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Derivations of Bd-Algebras were Constructed into Semigroups

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Abstract. Bantaojai et al. introduced the algebraic structure called Bd-algebras in 2022 as a nonempty set with the binary operation $*$ and the constant 0, satisfying some properties. In this article, we present the notions of left-right (right-left) derivations and derivations in Bd-algebras. After that, we examine some features of left-right (right-left) derivations and derivations in Bd-algebras. Finally, we show some semigroups that use the set of all derivations of Bd-algebras and some binary operations on that set.

1. Introduction

The notion of derivations has been extensively investigated in the framework of rings and near-rings (see, [10], [18], [26]). The concept of derivations in rings and near-rings theory was applied to BCI-algebras in 2004 by Jun and Xin [15], who achieved certain features. After that, the derivations were studied in other algebraic structures as follows. In 2016, Sawika et al. [28] presented the concepts of left-right (right-left) derivations and derivations of UP-algebras, and their properties were examined. In 2019, Kamaludin et al. [16] used the concept of derivations to investigate in BG-algebras. In 2020, the derivations of BF-algebras with different characteristics and the left and right derivatives of ideals in BF-algebras were studied by Dejen and Tegegne [11]. Additionally, an investigation on the general idea of derivation is also available. Zhan and Liu [32] introduced the concepts of left-right (right-left) f-derivations of BCI-algebras. Afterwards, Jana et al. [14] gave the concepts of f-derivations, left-right (right-left) derivations, and generalized derivations of KUS-algebras. The idea of generalized (regular) (α, β)-derivations of a BCI-algebra was first presented by Al-Roqi [4]. In 2021, Afriastuti et al. [1] defined the concept of (f, g)-derivations

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in BCH-algebras and investigated some of their properties. For studies on derivations in other algebraic structures, further research can be explored in various works (e.g., [3], [6], [12], [27]).

Returning to 1999, Neggers and Kim [23] determined a algebraic structure known as d-algebras, which is a generalization of BCK-algebras. Later, they also introduced the concept of B-algebras in 2002 [24]. In 2010, Al-Shehrie [5] presented the notion of derivations for the investigation in B-algebras, and the following year, Kandaraj and Chandramouleeswaran [17] similarly used the concept of derivations to explore in d-algebras. In 2014, Ardekani and Davvaz [7] introduced the notions of f-derivations and (f,g)-derivations of B-algebras and discussed some properties of (f,g)-derivations in commutative B-algebras. The concepts of outside and inside f_q-derivations as well as (right-left) and (left-right) f_q-derivations in B-algebras were introduced by Muangkarn et al. [20] in 2021. Recently, in 2024, Alnefaie [2] applied the concept of the reverse derivations of rings to explore in d-algebras. Moreover, the notion of fuzzy sets has been studied in the concept of derivations in the algebraic structure of d-algebras (see, [25], [31]). Other research studies that consider the concept of derivations in B-algebras and d-algebras can be further examined in the referenced research (e.g., [9], [19], [29], [30]).

Bantaojai et al. [8] combined some conditions of B-algebras and d-algebras to create a new algebra known as Bd-algebras in 2022. Later on, the Bd-algebras have been continuously analyzed (see, [13], [21], [22]). In this research, the concept of derivations will be studied in Bd-algebras. This work will define the concepts of left-right (right-left) derivations and derivations in Bd-algebras and investigate some properties of these concepts. Subsequently, we will prove that the set of all derivations of Bd-algebras may transform into semigroups under specific operations.

2. Preliminaries

In this section, we will review the essential concepts and basic examples necessary for application in the next section. An algebraic system $(S,*)$ that consists of a nonempty set S and the associative binary operation "$*$" on S is called a *semigroup*, meaning that for any $x,y,z \in S$, the condition $(x*y)*z = x*(y*z)$ holds. A semigroup $(S,*)$ is called *commutative* if $a*b = b*a$ for all $a,b \in S$.

Definition 2.1. *[8] An algebra $(X,*,0)$ is known as a Bd-algebra if it holds to the following axioms: for any $x,y \in X$, (i) $x*0 = x$, and (ii) if $x*y = 0$ and $y*x = 0$, then $x = y$.*

Example 2.1. *Let $X = \{0,a,b,c\}$ with the binary operation $*$ on X as defined by:*

$*$	0	a	b	c
0	0	b	a	c
a	a	0	c	b
b	b	c	0	a
c	c	a	b	0

It's not difficult to check that $(X,*,0)$ is a Bd-algebra.

