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On a Soft Quotient Structure over JU-Algebras

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Abstract. This article aims to investigate the soft quotient structure of JU-algebras constructed by intersection soft ideals of JU-algebras. We will introduce fundamental homomorphism structure applied to quotient over JU-algebras that are constructed using soft ideals. Additionally, we will provide a characterizations of positive implicative soft quotient JU-algebras.

1. Introduction

Algebraic studies revolve around classical and logical algebraic structures, with a myriad of crucial terminologies. Among these, ideals hold a pivotal role in understanding algebraic structures. Moreover, the exploration of soft quotients within algebraic structures has significant implications across various algebraic concepts. In 1999, Molodtsov [11] introduced soft sets, which were further extended by Acar et al. [1] in 2010 to encompass soft rings. Soft sets, a generalization of fuzzy sets, address uncertainty using a parameterized set family. Termed "soft" due to their boundary dependency on parameters, these structures serve as mathematical tools, akin to rough sets, in managing uncertainties inherent in information. Authors have extensively studied algebraic structures within set theory that tackle uncertainties. Exploring the soft and rough approximations is also used for several applications including decision-making problems. Numerous soft sets and related concepts and properties are available in works like [1, 4–6, 8]. Further explorations into intersection soft set theory and associated terminologies within algebraic structures are detailed in [9, 10, 14].

In [3], Moin et al. introduced the framework of JU-algebras along with their p-closure ideals. Subsequently, Usman et al. [2] introduced the concepts of pseudo-valuations and pseudo-metrics within JU-algebras. Daniel's work in [13] focused on elucidating concepts related to JU-filters,

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while [12] explored additional novel findings concerning JU-algebras. Ali. H. Hakami et.al. studied the graph structure constructed on the ideals of JU algebras [7].
This article introduces soft quotient JU-algebras using soft sets, induced by intersection soft ideals within a JU-algebra. We have explored a few fundamental theorems of quotient JU-algebras and provided proofs for some associated characterizations for positive implicative soft quotient JU-algebras.

2. Preliminaries

We will explore fundamental definition and concepts of JU-algebras, encompassing JU-subalgebras, JU-ideals, and soft sets in this section. We'll cover essential terminologies, offer examples, and present some associated results.

Definition 2.1. *[3] An algebra $(J, \diamond, 1)$ of type $(2,0)$ equipped with a given binary operation $\diamond$ is called a JU-algebra if the following conditions are satisfied for every elements $x, y, z \in J$,*
(J_1) $(y \diamond x) \diamond [(x \diamond z) \diamond (y \diamond z)] = 1$,
(J_2) $1 \diamond z = z$,
(J_3) $z \diamond y = y \diamond z = 1$ *implies* $z = y$.

We designate the element 1 as a fixed element associated to the JU algebra J. We use the symbol J for a JU algebra in simplified form instead of writing the complete structure $(J, \diamond, 1)$ to denote a JU-algebra. We establish a relation "≤" on J that is defined as $y \leq x \Leftrightarrow x \diamond y = 1$.

Lemma 2.1. *[3] For a JU-algebra J, the relation "≤" is a partial order on J that is, $(J, \leq)$ is a partial ordered.*
(J_4) $z \leq z$,
(J_5) *If* $z \leq y$ *and* $y \leq z$, *then* $z = y$,
(J_6) *If* $z \leq x$ *and* $x \leq y$, *then* $z \leq y$.

Example 2.1. *Take a set $J = \{1, j_1, j_2, j_3, j_4\}$ with a binary operation $\diamond$ on J is defined by the following rule given in the table*

$\diamond$	1	j_1	j_2	j_3	j_4
1	1	j_1	j_2	j_3	j_4
j_1	1	1	j_2	j_3	j_4
j_2	1	j_1	1	j_3	j_3
j_3	1	1	j_2	1	j_2
j_4	1	1	1	1	1

We can easily verify that the set J with given operation is a JU-algebra.

A few more basic properties of a JU-algebra is presented in the following result.

