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Solution and Fuzzy Stability of n-Dimensional Quadratic Functional Equation

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Abstract. In this paper, the authors have developed and established the solution of the n-dimensional quadratic functional equation within the context of a vector space, specifically focusing on its properties and behavior in fuzzy normed spaces. They not only provide an explicit form of the solution but also investigate the stability of this solution under various perturbations. By extending the classical stability results to the setting of fuzzy normed spaces, the authors explore how uncertainties, represented by fuzzy norms, affect the solution's stability.

1. Introduction

Over the past eight decades, the stability problems associated with various functional equations have been thoroughly explored by numerous researchers [1, 2, 14, 15, 22, 28, 30, 33, 35]. The term 'generalized Ulam-Hyers stability' has emerged from this rich historical context. This terminology has since been extended to encompass other types of functional equations as well. For more comprehensive definitions and discussions of these terminologies, one can refer to the works of [12, 16, 19, 21, 32].

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The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

and therefore the equation (1.1) is called quadratic functional equation.

The Hyers - Ulam stability theorem for the quadratic functional equation (1.1) was proved by F.Skof [34] for the functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2 be a Banach space. The result of Skof is still true if the relevant domain E_1 is replaced by an Abelian group and it was delt by P.W.Cholewa [10]. S.Czerwik [11] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). This result further generalized by Th.M.Rassais [31], C.Borelli and G.L.Forti [7].

K.W. Jun and H.M. Kim [17] introduced the following generalized quadratic and additive type functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f\left(x_i + x_j\right) \tag{1.2}$$

in real vector spaces. For $n = 3$, Pl.Kannappan proved that a function f satisfies the functional equation (1.2) if and only if there exists a symmetric bi-additive function A and an additive function B such that $f(x) = B(x,x) + A(x)$ (see [20]). The Hyers-Ulam stability for the equation (1.2) when $n = 3$ was proved by S.M. Jung [18]. The Hyers-Ulam-Rassias stability for the equation (1.2) when $n = 4$ was also investigated by I.S. Chang et al., [8].

In this paper, the authors established the solution in vector space and Fuzzy stability of n-dimensional quadratic functional equation

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = (-n^2 + 6n - 4)\sum_{i=1}^{n} f(x_i) + (n-4)\sum_{1\le i<j\le n} f\left(x_i + x_j\right) \tag{1.3}$$

where

$$x_{ij} = \begin{cases} -x_j & if \quad i = j, \\ x_j & if \quad i \neq j, \end{cases}$$

and n is a positive integer.

2. General Solution

In this section, the general solution of the functional equation (1.3) is presented.

Theorem 2.1. *Let X and Y be real vector spaces. The mapping $f : X \to Y$ satisfies the functional equation*

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2) \tag{2.1}$$

for all $x_1, x_2 \in X$ if and only if $f : X \to Y$ satisfies the functional equation (1.3) for all $x_1, \cdots, x_n \in X$.

Proof. Let $f : X \to Y$ satisfies the functional equation (1.3). Replacing $(x_2, x_3, \cdots, x_n)$ by $(0, 0, \cdots, 0)$ in (1.3), we get

$$f(-x_1) = f(x_1) \ \forall \ x_1 \in X.$$

Therefore f is an even function. Setting $(x_3, x_4, \cdots, x_n)$ by $(0, 0, \cdots, 0)$ in (1.3), we obtain

$$\begin{aligned} &f(-x_1 + x_2) + f(x_1 - x_2) + (n-2)f(x_1 + x_2) \\ &= (-n^2 + 6n - 4)\left[f(x_1) + f(x_2)\right] + (n-4)\left[f(x_1 + x_2) + (n-2)f(x_1) + (n-2)f(x_2)\right]. \end{aligned} \tag{2.2}$$

for all $x_1, x_2 \in X$. Using evenness of f and rearranging the functions in the above equation our result is desired.