Example 2.2. *Let $\mathbb{Z}$ be the set of integers. It is evident that $(\mathbb{Z}, -, 0)$ is a Bd-algebra under the usual subtraction on integers.*

3. Derivations in Bd-algebras

In this section, we introduce the concepts of left-right (right-left) derivations and derivations in Bd-algebras and investigate some properties of these concepts. In the end, we will present that the set of all derivations of Bd-algebras can form a semigroup under certain binary operations on the set of all derivations of Bd-algebras. Throughout this study, we assume a binary operation $\wedge$ on a Bd-algebra $(X, *, 0)$ by $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition 3.1. *Let $(X, *, 0)$ be a Bd-algebra and $d : X \to X$. A map d is called a left-right derivation ((l, r)-derivation) of X if $d(x * y) = (d(x) * y) \wedge (x * d(y))$ for all $x, y \in X$. A map d is called a right-left derivation ((r, l)-derivation) of X if $d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$. A map d is called a derivation of X if it is both an (l, r)-derivation and an (r, l)-derivation of X.*

Example 3.1. *Let $(X, *, 0)$ be a Bd-algebra as defined in Example 2.1. Next, we define a map $d : X \to X$ by for every $x \in X$,*

$$d(x) = \begin{cases} c & \text{if } x = 0, \\ b & \text{if } x = a, \\ a & \text{if } x = b, \\ 0 & \text{if } x = c. \end{cases}$$

It can be verified that d is an (l, r)-derivation and an (r, l)-derivation of X. It follows that d is also a derivation of X.

Example 3.2. *From Example 2.2, we have $(\mathbb{Z}, -, 0)$ is a Bd-algebra. Next, the self-map d of $\mathbb{Z}$ is defined by $d(x) = x - 1$ for all $x \in \mathbb{Z}$. Now, for every $x, y \in \mathbb{Z}$, we have*

$$\begin{aligned}(d(x) - y) \wedge (x - d(y)) &= (x - d(y)) - [(x - d(y)) - (d(x) - y)] \\ &= d(x) - y = (x - 1) - y = (x - y) - 1 = d(x - y).\end{aligned}$$

It follows that d is an (l, r)-derivation of $\mathbb{Z}$. On the other hand, for each $x, y \in \mathbb{Z}$, we get

$$\begin{aligned}(x - d(y)) \wedge (d(x) - y) &= (d(x) - y) - [(d(x) - y) - (x - d(y))] \\ &= x - d(y) = x - (y - 1) = x - y + 1 \neq (x - y) - 1 = d(x - y).\end{aligned}$$

This shows that d is not an (r, l)-derivation of $\mathbb{Z}$.

Example 3.3. *Let $X = \{0, a, b\}$ and the binary operation $*$ on X as shown in the following table:*

$*$	0	a	b
0	0	0	0
a	a	a	a
b	b	b	b

Then, $(X, *, 0)$ is a Bd-algebra. Define the map $d : X \to X$ by $d(x) = 0$ for all $x \in X$. It turns out that d is an (r, l)-derivation of X, but it is not an (l, r)-derivation of X, since $d(a * 0) \neq (d(a) * 0) \wedge (a * d(0))$.

Let $(X, *, 0)$ be a Bd-algebra. A map $d : X \to X$ is said to be *regular* if $d(0) = 0$.

Remark 3.1. *In Example 3.1, we obtain d is an (r, l)-derivation of a Bd-algebra $(X, *, 0)$. In addition, the (r, l)-derivation d of X is not regular, since $d(0) = c \neq 0$. Moreover, $d(0) = c \neq 0 = 0 \wedge d(0)$.*

By adding some conditions to the self-map d of a Bd-algebra $(X, *, 0)$, we can obtain the following theorems.

Theorem 3.1. *Let $(X, *, 0)$ be a Bd-algebra, and d ba a self-map of X. If d is an (l, r)-derivation of X with it is regular, then $d(x) = d(x) \wedge x$ for all $x \in X$.*

Proof. Assume that d is an (l, r)-derivation of X with it is regular. Let $x \in X$. Then, we have

$$\begin{aligned} d(x) &= d(x * 0) \\ &= (d(x) * 0) \wedge (x * d(0)) \\ &= (d(x) * 0) \wedge (x * 0) \\ &= d(x) \wedge x. \end{aligned}$$

Hence, $d(x) = d(x) \wedge x$. □

It can be seen that the converse of Theorem 3.1 is not always true, as shown in the following example.

