

## 2- $\mathcal{K}$ -Frames in 2-Hilbert Spaces

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**Abstract.** The aim of this article, we introduce a 2- $\mathcal{K}$ -frames in the 2-Hilbert space  $\mathcal{H}$ , where  $\mathcal{K}$  is a  $\theta$ -bounded linear operator on  $\mathcal{H}$  for a fixed element  $\theta$  in  $\mathcal{H}$ , and explore some properties of them. Also, we will characterize the 2- $\mathcal{K}$ -frames by the 2 pre- $\mathcal{K}$ -frame operators, the 2- $\mathcal{K}$ -frame operators, likewise to what is seen in the case of Hilbert spaces. In the rest of the article, we will set a definition of  $\theta$ -atomic systems for  $\mathcal{K}$ , and give some results concerning this notion. Finally we will find out a representation of each element in a closed range  $R(\mathcal{K})$ .

### 1. INTRODUCTION

Frames in Hilbert spaces have been introduced by Duffin and Schaeffer to study some deep problems in nonharmonic Fourier series in 1952 [5], reintroduced by Daubechies et al. [6], in 1986, and since then work on frames is going on. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [9], sampling ([7], [8]), coding and communications [14], filter bank theory [2], system modelling [15], and so on.

The concept of 2-inner product space was first introduced by Diminnie et al. [16]. A systematic presentation of the recent results related to this theory of 2-inner product space as well as an extensive list of the related references can be found in the book [3].

The 2 frames in 2-Hilbert space were introduced by Ali Akbar Arefijammaal and Ghadir Sadeghi [1], and some results on it are presented [13].

The notion of  $\mathcal{K}$ -frames was proposed by Găvruta to study the atomic systems with respect to a bounded linear operator  $\mathcal{K}$  on Hilbert spaces in [10], and its properties have been exposed in [16]. We can see easily that it generalizes the notion of frames in the sense that frames can be seen as  $\mathcal{K}$  frames by setting  $\mathcal{K}$  as the identity operator.

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The purpose of this article is to introduce the notion of a 2- $\mathcal{K}$ -frame in 2-Hilbert space, in the same way that the concept of  $\mathcal{K}$ -frame in Hilbert space following the approach developed in [16].

First of all we recall the basic definitions of 2-inner product spaces, 2-normed spaces, 2-Hilbert spaces, and some facts about 2-frames in 2-Hilbert spaces,  $\mathcal{K}$ -frames in Hilbert spaces where  $\mathcal{K}$  is a bounded linear operator, then try to define 2- $\mathcal{K}$ -frames in 2-Hilbert spaces where  $\mathcal{K}$  is a  $\theta$ -bounded linear operator on  $\mathcal{H}$  for a fixed element  $\theta$  in  $\mathcal{H}$ , we will give some characterizations of 2- $\mathcal{K}$ -frame by its 2 pre- $\mathcal{K}$ -frame and 2- $\mathcal{K}$ -frame operators, and finally give a representation of each element in a closed range  $R(\mathcal{K})$ .

## 2. PRELIMINARIES

Throughout this paper, we will adopt such notations:

- $\mathbb{K}$  denotes the field  $\mathbb{R}$  or  $\mathbb{C}$ .
- $(\cdot, \cdot)$  is the notation used for inner-product.
- $H$  is the notation used for a separable Hilbert space.
- $B(H)$  is a collection of all bounded linear operators from  $H$  to  $H$ .
- $R(K)$  and  $N(K)$  are the range and the kernel of  $K$  respectively.
- $K^+$  denotes the pseudo-inverse of  $K$ .
- $l_2(\mathbb{N})$  refers to the Hilbert space  $\{(u_n)_{n \in \mathbb{N}}; u_n \in \mathbb{C}, \sum_{n \in \mathbb{N}} |u_n|^2 < \infty\}$ .
- $\mathcal{X}$  is the notation used for linear space of dimension greater than 1 over the field  $\mathbb{K}$ .
- $(\cdot, \cdot|)$  is the notation used for 2-inner product on  $\mathcal{X}$ .
- $(\mathcal{H}, (\cdot, \cdot|))$  is the notation used for 2-Hilbert space.

Let us first introduce the basic definitions of 2-inner product spaces, 2-normed spaces from [1], [13], [12], and we review here the concepts of 2-frames for a 2-inner product space and some fundamental properties of them, from [1].

**Definition 2.1.** [1] Let  $\mathcal{X}$  be a linear space of dimension greater than 1 over the field  $\mathbb{K}$ .

A  $\mathbb{K}$  valued function  $(\cdot, \cdot|)$  defined on  $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$  is called a 2-inner product on  $\mathcal{X}$  if the following properties are verified

- $a_1)$   $(f, f|g) \geq 0$  and  $(f, f|g) = 0$  if and only if  $f, g$  are linearly dependent vectors.
- $a_2)$   $(f, f|g) = \overline{(g, g|f)}$  for all  $f, g \in \mathcal{X}$ .
- $a_3)$   $(h, f|g) = \overline{(f, h|g)}$  for all  $f, h, g \in \mathcal{X}$ .
- $a_4)$   $(\alpha f, h|g) = \alpha(f, h|g) \quad \forall \alpha \in \mathbb{K}$  and for all  $f, h, g \in \mathcal{X}$ .
- $a_5)$   $(f + f', h|g) = (f, h|g) + (f', h|g)$  for all  $f, f', h, g \in \mathcal{X}$ .