Conversely, assume that $f : X \to Y$ satisfies (2.1). Setting $x_1 = x_2 = 0$ in (2.1), we get $f(0) = 0$. Let $x_2 = 0$ in (2.1), we obtain $f(-x_1) = f(x_1)$ for all $x_1 \in X$. Therefore f is an even function. Replacing x_2 by x_1 and $2x_1$ respectively in (2.1), we get $f(2x_1) = 2^2 f(x_1)$ and $f(3x_1) = 3^2 f(x_1)$ for all $x_1 \in X$. In general for any positive integer n, we have $f(nx_1) = n^2 f(x_1)$ for all $x_1 \in X$.
Both sides multiplying by 2 and using evenness of f in equation (2.1), we obtain

$$f(-x_1 + x_2) + f(x_1 - x_2) = 4\left[f(x_1) + f(x_2)\right] - 2f(x_1 + x_2) \tag{2.3}$$

Replacing (x_1, x_2) by $(x_1, x_2 - x_3)$, we get,

$$f(x_1 + x_2 - x_3) + f(x_1 - x_2 + x_3) = 2f(x_1) + 2f(x_2 - x_3) \tag{2.4}$$

Again replacing (x_1, x_2) by $(x_2, x_1 - x_3)$, we arrive

$$f(x_1 + x_2 - x_3) + f(-x_1 + x_2 + x_3) = 2f(x_2) + 2f(x_1 - x_3) \tag{2.5}$$

Also, replacing (x_1, x_2) by $(x_1 - x_2, x_3)$, we obtain

$$f(x_1 - x_2 + x_3) + f(-x_1 + x_2 + x_3) = 2f(x_1 - x_2) + 2f(x_3) \tag{2.6}$$

Adding (2.4),(2.5) and (2.6), we have

$$\begin{aligned} &f(-x_1 + x_2 + x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) \\ &= (-3^2 + 6 \cdot 3 - 4)\sum_{i=1}^{3} f(x_i) + (3-4)\sum_{1 \le i < j \le 3} f\left(x_i + x_j\right) \end{aligned} \tag{2.7}$$

Similarly one can easily verify for four variables, we get

$$\begin{aligned} &f(-x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2 + x_3 + x_4) + f(x_1 + x_2 - x_3 + x_4) \\ &+ f(x_1 + x_2 + x_3 - x_4) = (-4^2 + 6 \cdot 4 - 4)\sum_{i=1}^{4} f(x_i) + (4-4)\sum_{1 \le i < j \le 4} f\left(x_i + x_j\right) \end{aligned} \tag{2.8}$$

Extending this result, for any positive integer n, we arrive (1.3). □

3. Fuzzy Stability Results

In this section, the authors present the basic definitions in fuzzy normed spaces and investigate the fuzzy stability of the generalized quadratic functional equation (1.3). A.K. Katsaras [23] introduced a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Several mathematicians have explored fuzzy norms on vector spaces from different perspectives [13, 25, 36]. In particular, T. Bag and S.K. Samanta [5], following the work of S.C. Cheng and J.N. Mordeson [9], proposed a fuzzy norm in such a way that the corresponding fuzzy metric is of the Kramosil and Michalek type [24]. They also established a decomposition theorem for a fuzzy norm, breaking it down into a family of crisp norms, and investigated some key properties of fuzzy normed spaces. [6].

We adopt the definition of fuzzy normed spaces as presented in [5] and [26, 27].

Definition 3.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(F1) $N(x,c) = 0$ *for* $c \leq 0$;

(F2) $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;

(F3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

(F4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

(F5) $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;

(F6) *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 3.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 3.2. *Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$.*

Definition 3.3. *A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 3.4. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 3.5. *A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$ then f is said to be continuous on X.*

The stability of various functional equations in fuzzy normed spaces was investigated in [3,26, 27].

Throughout this section, assume that $X, (Z, N')$ and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. Now use the following notation for a given mapping $f : X \to Y$

$$Df(x_1, \cdots, x_n) = \sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) - (-n^2 + 6n - 4)\sum_{i=1}^{n} f(x_i) - (n-4)\sum_{1\le i<j\le n} f\left(x_i + x_j\right)$$

where

$$x_{ij} = \begin{cases} -x_j & if \quad i = j, \\ x_j & if \quad i \neq j, \end{cases}$$

and n is a positive integer and for all $x_1, \cdots, x_n \in X$.

Now, the authors investigate the generalized Ulam-Hyers stability of the functional equation (1.3).