Example 3.4. *Let $X = \{0, a, b\}$ and the binary operation $*$ on X as shown in the following table:*

$*$	0	a	b
0	0	a	0
a	a	0	a
b	b	b	0

Then, $(X, *, 0)$ is a Bd-algebra. Next, define the self-map d of X by $d(x) = 0$ for all $x \in X$. It is clear that d is regular. Now, consider

$$d(0) \wedge 0 = d(0), d(a) \wedge a = d(a), \text{and } d(b) \wedge b = d(b).$$

Hence, $d(x) = d(x) \wedge x$ for all $x \in X$. However, the map d is not an (l, r)-derivation of X, because $d(0 * a) \neq (d(0) * a) \wedge (0 * d(a))$.

Theorem 3.2. *Let $(X, *, 0)$ be a Bd-algebra and d ba a self-map of X. If d is an (r, l)-derivation of X with it is regular, then $d(x) = x \wedge d(x)$ for all $x \in X$.*

Proof. Assume that d is an (r,l)-derivation of X with it is regular. Let $x \in X$. Thus, we get

$$\begin{aligned} d(x) &= d(x * 0) \\ &= (x * d(0)) \wedge (d(x) * 0) \\ &= (x * 0) \wedge (d(x) * 0) \\ &= x \wedge d(x). \end{aligned}$$

Hence, $d(x) = x \wedge d(x)$. □

The contrary of Theorem 3.2 is not universally valid, as seen in the subsequent example.

Example 3.5. *Let $X = \{0, a, b\}$ and the binary operation $*$ on X as shown in the following table:*

$*$	0	a	b
0	0	0	0
a	a	a	b
b	b	a	b

We obtain that $(X, *, 0)$ is a Bd-algebra. Next, define the self-map d of X by

$$d(x) = \begin{cases} 0 & \text{if } x = b, \\ a & \text{if } x = a, \\ b & \text{if } x = 0. \end{cases}$$

It's not difficult to check that $d(x) = x \wedge d(x)$ for all $x \in X$. Now, we consider $d(a * b) = 0 \neq a = (a * d(b)) \wedge (d(a) * b)$. This means that d is not an (r,l)-derivation of X.

The following will define the set of all derivations of the Bd-algebra $(X, *, 0)$ as follows

$$Der(X) := \{d : X \to X \mid d \text{ is a derivation of } X\}.$$

Let $(X, *, 0)$ be a Bd-algebra and $d_1, d_2 : X \to X$. Now, we recall the composition of d_1 and d_2 by $(d_1 \circ d_2)(x) = d_1(d_2(x))$ for all $x \in X$.

Example 3.6. *Let $X = \{0, a, b\}$ and the binary operation $*$ on X as shown in the following table:*

$*$	0	a	b
0	0	a	a
a	a	a	a
b	b	b	b

It is not difficult to check that $(X, *, 0)$ is a Bd-algebra, but it does not satisfy the condition $y * (y * x) = x$ for all $x, y \in X$ because $a * (a * b) \neq b$. Next, we define self-maps d_1 and d_2 of X as follows:

$$d_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x = b, \\ b & \text{if } x = a, \end{cases} \quad \text{and} \quad d_2(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x \in \{a, b\}. \end{cases}$$

By careful computations, it is evident that d_1 *and* d_2 *are* (r,l)*-derivations of* X*. However,* $d_1 \circ d_2$ *is not an* (r,l)*-derivation of* X*, since* $(d_1 \circ d_2)(0 * a) = b \neq a = (0 * (d_1 \circ d_2)(a)) \wedge ((d_1 \circ d_2)(0) * a)$*.*

From Example 3.6, it is seen that by putting the condition $y * (y * x) = x$ for all $x, y \in X$ into a Bd-algebra $(X, *, 0)$, the following theorems have been proven as true.

Theorem 3.3. *Let* $(X, *, 0)$ *be a Bd-algebra with* $y * (y * x) = x$ *for all* $x, y \in X$*. If* d_1 *and* d_2 *are* (l,r)*-derivations of* X*, then* $d_1 \circ d_2$ *is an* (l,r)*-derivation of* X*.*

