuitionistic fuzzy left ideal of $\mathbf{M}$. □

Similar to Theorem 3.7, we obtain the following theorem.

Theorem 3.8. *Let* $\mathbf{L}$, $\mathbf{M}$ *be ordered semigroups and let* $\theta : L \to M$ *be a reverse isotone epimorphism of ordered semigroup. If* $\theta^{-1}(f)$ *is an multi-intuitionistic fuzzy right ideal of* $\mathbf{L}$, *then* f *is an multi-intuitionistic fuzzy right ideal of* $\mathbf{M}$.

Combining Theorem 3.7 and Theorem 3.8, we have the following corollary.

Corollary 3.6. *Let* $\mathbf{L}$, $\mathbf{M}$ *be ordered semigroups and let* $\theta : L \to M$ *be a reverse isotone epimorphism of ordered semigroup. If* $\theta^{-1}(f)$ *is an multi-intuitionistic fuzzy ideal of* $\mathbf{L}$, *then* f *is an multi-intuitionistic fuzzy ideal of* $\mathbf{M}$.

Theorem 3.9. *Let* $\mathbf{L}$, $\mathbf{M}$ *be ordered semigroups and let* $\theta : L \to M$ *be a reverse isotone epimorphism of ordered semigroup. If* $\theta^{-1}(f)$ *is an multi-intuitionistic fuzzy interior ideal of* $\mathbf{L}$, *then* f *is an multi-intuitionistic fuzzy interior ideal of* $\mathbf{M}$.

Proof. Let $f = (f^+, f^*)$ be an multi-intuitionistic fuzzy set of M and let $\theta^{-1}(f)$ be an multi-intuitionistic fuzzy interior ideal of $\mathbf{L}$ and $a, b \in M$. Since θ is onto, there exist $x, y \in L$ such that $\theta(x) = a$ and $\theta(y) = b$ and then, for each $i \in \{1, \ldots, m\}$, we obtain

$$\begin{aligned}
(\pi_i \circ f^+)(a \circ_M b) &= (\pi_i \circ f^+)(\theta(x) \circ_M \theta(y)) \\
&= (\pi_i \circ f^+)(\theta(x \circ_L y)) \\
&= (\pi_i \circ \theta^{-1}(f^+))(x \circ_L y) \\
&\geq \min\{(\pi_i \circ \theta^{-1}(f^+))(x), (\pi_i \circ \theta^{-1}(f^+))(y)\} \\
&= \min\{(\pi_i \circ f^+)(\theta(x)), (\pi_i \circ f^+)(\theta(y))\} \\
&= \min\{(\pi_i \circ f^+)(a), (\pi_i \circ f^+)(b)\},
\end{aligned}$$

and

$$\begin{aligned}
(\pi_i \circ f^*)(a \circ_M b) &= (\pi_i \circ f^*)(\theta(x) \circ_M \theta(y)) \\
&= (\pi_i \circ f^*)(\theta(x \circ_L y))
\end{aligned}$$

$$
\begin{aligned}
&= (\pi_i \circ \theta^{-1}(f^*))(x \circ_L y) \\
&\leq \max\{(\pi_i \circ \theta^{-1}(f^*))(x), (\pi_i \circ \theta^{-1}(f^*))(y)\} \\
&= \max\{(\pi_i \circ f^*)(\theta(x)), (\pi_i \circ f^*)(\theta(y))\} \\
&= \max\{(\pi_i \circ f^*)(a), (\pi_i \circ f^*)(b)\}.
\end{aligned}
$$

This shows that f is an multi-intuitionistic fuzzy subsemigroup of $\mathbf{M}$.
Let $a, b, c \in M$. Then, since θ is onto, there exist $x, y, z \in L$ such that $a = \theta(x),\ b = \theta(y)$ and $c = \theta(z)$ and for each $i \in \{1, \ldots, m\}$, we have

$$
\begin{aligned}
(\pi_i \circ f^+)(a \circ_M b \circ_M c) &= (\pi_i \circ f^+)(\theta(x) \circ_M \theta(y) \circ_M \theta(z)) \\
&= (\pi_i \circ f^+)(\theta(x \circ_L y \circ_L z)) \\
&= (\pi_i \circ \theta^{-1}(f^+))(x \circ_L y \circ_L z) \\
&\geq (\pi_i \circ \theta^{-1}(f^+))(y) \\
&= (\pi_i \circ f^+)(\theta(y)) \\
&= (\pi_i \circ f^+)(b),
\end{aligned}
$$

and

$$
\begin{aligned}
(\pi_i \circ f^*)(a \circ_M b \circ_M c) &= (\pi_i \circ f^*)(\theta(x) \circ_M \theta(y) \circ_M \theta(z)) \\
&= (\pi_i \circ f^*)(\theta(x \circ_L y \circ_L z)) \\
&= (\pi_i \circ \theta^{-1}(f^*))(x \circ_L y \circ_L z) \\
&\leq (\pi_i \circ \theta^{-1}(f^*))(y) \\
&= (\pi_i \circ f^*)(\theta(y)) \\
&= (\pi_i \circ f^*)(b).
\end{aligned}
$$