Theorem 3.1. *Let $\beta \in \{-1, 1\}$. Let $\alpha : X^n \to Z$ be a mapping with $0 < \left(\frac{d}{2^2}\right)^{\beta} < 1$*

$$N'\left(\alpha\left(2^{\beta}x, 2^{\beta}x, 0, \ldots, 0\right), r\right) \ge N'\left(d^{\beta}\alpha\left(x, x, 0, \ldots, 0\right), r\right) \tag{3.1}$$

for all $x \in X$ and all $d > 0$, and

$$\lim_{n\to\infty} N'\left(\alpha\left(2^{\beta n}x_1, \ldots, 2^{\beta n}x_n\right), 2^{\beta 2n}r\right) = 1 \tag{3.2}$$

for all $x_1, \ldots, x_n \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N\left(Df(x_1, \ldots, x_n), r\right) \ge N'\left(\alpha(x_1, \ldots, x_n), r\right) \tag{3.3}$$

for all $r > 0$ and all $x_1, \ldots, x_n \in X$. Then the limit

$$Q(x) = N - \lim_{n\to\infty} \frac{f(2^{\beta n}x)}{2^{\beta 2n}} \tag{3.4}$$

exists for all $x \in X$ and the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (1.3) and

$$N\left(f(x) - Q(x), r\right) \ge N'\left(\alpha(x, x, 0, \ldots, 0), r|2^2 - d|\right) \tag{3.5}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\beta = 1$. Replacing $(x_1, x_2, x_3, \dots, x_n)$ by $(x, x, 0, \dots, 0)$ in (3.3), we get

$$N\Big(f(2x) - 2^2 f(x), r\Big) \geq N'\left(\alpha(x, x, 0, \dots, 0), r\right) \tag{3.6}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^n x$ in (3.6), we obtain

$$N\left(\frac{f(2^{n+1}x)}{2^2} - f(2^n x), \frac{r}{2^2}\right) \geq N'\left(\alpha(2^n x, 2^n x, 0, \dots, 0), r\right) \tag{3.7}$$

for all $x \in X$ and all $r > 0$. Using (3.1), $(F3)$ in (3.7), we arrive

$$N\left(\frac{f(2^{n+1}x)}{2^2} - f(2^n x), \frac{r}{2^2}\right) \geq N'\left(\alpha(x, x, 0, \dots, 0), \frac{r}{d^n}\right) \tag{3.8}$$

for all $x \in X$ and all $r > 0$. It is easy to verify from (3.8), that

$$N\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{f(2^n x)}{2^{2n}}, \frac{r}{2^2 \cdot 2^{2n}}\right) \geq N'\left(\alpha(x, x, 0, \dots, 0), \frac{r}{d^n}\right) \tag{3.9}$$

holds for all $x \in X$ and all $r > 0$. Replacing r by $d^n r$ in (3.9), we get

$$N\left(\frac{f(2^{n+1}x)}{2^{2(n+1)}} - \frac{f(2^n x)}{2^{2n}}, \frac{d^n\, r}{2^2 \cdot 2^{2n}}\right) \geq N'\left(\alpha(x, x, 0, \dots, 0), r\right) \tag{3.10}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{f(2^n x)}{2^{2n}} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{2(i+1)}} - \frac{f(2^i x)}{2^{2i}} \tag{3.11}$$

for all $x \in X$. From equations (3.10) and (3.11), we have

$$\begin{aligned} N\left(\frac{f(2^n x)}{2^{2n}} - f(x), \sum_{i=0}^{n-1} \frac{d^i\, r}{2^2 \cdot 2^{2i}}\right) &\geq min \bigcup_{i=0}^{n-1} \left\{\frac{f(2^{i+1}x)}{2^{2(i+1)}} - \frac{f(2^i x)}{2^{2i}}, \frac{d^i\, r}{2^2 \cdot 2^{2i}}\right\} \\ &\geq min \bigcup_{i=0}^{n-1} \{N'\left(\alpha(x, x, 0, \dots, 0), r\right)\} \\ &\geq N'\left(\alpha(x, x, 0, \dots, 0), r\right) \end{aligned} \tag{3.12}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^m x$ in (3.12) and using (3.1), $(F3)$, we obtain

$$N\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, \sum_{i=0}^{n-1} \frac{d^i\, r}{2^2 \cdot 2^{2i}}\right) \geq N'\left(\alpha(x, x, 0, \dots, 0), \frac{r}{d^m}\right) \tag{3.13}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing r by $d^m r$ in (3.13), we get

$$N\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, \sum_{i=m}^{m+n-1} \frac{d^i\, r}{2^2 \cdot 2^{2i}}\right) \geq N'\left(\alpha(x, x, 0, \dots, 0), r\right) \tag{3.14}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Using $(F3)$ in (3.14), we obtain