Proof. Let d_1 and d_2 be (l,r)-derivations of X. For every $x, y \in X$, we have

$$\begin{aligned}
(d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) \\
&= d_1\left[(d_2(x) * y) \wedge (x * d_2(y))\right] \\
&= d_1\left[(x * d_2(y)) * ((x * d_2(y)) * (d_2(x) * y))\right] \\
&= d_1(d_2(x) * y) \\
&= (d_1(d_2(x)) * y) \wedge (d_2(x) * d_1(y)) \\
&= (d_2(x) * d_1(y)) * [(d_2(x) * d_1(y)) * (d_1(d_2(x)) * y)] \\
&= d_1(d_2(x)) * y \\
&= (x * d_1(d_2(y))) * [(x * d_1(d_2(y))) * (d_1(d_2(x)) * y)] \\
&= (x * (d_1 \circ d_2)(y)) * [(x * (d_1 \circ d_2)(y)) * ((d_1 \circ d_2)(x) * y)] \\
&= ((d_1 \circ d_2)(x) * y) \wedge (x * (d_1 \circ d_2)(y)).
\end{aligned}$$

This shows that $d_1 \circ d_2$ is an (l,r)-derivation of X. □

Theorem 3.4. *Let* $(X, *, 0)$ *be a Bd-algebra with* $y * (y * x) = x$ *for all* $x, y \in X$*. If* d_1 *and* d_2 *are* (r,l)*-derivations of* X*, then* $d_1 \circ d_2$ *is an* (r,l)*-derivation of* X*.*

Proof. Assume that d_1 and d_2 are (l,r)-derivations of X. For each $x, y \in X$, we have

$$\begin{aligned}
(d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) \\
&= d_1\left[(x * d_2(y)) \wedge (d_2(x) * y)\right] \\
&= d_1\left[(d_2(x) * y) * ((d_2(x) * y) * (x * d_2(y)))\right] \\
&= d_1(x * d_2(y)) \\
&= (x * d_1(d_2(y))) \wedge (d_1(x) * d_2(y)) \\
&= (d_1(x) * d_2(y)) * [(d_1(x) * d_2(y)) * (x * d_1(d_2(y)))] \\
&= x * d_1(d_2(y)) \\
&= (d_1(d_2(x)) * y) * [(d_1(d_2(x)) * y) * (x * d_1(d_2(y)))] \\
&= ((d_1 \circ d_2)(x) * y) * [((d_1 \circ d_2)(x) * y) * (x * (d_1 \circ d_2)(y))] \\
&= (x * (d_1 \circ d_2)(y)) \wedge ((d_1 \circ d_2)(x) * y).
\end{aligned}$$

It turns out that $d_1 \circ d_2$ is an (r, l)-derivation of X. □

The following corollary is obtained by Theorem 3.3 and Theorem 3.4.

Corollary 3.1. *Let $(X, *, 0)$ be a Bd-algebra with $y * (y * x) = x$ for all $x, y \in X$. If d_1 and d_2 are derivations of X, then $d_1 \circ d_2$ is a derivation of X.*

Theorem 3.5. *Let $(X, *, 0)$ be a Bd-algebra with $y * (y * x) = x$ for all $x, y \in X$. If d_1 and d_2 are derivations of X, then $d_1 \circ d_2 = d_2 \circ d_1$.*

Proof. Let $x, y \in X$. Since d_1 is an (r, l)-derivation and d_2 is an (l, r)-derivation of X, we have

$$\begin{aligned}(d_1 \circ d_2)(x * y) &= d_1(d_2(x * y)) \\ &= d_1 \left[(d_2(x) * y) \wedge (x * d_2(y))\right] \\ &= d_1 \left[(x * d_2(y)) * ((x * d_2(y)) * (d_2(x) * y))\right] \\ &= d_1(d_2(x) * y) \\ &= (d_2(x) * d_1(y)) \wedge (d_1(d_2(x)) * y) \\ &= (d_1(d_2(x)) * y) * \left[(d_1(d_2(x)) * y) * (d_2(x) * d_1(y))\right] \\ &= d_2(x) * d_1(y)\end{aligned}$$

and

$$\begin{aligned}(d_1 \circ d_2)(x * y) &= d_2(d_1(x * y)) \\ &= d_2 \left[(x * d_1(y)) \wedge (d_1(x) * y)\right] \\ &= d_2 \left[(d_1(x) * y) * ((d_1(x) * y) * (x * d_1(y)))\right] \\ &= d_2(x * d_1(y)) \\ &= (d_2(x) * d_1(y)) \wedge (x * d_2(d_1(y))) \\ &= (x * d_2(d_1(y))) * \left[(x * d_2(d_1(y))) * (d_2(x) * d_1(y))\right] \\ &= d_2(x) * d_1(y).\end{aligned}$$