Let $a, b \in M$ be such that $a \preceq_M b$. Since θ is onto, there exist $x, y \in L$ such that $\theta(x) = a$ and $b = \theta(y) = b$. Then, since θ is reverse isotone, so $x \preceq_L y$ and for each $i \in \{1, \ldots, m\}$, we obtain

$$
\begin{aligned}
(\pi_i \circ f^+)(a) &= (\pi_i \circ f^+)(\theta(x)) \\
&= (\pi_i \circ \theta^{-1}(f^+))(x) \\
&\geq (\pi_i \circ \theta^{-1}(f^+))(y) \\
&= (\pi_i \circ f^+)(\theta(y)) \\
&= (\pi_i \circ f^+)(b),
\end{aligned}
$$

and

$$
\begin{aligned}
(\pi_i \circ f^*)(a) &= (\pi_i \circ f^*)(\theta(x)) \\
&= (\pi_i \circ \theta^{-1}(f^*))(x) \\
&\leq (\pi_i \circ \theta^{-1}(f^*))(y)
\end{aligned}
$$

$$
\begin{aligned}
&= (\pi_i \circ f^*)(\theta(y)) \\
&= (\pi_i \circ f^*)(b).
\end{aligned}
$$

It is complete to prove that f is an multi-intuitionistic fuzzy interior ideal of $\mathbf{M}$. □

4. Conclusions

In this present paper, we considered on the structures of ordered semigroups. We introduced the notion of multi-intuitionistic fuzzy sets in ordered semigroups. The concepts of multi-intuitionistic fuzzy subsemigroups, multi-intuitionistic fuzzy left (right, two-sided, interior) ideals of an ordered semigroup are introduced and some algebraic properties of multi-intuitionistic fuzzy subsemigroups and such their multi-intuitionistic fuzzy ideals are studied. Moreover, the relationships among their multi-intuitionistic fuzzy ideals are investigated. In regular, intra-regular, and semisimple ordered semigroups, the concept of multi-intuitionistic fuzzy interior ideals and multi-intuitionistic fuzzy ideals coincide are proved. Finally, the new multi-intuitionistic fuzzy sets are considered. In our future work, we will use this concept to study in many type of structures, for example ordered hypersemigroups and ordered semirings etc.

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References

[1] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[2] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[3] I. Cristea, B. Davvaz, Atanassov's Intuitionistic Fuzzy Grade of Hypergroups, Inf. Sci. 180 (2010), 1506–1517. https://doi.org/10.1016/j.ins.2010.01.002.

[4] B. Davvaz, W. Dudek, Y. Jun, Intuitionistic Fuzzy H_v-Submodules, Inf. Sci. 176 (2006), 285–300. https://doi.org/10.1016/j.ins.2004.10.009.

[5] Y.B. Jun, K.H. Kim, Intuitionistic Fuzzy Ideals in BCK-Algebras, Int. J. Math. Math. Sci. 24 (2000), 839–849.

[6] S. Yamak, O. Kazanci, B. Davvaz, Divisible and Pure Intuitionistic Fuzzy Subgroups and Their Properties, Int. J. Fuzzy Syst. 10 (2008), 298–307.

[7] N. Kuroki, Fuzzy Bi-Ideals in Semigroups, Comment. Math. Univ. St. Pauli. 28 (1979), 17–21.

[8] N. Kehayopulu, M. Tsingelis, Fuzzy Sets in Ordered Groupoids, Semigroup Forum 65 (2002), 128–132. https://doi.org/10.1007/s002330010079.

[9] J. Chen, S. Li, S. Ma, X. Wang, m-Polar Fuzzy Sets: An Extension of Bipolar Fuzzy Sets, Sci. World J. 2014 (2014), 416530. https://doi.org/10.1155/2014/416530.

[10] O. Kazancı, S. Hoskova-Mayerova, B. Davvaz, Multipolar Fuzzy Hyperideals in Ordered Semihypergroups, Mathematics 10 (2022), 3424. https://doi.org/10.3390/math10193424.

[11] N. Kehayopulu, M. Tsingelis, Fuzzy Ideals in Ordered Semigroups, Quasigroups Relat. Syst. 15 (2007), 279-289.

[12] N. Kehayopulu, M. Tsingelis, Fuzzy Interior Ideals in Ordered Semigroups, Lobachevskii J. Math. 21 (2006), 65-71. http://eudml.org/doc/225385.