Lemma 2.2. *[3] A JU-algebra J also satisfy the following conditions for all $x, y, z \in J$.*
(J_7) $z \diamond z = 1$,

(J_8) $x \diamond (y \diamond z) = y \diamond (x \diamond z)$,
(J_9) *If* $(z \diamond y) \diamond y = 1$, *then* J *becomes a KU-algebra,*
(J_{10}) $(y \diamond z) \diamond 1 = (y \diamond 1) \diamond (z \diamond 1)$.

Definition 2.2. *[3] For a JU-algebra J, a non-empty subset K of is said to be a JU-subalgebra of the algebra J if* $a \diamond b \in K$, *for all* $a, b \in K$.

Definition 2.3. *[3] For a JU-algebra J, a non-empty subset S of J is said to be a JU-ideal of the JU-algebra J if the given axioms holds.*
(1) $1 \in S$,
(2) $z \diamond (y \diamond a) \in S, y \in S \implies z \diamond x \in S$, *for any* $a, b, c \in J$.

Example 2.2. *Consider a set* $J = \{1, j_1, j_2, j_3, j_4, j_5\}$ *with a binary operation* $\diamond$ *that is defined in the following table:*

$\diamond$	1	j_1	j_2	j_3	j_4	j_5
1	1	j_1	j_2	j_3	j_4	j_5
j_1	1	1	j_2	j_2	j_4	j_5
j_2	1	1	1	j_1	j_4	j_5
j_3	1	1	1	1	j_4	j_5
j_4	1	1	1	j_1	1	j_5
j_5	1	1	1	1	1	1

It is trivial to verify that $(J, \diamond, 1)$ *is a JU-algebra and the subsets* $K = \{1, j_1\}$ *and* $L = \{1, j_1, j_2, j_3, j_4\}$ *forms JU-ideals of the JU-algebra J.*

An *implicative* JU-algebra J is an algebra that satisfy the condition $a = (a \diamond b) \diamond a$ for all $a, b \in J$. Moreover, if $(c \diamond b) \diamond (c \diamond a) = c \diamond (b \diamond a)$ for all $a, b, c \in J$, then we call J a *positive implicative* JU-algebra.
An important characterization of a positive implicative algebra is given in the following sections.

Definition 2.4. [11] For a parameters set P and a fix universal set X if we define a function $\mathfrak{J} : P \to \mathcal{P}(X)$, then the pair $(\mathfrak{J}, P)$ is known as a *soft set* over the set X.
For an element $\flat \in P$, $\mathfrak{J}(\flat)$ is called the set $\flat$-approximate elements for the soft set $(\mathfrak{J}, P)$.

Amongst the several examples in different spaces or other structures, we have considered an example of a soft set in topological spaces as below.

Example 2.3. *If* (J, ℓ) *is a topological space with a collection of subsets* ℓ *of the set J considered as an open sets of J. Then, the set of open neighborhoods* $N(a)$ *of point a, where* $N(a) = \{W \in \ell | \ a \in W\}$, *can be taken as the soft set for* $(N(a), \ell)$.

Definition 2.5. [11] For a nonempty subset X of the set of parameters P. A soft set $(\mathfrak{J}, P)$ over the set J that satisfy: $\mathfrak{J}(a) = \emptyset$ for all $a \notin X$ is named as the X-soft set over the universal set J and it is denoted by $\mathfrak{J}_X$. An X-soft set $\mathfrak{J}_X$ over the set J is a function $\mathfrak{J}_X : P \to \mathcal{P}(J)$ with $\mathfrak{J}_X(a) = \emptyset$ for

every $a \notin X$.
Therefore, a soft set over the set J can also be considered as the set of ordered pairs

$$\mathfrak{I}_X = \{(a, \mathfrak{I}_X(a)) : \ a \in P, \mathfrak{I}_X(a) \in \mathcal{P}(J)\}.$$

3. Soft Quotient JU-Algebras Over Int-Soft Ideals

In this whole section we will consider J as a simple notation to represent a JU-algebra and the pair $(\mathfrak{I}, P)$ will be considered as a soft set, as defined in the above section.