Hence, $(d_1 \circ d_2)(x * y) = (d_2 \circ d_1)(x * y)$. Now, letting $y = 0$, it follows $(d_1 \circ d_2)(x) = (d_2 \circ d_1)(x)$. Therefore, $d_1 \circ d_2 = d_2 \circ d_1$. □

Theorem 3.6. *Let $(X, *, 0)$ be a Bd-algebra with $y * (y * x) = x$ for all $x, y \in X$, and let $d_1, d_2, d_3 \in Der(X)$. Then $(d_1 \circ d_2) \circ d_3 = d_1 \circ (d_2 \circ d_3)$.*

Proof. Let $x, y \in X$. Then, we have

$$\begin{aligned}((d_1 \circ d_2) \circ d_3)(x * y) &= (d_1 \circ d_2)(d_3(x * y)) \\ &= (d_1 \circ d_2) \left[(d_3(x) * y) \wedge (x * d_3(y))\right] \\ &= (d_1 \circ d_2) \left[(x * d_3(y)) * ((x * d_3(y)) * (d_3(x) * y))\right]\end{aligned}$$

$$= (d_1 \circ d_2)(d_3(x) * y)$$
$$= d_1(d_2(d_3(x) * y))$$

and

$$\begin{aligned}(d_1 \circ (d_2 \circ d_3))(x * y) &= d_1((d_2 \circ d_3)(x * y)) \\ &= d_1(d_2(d_3(x * y))) \\ &= d_1(d_2((d_3(x) * y) \wedge (x * d_3(y)))) \\ &= d_1(d_2((x * d_3(y)) * ((x * d_3(y)) * (d_3(x) * y)))) \\ &= d_1(d_2(d_3(x) * y)).\end{aligned}$$

Hence, $((d_1 \circ d_2) \circ d_3)(x * y) = (d_1 \circ (d_2 \circ d_3))(x * y)$. By letting $y = 0$, we have $((d_1 \circ d_2) \circ d_3)(x) = (d_1 \circ (d_2 \circ d_3))(x)$. This means that $(d_1 \circ d_2) \circ d_3 = d_1 \circ (d_2 \circ d_3)$. □

The following theorem can be summarized up from the inclusion of Corollary 3.1, Theorem 3.5, and Theorem 3.6.

Theorem 3.7. *Let $(X, *, 0)$ be a Bd-algebra with $y * (y * x) = x$ for all $x, y \in X$. Then $(Der(x), \circ)$ forms a commutative semigroup.*

Let $(X, *, 0)$ be a Bd-algebra and $d_1, d_2 : X \to X$. Define the operation $\wedge$ by $(d_1 \wedge d_2)(x) = d_1(x) \wedge d_2(x)$ for all $x \in X$.

Example 3.7. *Let $X = \{0, a, b\}$ and consider the binary operation $*$ defined on X as shown in the following table:*

$*$	0	a	b
0	0	b	b
a	a	a	b
b	b	a	b

Verifying that $(X, *, 0)$ forms a Bd-algebra is straightforward; nonetheless, it is unsuccessful to satisfy the condition $y * (y * x) = x$ for every $x, y \in X$, as proved by $a * (a * 0) \neq 0$. Now, we define self-maps d_1 and d_2 of X by

$$d_1(x) = x \text{ for all } x \in X \text{ and } d_2(x) = 2 \text{ for all } x \in X,$$

respectively. It is obvious that the self-maps d_1 and d_2 are (r, l)-derivations of X. However, $d_1 \wedge d_2$ is not an (r, l)-derivation of X, since $(d_1 \wedge d_2)(a * 0) = a \neq b = (a * (d_1 \wedge d_2)(0)) \wedge ((d_1 \wedge d_2)(a) * 0)$.

From Example 3.7, it is clear that if we apply the condition $y * (y * x) = x$ for any $x, y \in X$ within a Bd-algebra $(X, *, 0)$, then the following theorems are valid.

Theorem 3.8. *Let $(X, *, 0)$ be a Bd-algebra with $y * (y * x) = x$ for all $x, y \in X$. If d_1 and d_2 are (l, r)-derivations of X, then $d_1 \wedge d_2$ is also an (l, r)-derivation of X.*