$(\mathcal{X}, (\cdot, \cdot|))$  is called a 2-inner product space.

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$|(f, h|g)|^2 \leq (f, f|g)(h, h|g) \quad \forall f, h, g \in \mathcal{X}. \quad (2,1)$$

**Definition 2.2.** [1] Let  $(\mathcal{X}, (\cdot, \cdot))$  be a 2-inner product space. A function  $\|\cdot, \cdot\|$  is defined on  $\mathcal{X} \times \mathcal{X}$  by

$$\|f, g\|^2 = (f, f|g), \quad \forall f, g \in \mathcal{X}. \quad (2.2)$$

and satisfies the following conditions (b1)-(b4) is called a 2-norm.

b<sub>1</sub>)  $\|f, g\| \geq 0$  and  $\|f, g\| = 0$  if and only if  $f, g$  are linearly dependent vectors.

b<sub>2</sub>)  $\|f, g\| = \|g, f\|$  for all  $f, g \in \mathcal{X}$ .

b<sub>3</sub>)  $\|\alpha f, g\| = |\alpha| \|f, g\| \quad \forall \alpha \in \mathbb{K}$  and for all  $f, g \in \mathcal{X}$ .

b<sub>4</sub>)  $\|f + f', g\| \leq \|f, g\| + \|f', g\|$  for all  $f, f', g \in \mathcal{X}$ .

The pair  $(\mathcal{X}, \|\cdot, \cdot\|)$  is called a linear 2-normed space.

Whenever a 2-inner product space  $(\mathcal{X}, (\cdot, \cdot))$  is given, we consider it as a linear 2-normed space  $(\mathcal{X}, \|\cdot, \cdot\|)$  with the 2-norm defined by (2.2).

**Lemma 2.3.** [1] Let  $(\mathcal{X}, (\cdot, \cdot))$  be a 2-inner product space and  $f, g \in \mathcal{X}$ . Then,

$$\|f, g\| = \sup\{|(f, h|g)|; h \in \mathcal{X}, \|h, g\| = 1\}. \quad (2.3)$$

**Example 1.** [1] If  $(\mathcal{X}, (\cdot, \cdot))$  is a an inner product space, the  $\mathbb{K}$  valued function defined on  $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$  by: for all  $f, h, g \in \mathcal{X}$

$$(f, h|g) = \begin{vmatrix} (f, h) & (f, g) \\ (g, h) & (g, g) \end{vmatrix} = (f, h)(g, g) - (g, h)(f, g). \quad (2.4)$$

is called the standard 2-inner product defined on  $\mathcal{X}$ .

We introduce the definition of a 2-Hilbert space from [1] as follows.

**Definition 2.4.** [1] Let  $(\mathcal{X}, (\cdot, \cdot))$  be a 2-inner product space,  $\theta$  is a fixed element in  $\mathcal{X}$ .

A sequence  $\{f_n\}_{n \in \mathbb{N}}$  of  $\mathcal{X}$  is said to be convergent if there exists an element  $f \in \mathcal{X}$  such that  $\lim_{n \rightarrow +\infty} \|f_n - f, \theta\| = 0$ , for all  $\theta \in \mathcal{X}$ .

A sequence  $\{f_n\}_{n \in \mathbb{N}}$  of  $\mathcal{X}$  is said to be a Cauchy sequence if for every  $\varepsilon > 0$  there exists  $N > 0$  such that for every  $m, n \geq N$ ,  $0 < \|f_n - f_m, \theta\| < \varepsilon$ , for all  $\theta \in \mathcal{X}$ .

A 2-inner product space  $\mathcal{X}$  is said a 2-Hilbert space if it is complete. That is, every Cauchy sequence in  $\mathcal{X}$  is convergent in this space.

**Definition 2.5.** [1] Let  $(\mathcal{H}, (\cdot, \cdot))$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ .

A sequence  $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$  is called a 2-frame associated to  $\theta$  if there exist two constants  $0 < E \leq F < \infty$  satisfying:

$$E\|f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(f, f_n|\theta)|^2 \leq F\|f, \theta\|^2, \quad \forall f \in \mathcal{H}. \quad (2.5)$$

The numbers  $E, F$  are called the lower and upper 2-frame bounds respectively.

A sequence  $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$  is called a tight 2-frame associated to  $\theta$  if  $E = F$ .

A sequence  $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$  is called a 2-Bessel sequence associated to  $\theta$  if only the right inequality of (2.5) holds.