Definition 3.1. For a JU-algebra J, a soft set $(\mathfrak{I}, J)$ over a universal set X is called an *intersection soft ideal* over the set J if the following condition holds.

$$\mathfrak{I}(a) \subseteq \mathfrak{I}(1) \quad \forall\, a \in J, \tag{3.1}$$

$$\mathfrak{I}(b \diamond a) \cap \mathfrak{I}(b) \subseteq \mathfrak{I}(a) \quad \forall\, a, b \in J. \tag{3.2}$$

For simplicity, we may write "int-soft" ideal instead of "intersection soft" ideal.

Lemma 3.1. *An int-soft ideal $(\mathfrak{I}, J)$ of the JU-algebra J carries the following properties.*

$$a \leq b \Rightarrow \mathfrak{I}(b) \subseteq \mathfrak{I}(a) \ \ \forall\, a, b \in J, \tag{3.3}$$

$$b \diamond a \leq c \Rightarrow \mathfrak{I}(b) \cap \mathfrak{I}(c) \subseteq \mathfrak{I}(a) \ \ \forall\, a, b, c \in J. \tag{3.4}$$

For an int-soft ideal $(\mathfrak{I}, P)$ over the set J and for any $a, b \in J$, we define a binary operation "$\Diamond$" on J as, $a \Diamond b \Leftrightarrow \mathfrak{I}(b \diamond a) = \mathfrak{I}(a \diamond b) = \mathfrak{I}(1)$ for any $a, b \in J$.

Lemma 3.2. *The defined relation $\Diamond$ on J is an equivalence relation.*

Proof. It is trivial to verify by the definition of the relation $\Diamond$ that it is a reflexive and a symmetric relation.
For transitive property, take any three elements $a, b, c \in J$ such that $a \Diamond b$, $b \Diamond c$, then

$$\mathfrak{I}(b \diamond a) = \mathfrak{I}(a \diamond b) = \mathfrak{I}(1) \text{ and } \mathfrak{I}(c \diamond b) = \mathfrak{I}(b \diamond c) = \mathfrak{I}(1).$$

As $(b \diamond a) \diamond (c \diamond a) \leq c \diamond b$ *and* $(b \diamond c) \diamond (a \diamond c) \leq a \diamond b$, by using the Equation (3.4) we get that

$$\mathfrak{I}(1) = \mathfrak{I}(b \diamond a) \cap \mathfrak{I}(c \diamond b) \subseteq \mathfrak{I}(c \diamond a) \subseteq \mathfrak{I}(1) \text{ and } \mathfrak{I}(1) = \mathfrak{I}(b \diamond c) \cap \mathfrak{I}(a \diamond b) \subseteq \mathfrak{I}(a \diamond c) \subseteq \mathfrak{I}(1).$$

Since $\mathfrak{I}(c \diamond a) = \mathfrak{I}(1) = \mathfrak{I}(a \diamond c)$, it follows that $a \Diamond c$. Hence the given relation $\Diamond$ forms an equivalence relation on the set J. □

Lemma 3.3. *Consider a JU-algebra J and for any elements $a, b, c \in J$, if $a \Diamond b$, then $c \diamond a \Diamond c \diamond b$ and $a \diamond c \Diamond b \diamond c$.*

Proof. For any three elements $a, b, c \in J$ such that $a \Diamond b$. Then $\mathfrak{I}(b \diamond a) = \mathfrak{I}(a \diamond b) = \mathfrak{I}(1)$.
As $(c \diamond b) \diamond (c \diamond a) \leq b \diamond a$ and $(c \diamond a) \diamond (c \diamond b) \leq a \diamond b$, now by using the Equation (3.3), we get that

$$\mathfrak{I}(b \diamond a) = \mathfrak{I}(1) \subseteq \mathfrak{I}((c \diamond b) \diamond (c \diamond a)) \subseteq J(1) \text{ and } J(1) = \mathfrak{I}(a \diamond b) \subseteq \mathfrak{I}((c \diamond a) \diamond (c \diamond b)) \subseteq \mathfrak{I}(1).$$

It infers that, $\mathfrak{I}((c \diamond b) \diamond (c \diamond a)) = \mathfrak{I}(1) = \mathfrak{I}((c \diamond a) \diamond (c \diamond b))$, and hence we have $c \diamond a \Diamond c \diamond b$.
In the similar way, one can easily see that $a \diamond c \Diamond b \diamond c$. □

By using the Lemma 3.2 and the Lemma 3.3, we obtained the following result.