Proof. Assume that d_1 and d_2 are (l, r)-derivations of X. Let $x, y \in X$. Then, we have

$$\begin{aligned}(d_1 \wedge d_2)(x * y) &= d_1(x * y) \wedge d_2(x * y) \\ &= [(d_1(x) * y) \wedge (x * d_1(y))] \wedge [(d_2(x) * y) \wedge (x * d_2(y))] \\ &= [(x * d_1(y)) * ((x * d_1(y)) * (d_1(x) * y))] \wedge [(x * d_2(y)) * ((x * d_2(y)) * (d_2(x) * y))] \\ &= (d_1(x) * y) \wedge (d_2(x) * y) \\ &= (d_2(x) * y) * ((d_2(x) * y) * (d_1(x) * y)) \\ &= d_1(x) * y \\ &= [d_2(x) * (d_2(x) * d_1(x))] * y \\ &= (d_1(x) \wedge d_2(x)) * y \\ &= (d_1 \wedge d_2)(x) * y \\ &= (x * (d_1 \wedge d_2)(y)) * [(x * (d_1 \wedge d_2)(y)) * ((d_1 \wedge d_2)(x) * y)] \\ &= ((d_1 \wedge d_2)(x) * y) \wedge (x * (d_1 \wedge d_2)(y)).\end{aligned}$$

Hence, $d_1 \wedge d_2$ is an (l, r)-derivation of X. □

Theorem 3.9. *Let $(X, *, 0)$ be a Bd-algebra with $y * (y * x) = x$ for all $x, y \in X$. If d_1 and d_2 are (r, l)-derivations of X, then $d_1 \wedge d_2$ is also an (r, l)-derivation of X.*

Proof. Assume that d_1 and d_2 are (r, l)-derivations of X. Let $x, y \in X$. Then, we have

$$\begin{aligned}(d_1 \wedge d_2)(x * y) &= d_1(x * y) \wedge d_2(x * y) \\ &= [(x * d_1(y)) \wedge (d_1(x) * y)] \wedge [(x * d_2(y)) \wedge (d_2(x) * y)] \\ &= [(d_1(x) * y) * ((d_1(x) * y) * (x * d_1(y)))] \wedge [(d_2(x) * y) * ((d_2(x) * y) * (x * d_2(y)))] \\ &= (x * d_1(y)) \wedge (x * d_2(y)) \\ &= (x * d_2(y)) * ((x * d_2(y)) * (x * d_1(y))) \\ &= x * d_1(y) \\ &= x * [d_2(y) * (d_2(y) * d_1(y))] \\ &= x * (d_1(y) \wedge d_2(y)) \\ &= x * (d_1 \wedge d_2)(y) \\ &= ((d_1 \wedge d_2)(x) * y) * [((d_1 \wedge d_2)(x) * y) * (x * (d_1 \wedge d_2)(y))] \\ &= (x * (d_1 \wedge d_2)(y)) \wedge ((d_1 \wedge d_2)(x) * y).\end{aligned}$$

This implies that $d_1 \wedge d_2$ is an (r, l)-derivation of X. □

According to Theorems 3.8 and 3.9, the following corollary is obtained.

Corollary 3.2. *Let $(X, *, 0)$ be a Bd-algebra with $y * (y * x) = x$ for all $x, y \in X$. If d_1 and d_2 are derivations of X, then $d_1 \wedge d_2$ is also a derivation of X.*

Theorem 3.10. *Let $(X,*,0)$ be a Bd-algebra with $y*(y*x)=x$ for all $x,y\in X$. Then $(d_1\wedge d_2)\wedge d_3 = d_1\wedge(d_2\wedge d_3)$ for all $d_1,d_2,d_3\in Der(X)$.*

Proof. Let $d_1,d_2,d_3\in Der(X)$ and $x,y\in X$. Then, we have

$$\begin{aligned}((d_1\wedge d_2)\wedge d_3)(x*y) &= (d_1\wedge d_2)(x*y)\wedge d_3(x*y)\\ &= d_3(x*y)*[d_3(x*y)*(d_1\wedge d_2)(x*y)]\\ &= (d_1\wedge d_2)(x*y)\\ &= d_1(x*y)\wedge d_2(x*y).\end{aligned}$$

On the other hand,

$$\begin{aligned}(d_1\wedge(d_2\wedge d_3))(x*y) &= d_1(x*y)\wedge(d_2\wedge d_3)(x*y)\\ &= d_1(x*y)\wedge[d_2(x*y)\wedge d_3(x*y)]\\ &= d_1(x*y)\wedge[d_3(x*y)*(d_3(x*y)*d_2(x*y))]\\ &= d_1(x*y)\wedge d_2(x*y).\end{aligned}$$

Thus, $((d_1\wedge d_2)\wedge d_3)(x*y) = (d_1\wedge(d_2\wedge d_3))(x*y)$. Take $y=0$, we obtain $((d_1\wedge d_2)\wedge d_3)(x) = (d_1\wedge(d_2\wedge d_3))(x)$. We conclude that $(d_1\wedge d_2)\wedge d_3 = d_1\wedge(d_2\wedge d_3)$. □

From Corollary 3.2 and Theorem 3.10, the conclusion of the following theorem can be derived.