$$N\left(\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}, r\right) \geq N'\left(\alpha(x, x, 0, \ldots, 0), \frac{r}{\sum_{i=m}^{m+n-1} \frac{d^i}{2^2 \cdot 2^{2i}}}\right) \tag{3.15}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < d < 2^2$ and

$$\sum_{i=0}^{n} \left(\frac{d}{2^2}\right)^i < \infty,$$

the cauchy criterion for convergence and $(F5)$ implies that

$$\left\{\frac{f(2^n x)}{2^{2n}}\right\}$$

is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $Q(x) \in Y$. So one can define the mapping $Q : X \to Y$ by

$$Q(x) = N - \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$$

for all $x \in X$. Letting $m = 0$ in (3.15), we get

$$N\left(\frac{f(2^n x)}{2^{2n}} - f(x), r\right) \geq N'\left(\alpha(x, x, 0, \ldots, 0), \frac{r}{\sum_{i=0}^{n-1} \frac{d^i}{2^2 \cdot 2^{2i}}}\right) \tag{3.16}$$

for all $x \in X$ and all $r > 0$. Letting $n \to \infty$ in (3.16) and using $(F6)$, we arrive

$$N\left(f(x) - Q(x), r\right) \geq N'\left(\alpha(x, x, 0, \ldots, 0), r(2^2 - d)\right)$$

for all $x \in X$ and all $r > 0$. To prove Q satisfies the (1.3), replacing $(x_1, \ldots, x_n)$ by $(2^n x_1, \ldots, 2^n x_n)$ in (3.3), respectively , we obtain

$$N\left(\frac{1}{2^{2n}} Df(2^n x_1, \ldots, 2^n x_n), r\right) \geq N'\left(\alpha(2^n x_1, \ldots, 2^n x_n), 2^{2n} r\right) \tag{3.17}$$

for all $r > 0$ and all $x_1, \ldots, x_n \in X$. Now,

$$N\left(\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right) - (-n^2 + 6n - 4)\sum_{i=1}^{n} Q(x_i) - (n-4)\sum_{1\leq i<j\leq n} Q\left(x_i + x_j\right), r\right)$$

$$\geq min\left\{N\left(\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right) - \frac{1}{2^{2n}}\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} 2^n x_{ij}\right), \frac{r}{4}\right),\right.$$

$$N\left(-(-n^2 + 6n - 4)\sum_{i=1}^{n} Q(x_i) + \frac{1}{2^{2n}}(-n^2 + 6n - 4)\sum_{i=1}^{n} f(2^n x_i), \frac{r}{4}\right),$$

$$N\left(-(n-4)\sum_{1\le i<j\le n} Q\left(x_i+x_j\right)+\frac{1}{2^{2n}}(n-4)\sum_{1\le i<j\le n} f\left(2^n(x_i+x_j)\right),\frac{r}{4}\right),$$

$$N\left(\frac{1}{2^{2n}}\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} 2^n x_{ij}\right)-\frac{1}{2^{2n}}(-n^2+6n-4)\sum_{i=1}^{n} f(2^n x_i)\right.$$

$$\left.\left. -\frac{1}{2^{2n}}(n-4)\sum_{1\le i<j\le n} f\left(2^n(x_i+x_j)\right),\frac{r}{4}\right)\right\} \tag{3.18}$$

for all $x_1,\ldots,x_n \in X \in X$ and all $r>0$. Using (3.17) and $(F5)$ in (3.18), we arrive

$$\begin{aligned} N&\left(\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right)-(-n^2+6n-4)\sum_{i=1}^{n} Q(x_i)-(n-4)\sum_{1\le i<j\le n} Q\left(x_i+x_j\right),r\right)\\ &\ge min\left\{1,1,1,N'\left(\alpha(2^n x_1,\ldots,2^n x_n),2^{2n}r\right)\right\}\\ &\ge N'\left(\alpha(2^n x_1,\ldots,2^n x_n),2^{2n}r\right) \end{aligned} \tag{3.19}$$

for all $x_1,\ldots,x_n \in X \in X$ and all $r>0$. Letting $n\to\infty$ in (3.19) and using (3.2), we see that

$$N\left(\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right)-(-n^2+6n-4)\sum_{i=1}^{n} Q(x_i)-(n-4)\sum_{1\le i<j\le n} Q\left(x_i+x_j\right),r\right)=1$$