Lemma 3.4. *If $a\Diamond b$ and $m\Diamond n$ for any elements $a,b,m,n \in J$, then $m \diamond a \Diamond n \diamond b$.*

If we compile all the above results given in the Summarizing the Lemma 3.2, Lemma 3.3 and Lemma 3.4, we conclude that the operation $\Diamond$ is a in fact an equivalence relation on J with equivalence classes

$$\mathfrak{J}_a := \{b \in J | b\Diamond a\}.$$

The set $J/\mathfrak{J} := \{\mathfrak{J}_a | a \in J\}$ represents the collections of all equivalence classes of J.
Clearly, the given operation is well-defined by Lemma 3.4.

Theorem 3.1. *If $(\mathfrak{J}, J)$ is an intersection soft ideal of J, then the soft quotient algebra*

$$\mathcal{Q}/\mathfrak{J} := (J/\mathfrak{J}, \star, \mathfrak{J}_1)$$

is also a JU-algebra.

Proof. For any three elements $\mathfrak{J}_x, \mathfrak{J}_y, \mathfrak{J}_z \in J/\mathfrak{J}$. Then

$$(\mathfrak{J}_y \star \mathfrak{J}_x) \star ((\mathfrak{J}_x \star \mathfrak{J}_z) \star (\mathfrak{J}_y \star \mathfrak{J}_z)) = \mathfrak{J}_{(y\diamond x)\diamond((x\diamond z)\diamond(y\diamond z))} = \mathfrak{J}_1,$$

$$\mathfrak{J}_1 \star \mathfrak{J}_z = \mathfrak{J}_{1\diamond z} = \mathfrak{J}_1, \quad \forall z \in J,$$

and if $\mathfrak{J}_z \star \mathfrak{J}_y = \mathfrak{J}_1$ and $\mathfrak{J}_y \star \mathfrak{J}_z = \mathfrak{J}_1$, then $\mathfrak{J}_{z\diamond y} = \mathfrak{J}_1 = \mathfrak{J}_{y\diamond z}$. Implies,

$$\mathfrak{J}(y \diamond z) = \mathfrak{J}(z \diamond y) = \mathfrak{J}(1),$$

therefore,

$$z\Diamond y \Rightarrow \mathfrak{J}_z = \mathfrak{J}_y$$

. Hence $\mathcal{Q}/\mathfrak{J} := (J/\mathfrak{J}, \star, \mathfrak{J}_1)$ is a JU-algebra. □

Theorem 3.2. *Consider two soft quotient JU-algebras $\mathcal{Q} := (J, \diamond_J, 1_J)$ and $\mathcal{L} := (L, \diamond_L, 1_L)$ and there is an epimorphism map $\curlywedge : J \to L$. If the pair $(\bar{j}, L)$ is an int-soft ideal of $\mathcal{L}$, then the corresponding pair $(\bar{j} \diamond \curlywedge, J)$ is also an intersection soft ideal of $\mathcal{Q}$.*

Proof. For an element $a \in J$, we get,

$$(\bar{j} \diamond \curlywedge)(a) = \bar{j}(\curlywedge(a)) \subseteq \bar{j}(1_L) = \bar{j}(\curlywedge(1_J)) = (\bar{j} \diamond \curlywedge)(1_J)$$

and

$$\bar{j}(\curlywedge(a) \diamond_L b) \cap \bar{j}(b) \subseteq \bar{j}(\curlywedge(a)) = (\bar{j} \diamond \curlywedge)(a) \text{ for all } b \in L.$$

Consider x to be a pre-image of b associated with function $\curlywedge$. Then

$$\begin{aligned}\bar{j}(\curlywedge(a) \diamond_L b) \cap \bar{j}(b) &= \bar{j}(\curlywedge(a) \diamond_L \curlywedge(x)) \cap \bar{j}(\curlywedge(x)) \\ &= \bar{j}(\curlywedge(a \diamond_J x)) \cap \bar{j}(\curlywedge(x)) \\ &= (\bar{j} \diamond \curlywedge)(a \diamond_J x) \cap (\bar{j} \diamond \curlywedge)(x) \\ &\subseteq (\bar{j} \diamond \curlywedge)(a).\end{aligned}$$

Clearly, it implies that $(\bar{j} \diamond \leftthreetimes, J)$ is also an intersection soft ideal of J. □

The following result is an analogue to the classic fundamental theorem of homomorphism for soft quotient JU-algebras.