Theorem 3.11. *Let $(X,*,0)$ be a Bd-algebra with $y*(y*x)=x$ for all $x,y\in X$. Then $(Der(X),\wedge)$ forms a semigroup.*

Nevertheless, the semigroup $(Der(X),\wedge)$ in Theorem 3.11 does not necessarily have the commutative property, as shown in the following example.

Example 3.8. *By thorough examination, we found that the Bd-algebra $(X,*,0)$ in Example 2.1 satisfies the condition $y*(y*x)=x$ for all $x,y\in X$. From Example 3.1, we get the self-map d of X is a derivation of X. Next, we define a self-map d' of X by $d'(x)=x$ for all $x\in X$. It is clear that d' is a derivation of X. Thus, $d,d'\in Der(X)$. Now, we consider $(d\wedge d')(b)=d(b)\wedge d'(b)=a\wedge b=b*(b*a)=a$ and $(d'\wedge d)(b)=d'(b)\wedge d(b)=b\wedge a=a*(a*b)=b$. That is, $(d\wedge d')(b)\neq(d'\wedge d)(b)$. This proves that the operation $\wedge$ on $Der(X)$ does not have commutative properties.*

4. Conclusion

In this paper, we introduced the notions of (l,r)-derivations, (r,l)-derivations, and derivations in *Bd*-algebras. Then, we explored some features of (l,r)-derivations, (r,l)-derivations, and derivations in *Bd*-algebras. Subsequently, we defined $Der(X)$ as the set of all derivations of a *Bd*-algebra. Finally, under the composite function $\circ$ on $Der(X)$, we provided the algebraic structure $(Der(X),\circ)$ that forms a commutative semigroup. Furthermore, for any x in X, we have demonstrated that the binary operation $\wedge$ on $Der(X)$ is defined by $(d_1\wedge d_2)(x)=d_1(x)\wedge d_2(x)$ for all $d_1,d_2\in Der(X)$, making the algebraic structure $(Der(x), \wedge)$ a semigroup without commutativity. In our upcoming work, we will introduce and examine the characteristics of (l,r)-f-derivations, (r,l)-f-derivations, f-derivations, (l,r)-(f,g)-derivations, and (r,l)-(f,g)-derivations in Bd-algebras.

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References

[1] S. Afriastuti, S. Gemawati, S. Syamsudhuha, On (f,g)-Derivation in BCH-Algebra, Sci. Technol. Commun. J. 1 (2021), 89–94. https://doi.org/10.59190/stc.v1i3.196.

[2] K. Alnefaie, On Reverse Derivations in d-Algebras, Eur. J. Pure Appl. Math. 17 (2024), 362–371. https://doi.org/10.29020/nybg.ejpam.v17i1.5025.

[3] R.M. Al-Omary, On (α,β)-Derivations in d-Algebras, Boll. Un. Mat. Ital. 12 (2019), 549–556. https://doi.org/10.1007/s40574-018-00190-5.

[4] A.M. Al-Roqi, On Generalized (α,β)-Derivations in BCI-Algebras, J. Appl. Math. Inform. 32 (2014), 27–38. https://doi.org/10.14317/JAMI.2014.027.

[5] N. Al-Shehrie, Derivations of B-Algebras, J. King Abdulaziz Univ. Sci. 22 (2010), 71–83. https://doi.org/10.4197/Sci.22-1.5.

[6] L. Kamali Ardekani, B. Davvaz, On Generalized Derivations of BCI-Algebras and Their Properties, J. Math. 2014 (2014), 207161. https://doi.org/10.1155/2014/207161.

[7] L.K. Ardekani, B. Davvaz, On (f,g)-Derivations of B-Algebras, Mat. Vesn. 66 (2014), 125–132.

[8] T. Bantaojai, C. Suanoom, J. Phuto, A. Iampan, On Bd-Algebras, Int. J. Math. Comput. Sci. 17 (2022), 731–737.

[9] T. Bantaojai, C. Suanoom, J. Phuto, A. Iampan, A Bi-Endomorphism Induced a New Type of Derivations on B-Algebras, Ital. J. Pure Appl. Math. 48 (2022), 336–348.

[10] H.E. Bell, G. Mason, On Derivations in Near-Rings and Near-Fields, North-Holland, 1987.