for all $x_1,\ldots,x_n \in X$ and all $r>0$. Using $(F2)$ in the above inequality gives

$$\sum_{i=1}^{n} Q\left(\sum_{j=1}^{n} x_{ij}\right)=(-n^2+6n-4)\sum_{i=1}^{n} Q(x_i)+(n-4)\sum_{1\le i<j\le n} Q\left(x_i+x_j\right)$$

for all $x_1,\ldots,x_n \in X$. Hence Q satisfies the quadratic functional equation (1.3). In order to prove $Q(x)$ is unique, we let $Q'(x)$ be another quadratic functional equation satisfying (1.3) and (3.5). Hence,

$$\begin{aligned} N(Q(x)-Q'(x),r) &= N\left(\frac{Q(2^n x)}{2^{2n}}-\frac{Q'(2^n x)}{2^{2n}},r\right)\\ &\ge min\left\{N\left(\frac{Q(2^n x)}{2^{2n}}-\frac{f(2^n x)}{2^{2n}},\frac{r}{2}\right),N\left(\frac{f(2^n x)}{2^{2n}}-\frac{Q'(2^n x)}{2^{2n}},\frac{r}{2}\right)\right\}\\ &\ge N'\left(\alpha(2^n x,2^n x,0,\ldots,0),\frac{r\,2^{2n}(2^2-d)}{2}\right)\\ &\ge N'\left(\alpha(x,x,0,\ldots,0),\frac{r\,2^{2n}(2^2-d)}{2d^n}\right) \end{aligned}$$

for all $x\in X$ and all $r>0$. Since

$$\lim_{n\to\infty}\frac{r\,2^{2n}(2^2-d)}{2d^n}=\infty,$$

we obtain

$$\lim_{n\to\infty} N'\left(\alpha(x,x,0,\ldots,0),\frac{r\,2^{2n}(2^2-d)}{2d^n}\right)=1.$$

Thus

$$N(Q(x) - Q'(x), r) = 1$$

for all $x \in X$ and all $r > 0$, hence $Q(x) = Q'(x)$. Therefore $Q(x)$ is unique.

For $\beta = -1$, we can prove the result by a similar method. This completes the proof of the theorem. □

From Theorem 3.1, we obtain the following corollaries concerning the Hyers-Ulam-Rassias and JMRassias stabilities for the functional equation (1.3).

Corollary 3.1. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, \dots, x_n), r\right) \geq N'\left(\epsilon \sum_{i=1}^{n} \|x_i\|^p, r\right) \tag{3.20}$$

for all $r > 0$ and all $x_1, \dots, x_n \in X$, where ϵ, p are constants with $\epsilon > 0$ and either $p < 2$ or $p > 2$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), r\right) \geq N'\left(\epsilon \|x\|^p, r|2^2 - 2^p|\right) \tag{3.21}$$

for all $x \in X$ and all $r > 0$.

Proof. If we define $\varphi(x_1, \dots, x_n) = \epsilon \sum_{i=1}^{n} \|x_i\|^p$, then the corollary is followed from Theorem 3.1 by $d = 2^p$ □

Corollary 3.2. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, \dots, x_n), r\right) \geq N'\left(\epsilon \left\{\prod_{i=1}^{n} \|x_i\|^p + \left(\epsilon \sum_{i=1}^{n} \|x_i\|^{np}\right)\right\}, r\right) \tag{3.22}$$

for all $r > 0$ and all $x, y \in X$, where ϵ, p are constants with $\epsilon > 0$ and either $p < \frac{1}{n}$ or $p > \frac{1}{n}$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), r\right) \geq N'\left(\epsilon \|x\|^{2p}, r|2^2 - 2^{np}|\right) \tag{3.23}$$

for all $x \in X$ and all $r > 0$.

Proof. If we define $\varphi(x_1, \dots, x_n) = \epsilon \left\{\prod_{i=1}^{n} \|x_i\|^p + \left(\epsilon \sum_{i=1}^{n} \|x_i\|^{np}\right)\right\}$, then the corollary is followed from Theorem 3.1 by $d = 2^{np}$. □

4. Conclusion

This article proves the solution to the quadratic functional equation in vector space and establishes key stability results. It addresses Hyers-Ulam stability for approximate solutions under small perturbations and explores Rassias stability with refined perturbation conditions. These results deepen the understanding of quadratic functional equations in fuzzy normed spaces and have significant implications for both theoretical research and practical applications in stability theory.

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