Theorem 3.3. *Consider two soft quotient JU-algebras $\mathcal{Q} := (J, \diamond_J, 1_J)$ and $\mathcal{L} := (L, \diamond_L, 1_L)$ and there is an epimorphism map $\leftthreetimes : J \to L$. If $(\bar{j}, L)$ defines an intersection soft ideal over the algebra L, then the corresponding soft quotient algebra*

$$\mathcal{Q}/(\bar{j} \diamond \leftthreetimes) := (J/(\bar{j} \diamond \leftthreetimes), \star_J, (\bar{j} \diamond \leftthreetimes)_{1_J})$$

is also isomorphic to the soft quotient

$$\mathcal{L}/\bar{j} := (L/\bar{j}, \star_L, \bar{j}_{1_L})$$

.

Proof. We know that, $\mathcal{Q}/(\bar{j} \diamond \leftthreetimes) := (J/(\bar{j} \diamond \leftthreetimes), \star_J, (\bar{j} \diamond \leftthreetimes)_{1_J})$ and $\mathcal{L}/\bar{j} := (L/\bar{j}, \star_Y, \bar{j}_{1_L})$ are JU-algebras by using the Theorem 3.1 and Theorem 3.2. Now we have a well define map

$$\aleph : J/(\bar{j} \diamond \leftthreetimes) \to L/\bar{j} \text{ such that } \aleph\left((\bar{j} \diamond \leftthreetimes)_x\right) = \bar{j}_{\leftthreetimes(x)} \text{ for every } x \in J.$$

As if $(\bar{j} \diamond \leftthreetimes)_x = (\bar{j} \diamond \leftthreetimes)_y$ for all $x, y \in J$ we have

$$\begin{aligned}
(\bar{j}(\leftthreetimes(y) \diamond_L \leftthreetimes(x))) &= \bar{j}(\leftthreetimes(y \diamond_J x)) \\
&= (\bar{j} \diamond \leftthreetimes)(y \diamond_J x) \\
&= (\bar{j} \diamond \leftthreetimes)(1_J) \\
&= \bar{j}(\leftthreetimes(1_J)) = \bar{j}(1_L)
\end{aligned}$$

and

$$\begin{aligned}
\bar{j}(\leftthreetimes(y) \diamond_L \leftthreetimes(x)) &= \bar{j}(\leftthreetimes(y \diamond_J x)) \\
&= (\bar{j} \diamond \leftthreetimes)(y \diamond_J x) \\
&= (\bar{j} \diamond \leftthreetimes)(1_J) \\
&= \bar{j}(\leftthreetimes(1_J)) \\
&= \bar{j}(1_L),
\end{aligned}$$

and hence

$$\bar{j}_{\leftthreetimes(x)} = \bar{j}_{\leftthreetimes(y)}.$$

Now for any two elements $(\bar{j} \diamond \leftthreetimes)_x, (\bar{j} \diamond \leftthreetimes)_y \in J/(\bar{j} \diamond \leftthreetimes)$, we see that

$$\begin{aligned}
\aleph(\bar{j} \diamond \leftthreetimes)_y \star_J (\bar{j} \diamond \leftthreetimes)_x &= \aleph(\bar{j} \diamond \leftthreetimes)_{y \diamond_J x} = \bar{j} \leftthreetimes (y \diamond_J x) \\
&= \bar{j}_{\leftthreetimes(y)} \diamond_L \leftthreetimes(x) = \bar{j}_{\leftthreetimes(y)} \star_L \bar{j}_{\leftthreetimes(x)} \\
&= \aleph((\bar{j} \diamond \leftthreetimes)_y) \star_L \aleph(\bar{j} \diamond \leftthreetimes)_x.
\end{aligned}$$