[11] G.T. Dejen, F.G. Tegegne, Derivations of BF-Algebras, Int. J. Innov. Sci. Res. Technol. 5 (2020), 329–333.

[12] K.K. Dey, A.C. Paul, Generalized Derivations of Prime Gamma Rings, GANIT: J. Bangladesh Math. Soc. 33 (2014), 33–39. https://doi.org/10.3329/ganit.v33i0.17654.

[13] A. Iampan, W. Nakkhasen, Prime Fuzzy Bd-Ideals of Bd-Algebras, Int. J. Math. Comput. Sci. 19 (2024), 941–948.

[14] C. Jana, T. Senapati, M. Pal, Derivation, f-Derivation and Generalized Derivation of KUS-Algebras, Cogent Math. 2 (2015), 1064602. https://doi.org/10.1080/23311835.2015.1064602.

[15] Y. Jun, On Derivations of BCI-Algebras, Inf. Sci. 159 (2004), 167–176. https://doi.org/10.1016/j.ins.2003.03.001.

[16] Kamaludin, S. Gemawati, Kartini, Derivations in BG-Algebras, Int. J. Algebra 13 (2019), 249–257. https://doi.org/10.12988/ija.2019.9620.

[17] N. Kandaraj, M. Chandramouleeswaran, Derivations on d-Algebras, Int. J. Math. Sci. Appl. 1 (2011), 1–8.

[18] K. Kaya, Prime Rings with α Derivations, Hacettepe Bull. Mater. Sci. Eng. 16-17 (1987-1988), 63–71.

[19] Y.H. Kim, Some Derivations on D-Algebras, Int. J. Fuzzy Logic Intell. Syst. 18 (2018), 298–302. https://doi.org/10.5391/IJFIS.2018.18.4.298.

[20] P. Muangkarn, C. Suanoom, P. Pengyim, A. Iampan, f_q-Derivations of b-Algebras, J. Math. Comput. Sci. 11 (2021), 2047–2057.

[21] W. Nakkhasen, S. Phimkota, K. Phoemkhuen, A. Iampan, Characterizations of Fuzzy Bd-Ideals in Bd-Algebras, Int. J. Math. Comput. Sci. 19 (2024), 757–764.

[22] W. Nakkhasen, P. Srisarakham, W. Sakuntanat, A Study on Fuzzy Bd-Subalgebras of Bd-Algebras, ICIC Express Lett. 18 (2024), 777–783. https://doi.org/10.24507/icicel.18.08.777.

[23] J. Neggers, H.S. Kim, On d-Algebras, Math. Slovaca 49 (1999), 19–26. https://eudml.org/doc/31914.

[24] J. Neggers, H.S. Kim, On B-algebras, Math. Vesn. 54 (2002), 21–29.

[25] A. Oli, G. Tefera, Fuzzy Derivations of D-Ideals of d-Algebras and Cartesian Product of Fuzzy Derivation of d-Ideals of d-Algebras, Appl. Artif. Intell. 37 (2023), 2157938. https://doi.org/10.1080/08839514.2022.2157938.

[26] E. Posner, Derivations in prime rings, Proc. Am. Math. Soc. 8 (1957), 1093-1100.

[27] C. Ramadhona, S. Gemawati, Syamsudhuha, Generalized f-Derivation of BP-Algebras, Int. J. Math. Trends Technol. 66 (2020), 80–86.

[28] K. Sawika, R. Intasan, A. Kaewwasri, A. Iampan, Derivations of UP-Algebras, Korean J. Math. 24 (2016), 345–367. https://doi.org/10.11568/KJM.2016.24.3.345.

[29] A. Senbakar, D. Yilmaz, On Generalized (α, β)-Derivations in d-Algebras, Bull. Int. Math. Virtual Inst. 13 (2023), 239–247.

[30] E. Yattaqi, S. Gemawati, I. Hasbiyati, On Left f_q-Derivations of B-Algebras, Sci. Technol. Commun. J. 2 (2021), 1–8. https://doi.org/10.59190/stc.v2i1.197.

[31] D. Yılmaz, A Note on Fuzzy Derivations of Some Ideals on d-Algebras, Theor. Comput. Sci. 960 (2023), 113924. https://doi.org/10.1016/j.tcs.2023.113924.

[32] J. Zhan, Y.L. Liu, On f-derivations of BCI-algebras, Int. J. Math. Math. Sci. 2005 (2005), 1675–1684. https://doi.org/10.1155/IJMMS.2005.1675.