It concludes that the map $\boldsymbol{\aleph}$ is in fact a homomorphism. Since the map $\measuredangle$ is a surjective map, therefore, for any $\bar{j}_a \in L/\bar{j}$, there exists an element $x \in J$ such that $\measuredangle(x) = a$. It shows that $\boldsymbol{\aleph}((\bar{j} \diamond \measuredangle)_x) = \bar{j}_{\measuredangle(x)} = \bar{j}_a$, and implies that the map $\boldsymbol{\aleph}$ is also a surjective map.
In order to complete the proof, we only have to show that the map $\boldsymbol{\aleph}$ is also injective. For the purpose, consider any two elements $a, b \in J$ such that $\bar{j}_{\measuredangle(a)} = \bar{j}_{\measuredangle(b)}$. Then

$$\begin{aligned}(\bar{j} \diamond \measuredangle)(b \diamond_J a) &= \bar{j}(\measuredangle(b \diamond_J a)) \\ &= \bar{j}(\measuredangle(b) \diamond_L \measuredangle(a)) \\ &= \bar{j}(1_L) \\ &= \bar{j}(\measuredangle(1_J)) \\ &= (\bar{J} \diamond \measuredangle)(1_J)\end{aligned}$$

and

$$\begin{aligned}(\bar{j} \diamond \measuredangle)(b \diamond_J a) &= \bar{j}(\measuredangle(b \diamond_J a)) \\ &= \bar{j}(\measuredangle(b) \diamond_L \measuredangle(a)) \\ &= \bar{j}(1_L) \\ &= \bar{j}(\measuredangle(1_J)) \\ &= (\bar{j} \diamond \measuredangle)(1_J).\end{aligned}$$

Implies that $(\bar{j} \diamond \measuredangle)_a = (\bar{j} \diamond \measuredangle)_b$ and hence $\boldsymbol{\aleph}$ is injective. □

A canonical homomorphism $\sigma : J \to J/\mathfrak{I}$ such that $\sigma(x) = \mathfrak{I}_x$ is a natural homomorphism of J onto $J/\mathfrak{I}$. Similarly, if we consider a canonical homomorphisms

$$\sigma_J : J \to J/\bar{j} \diamond \measuredangle \text{ and } \sigma_L : L \to L/\bar{j}$$

in Theorem 3.3, then one can easily verify that

$$\boldsymbol{\aleph} \diamond \sigma_J = \sigma_L \diamond \measuredangle.$$

A characterization for a positive implicative soft quotient JU-algebra is given in the following results.

Theorem 3.4. *Consider an intersection soft ideal $(\mathfrak{I}, J)$ of a JU-algebra $\boldsymbol{Q}$. The corresponding soft quotient JU-algebra*

$$\boldsymbol{Q}/\mathfrak{I} := (J/\mathfrak{I}, \star, \mathfrak{I}_1)$$

is positive implicative if and only if it satisfy the following condition for all $a, b \in J$.

$$\mathfrak{I}(b \diamond (b \diamond a)) = \mathfrak{I}(b \diamond a). \tag{3.5}$$

Proof. Let $Q/\mathfrak{I} := (J/\mathfrak{I}, \star, \mathfrak{I}_1)$ be a given positive implicative JU-algebra and $a, b \in J$. Then $\mathfrak{I}_{b^{\diamond}a} = \mathfrak{I}_a \star \mathfrak{I}_b = (\mathfrak{I}_a \star \mathfrak{I}_b) \star \mathfrak{I}_b = \mathfrak{I}_{(a \diamond_b) \diamond_b}$, and therefore $\mathfrak{I}((b \diamond (b \diamond a)) \diamond (b \diamond a)) = \mathfrak{I}(1)$. As $(\mathfrak{I}, J)$ is in fact an int-soft ideal of the algebra Q, by considering relations (3.1) and (3.2), we obtain that

$$\begin{aligned}
&\mathfrak{I}((b \diamond (b \diamond a) \diamond (b \diamond a))) \cap \mathfrak{I}(b \diamond (b \diamond a)) \\
&= \mathfrak{I}(1) \cap \mathfrak{I}(b \diamond (b \diamond a)) \\
&= \mathfrak{I}(b \diamond (b \diamond a) \\
&\subseteq \mathfrak{I}(b \diamond a).
\end{aligned}$$

Now by Equation (3.3),

$$(b \diamond (b \diamond a)) \leq b \diamond a \Rightarrow \mathfrak{I}(b \diamond a) \subseteq \mathfrak{I}(b \diamond (b \diamond a))$$

implies the required relation given in Equation (3.5).
Conversely, let us consider that $(\mathfrak{I}, J)$ is an int-soft ideal of a JU-algebra Q which satisfies given Equation (3.5). For any $a, b \in J$, let $k := b \diamond (b \diamond a) \Rightarrow b \diamond (b \diamond (k \diamond a)) = 1$, and therefore, $\mathfrak{I}(1) = \mathfrak{I}(b \diamond (b \diamond (k \diamond a))) = \mathfrak{I}(b \diamond (k \diamond a))$. Now by Equation (3.5),

$$\mathfrak{I}((b \diamond (b \diamond a)) \diamond (b \diamond a)) = \mathfrak{I}(1) = \mathfrak{I}(b \diamond a) \diamond (b \diamond (b \diamond a))$$

because $(b \diamond a) \diamond (b \diamond (b \diamond a)) = 1$. Hence $\mathfrak{I}_a \star \mathfrak{I}_b = \mathfrak{I}_{a \diamond_L} = \mathfrak{I}_{(a \diamond b) \diamond b} = (\mathfrak{I}_a \star \mathfrak{I}_b) \star \mathfrak{I}_b$.
It implies that that $Q/\mathfrak{I} = (J/\mathfrak{I}, \star, \mathfrak{I}_1)$ is a positive implicative, as required. □

For any $a, b \in J$, we define $a \otimes b = (b \diamond a) \diamond a$. We call a JU-algebra J as a *commutative* JU-algebra if $a \otimes b = b \otimes a$ for all $a, b \in J$. A commutative and positive implicative JU-algebra is called *implicative.* A characterization is presented for a commutative soft quotient JU-algebra in the following result and it will also characterize the implicative soft quotient JU-algebras.

Theorem 3.5. *Consider an int-soft ideal $(\mathfrak{I}, J)$ of a JU-algebra Q. Then the corresponding soft quotient JU-algebra $Q/\mathfrak{I} := (J/\mathfrak{I}, \star, \mathfrak{I}_1)$ is commutative if and only if*

$$\mathfrak{I}(y \diamond x) = \mathfrak{I}(((x \diamond y) \diamond y) \diamond x) \text{ for all } x, y \in J. \tag{3.6}$$

Proof. Let us assume that the soft quotient $Q/\mathfrak{I} := (J/\mathfrak{I}, \star, \mathfrak{I}_1)$ is a commutative JU-algebra and take any two elements $x, y \in J$.
Then

$$\begin{aligned}
&\mathfrak{I}_x \otimes \mathfrak{I}_y = \mathfrak{I}_y \otimes \mathfrak{I}_x \\
&\Rightarrow (\mathfrak{I}_y \star \mathfrak{I}_x) \star \mathfrak{I}_x = (\mathfrak{I}_x \star \mathfrak{I}_y) \star \mathfrak{I}_y \\
&\Rightarrow \mathfrak{I}_{(y \diamond x) \diamond x} = \mathfrak{I}_{(x \diamond y) \diamond y} \\
&\Rightarrow \mathfrak{I}(((x \diamond y) \diamond y) \diamond (y \diamond x \diamond x)) = \mathfrak{I}(1).
\end{aligned}$$

Now it follows from (J_8) and the relation (3.1), it gives that

$$\mathfrak{I}(y \diamond x) \subseteq \mathfrak{I}(1) = \mathfrak{I}(((x \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x)) = \mathfrak{I}((y \diamond x) \diamond ((x \diamond y) \diamond y) \diamond x)$$
$$\Rightarrow \mathfrak{I}(y \diamond x) = \mathfrak{I}((y \diamond x) \diamond (((x \diamond y) \diamond y) \diamond x)) \cap \mathfrak{I}(y \diamond x).$$

Now by using relation (3.2),

$$\mathfrak{I}(y \diamond x) = \mathfrak{I}((y \diamond x) \diamond (((x \diamond y) \diamond y) \diamond x)) \cap \mathfrak{I}(y \diamond x) \subseteq \mathfrak{I}(((x \diamond y) \diamond y) \diamond x) \subseteq \mathfrak{I}(y \diamond x).$$

Therefore, $\mathfrak{I}(y \diamond x) = \mathfrak{I}(((x \diamond y) \diamond y) \diamond x)$ for all $x, y \in J$.
Conversely, assume that $(\mathfrak{I}, J)$ be an int-soft ideal of a JU-algebra $\mathcal{Q}$ that satisfy the Equation (3.6).
For simplicity, we assign $m := y \diamond x$ and $n := y \diamond x \diamond x$ for $x, y \in J$.
Then $(y \diamond (m \diamond x)) = 1$, implies $\mathfrak{I}(y \diamond (m \diamond x)) = \mathfrak{I}(1)$.

$$\begin{aligned}
&\mathfrak{I}(((((y \diamond x) \diamond x) \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x))\\
&= \mathfrak{I}((((m \diamond x) \diamond y) \diamond y) \diamond (m \diamond m))\\
&= \mathfrak{I}(((m \diamond y) \diamond y) \diamond m) = \mathfrak{I}(y \diamond a)\\
&= \mathfrak{I}(y \diamond (m \diamond x)) = \mathfrak{I}(1).
\end{aligned}$$

Note that,

$$\begin{aligned}
&(((((y \diamond x) \diamond x) \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x)) \diamond (((x \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x))\\
&\leq ((x \diamond y) \diamond y) \diamond ((((y \diamond x) \diamond x) \diamond y) \diamond y)\\
&\leq (((y \diamond x) \diamond x) \diamond y) \diamond (x \diamond y)\\
&\leq x \diamond ((y \diamond x) \diamond x) = 1.
\end{aligned}$$

It follows from (3.4) that,

$$\mathfrak{I}(1) = \mathfrak{I}(((((y \diamond x) \diamond x) \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x)) \cap \mathfrak{I}(1) \supseteq \mathfrak{I}(((x \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x)).$$

Thus $\mathfrak{I}(((x \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x)) = \mathfrak{I}(1)$.
In a similar way, we have,

$$\mathfrak{I}(((y \diamond x) \diamond x) \diamond ((x \diamond y) \diamond y)) = \mathfrak{I}(1).$$

Hence we conclude that

$$\begin{aligned}
&\mathfrak{I}(((y \diamond x) \diamond x) \diamond ((x \diamond y) \diamond y)) = \mathfrak{I}(((x \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x)) = \mathfrak{I}(1)\\
&\Rightarrow ((y \diamond x) \diamond x) \diamond ((x \diamond y) \diamond y)) \Diamond ((x \diamond y) \diamond y) \diamond ((y \diamond x) \diamond x) \text{ in } J\\
&\Rightarrow \mathfrak{I}_{(x \diamond y) \diamond y} = \mathfrak{I}_{(y \diamond x) \diamond x}\\
&\Rightarrow (\mathfrak{I}_x \star \mathfrak{I}_y) \star \mathfrak{I}_y = (\mathfrak{I}_y \star \mathfrak{I}_x) \star \mathfrak{I}_x\\
&\Rightarrow \mathfrak{I}_y \otimes \mathfrak{I}_x = \mathfrak{I}_x \otimes \mathfrak{I}_y.
\end{aligned}$$

It implies $\mathcal{Q}/\mathfrak{I} := (X/\mathfrak{I}, \star, \mathfrak{I}_1)$ is a commutative JU-algebra. □

Conclusion

In this article, we have constructed an intersection soft ideals over JU-algebras and studied a soft quotient structure of JU-algebras by using these intersection soft ideals. In particular, we have presented a few fundamental results of soft quotient algebras in the frame of intersection soft ideals. By using these results, we have also characterized positive implicative JU-algebras.

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