For the rest of this paper,  $\mathcal{H}$  will denote 2-Hilbert space.  $\mathcal{L}_\theta$  denotes the subspace of  $\mathcal{H}$  generated by a fixed element  $\theta \in \mathcal{H}$ ,  $\mathcal{M}_\theta$  denotes the algebraic complement of  $\mathcal{L}_\theta$  in  $\mathcal{H}$  i.e  $\mathcal{L}_\theta \oplus \mathcal{M}_\theta = \mathcal{H}$ , and  $(\mathcal{M}_\theta, \|\cdot\|_\theta)$  is a norm space where,  $\|f\|_\theta = \sqrt{(f, f|\theta)}$ ,  $f \in \mathcal{M}_\theta$ .

We also denote by  $\mathcal{H}_\theta$  the completion of the inner product space  $\mathcal{M}_\theta$  See more Lemma 5.1.2 in [4].

By the below theorem, any question about 2-frames in a 2-Hilbert space can be solved as a question about frames in a Hilbert space.

**Theorem 2.6.** [1] Let  $(\mathcal{H}, (\cdot, \cdot))$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ . A sequence  $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$  is a 2-frame associated to  $\theta$  with bounds  $E$  and  $F$  if and only if  $\mathcal{F}$  is a frame for the Hilbert space  $\mathcal{H}_\theta$  with the same bounds.

Given a 2-Bessel sequence  $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$  associated to  $\theta$ , the 2-frame operator is defined by

$$S_\theta : \mathcal{H}_\theta \rightarrow \mathcal{H}_\theta, \quad \text{where}$$

$$S_\theta f = \sum_{n \in \mathbb{N}} (f, f_n|\theta) f_n, \quad \forall f \in \mathcal{H}_\theta.$$

This operator  $S_\theta$  is a bounded, invertible, and self-adjoint operator. The result leads to the following decomposition:

$$f = S_\theta S_\theta^{-1} f = \sum_{n \in \mathbb{N}} (S_\theta^{-1} f, f_n|\theta) f_n, \quad \forall f \in \mathcal{H}_\theta. \quad (2.6)$$

In the following it is necessary to recall the definition and some properties of K-frames in Hilbert spaces from [16].

**Definition 2.7.** [16] Let  $(H, (\cdot, \cdot))$  be a Hilbert space,  $K$  is a bounded linear operator on  $H$  as an element in  $B(H)$ , and  $K^*$  its adjoint operator. A sequence  $F = \{x_n\}_{n \in \mathbb{N}} \subseteq H$  is a K-frame if there exist  $0 < E \leq F < \infty$  satisfying:

$$E\|K^*x\|^2 \leq \sum_{n \in \mathbb{N}} |(x, x_n)|^2 \leq F\|x\|^2, \quad \forall x \in H. \quad (2.7)$$

The numbers  $E, F$  are called lower and upper K-frame bounds respectively.

$F = \{x_n\}_{n \in \mathbb{N}} \subseteq H$  is called a Bessel sequence if only the right inequality of (2.7) holds.

$F = \{x_n\}_{n \in \mathbb{N}} \subseteq H$  is just an ordinary frames for  $H$  if  $K = I_H$ , where  $I_H$  is the identity operator on  $H$ .

Given a  $K$ -frame  $F = \{x_n\}_{n \in \mathbb{N}}$  for  $H$  where  $K \in B(H)$ . The  $K$ -frame operator is defined by

$$S : H \rightarrow H, \quad \text{where}$$

$$Sx = \sum_{n \in \mathbb{N}} (x, x_n)x_n \quad \forall x \in H.$$

This frame operator of  $K$ -frame is not invertible in general, but when we suppose that  $K$  has a closed range, there exists  $K^+$  a pseudo-inverse of  $K$  such that

$$KK^+x = x, \quad \forall x \in R(K)$$

in which case

$$S : R(K) \longrightarrow S(R(K))$$

is a homeomorphism, furthermore, we have

$$\frac{1}{B}\|x\| \leq \|S^{-1}(x)\| \leq \frac{1}{A}\|K^+\|^2\|x\|, \quad \forall x \in S(R(K)).$$

In the following we will assume that  $R(K)$  is closed, since this can assure that the pseudo-inverse  $K^+$  of  $K$  exists.

**Definition 2.8.** [16] Let  $(H, (\cdot, \cdot))$  be a Hilbert space,  $K$  a bounded linear operator on  $H$  as an element in  $B(H)$ . A sequence  $F = \{x_n\} \subseteq H$  is called an atomic system for  $K$  if the following conditions are satisfied.

- 1)  $F = \{x_n\}_{n \in \mathbb{N}}$  is a Bessel sequence for  $H$ ;
- 2) For any  $x \in H$  there exists  $a_x = \{a_n\}_{n \in \mathbb{N}} \in l_2(\mathbb{N})$ ; where  $\|a_x\|_{l_2(\mathbb{N})} \leq C\|x\|$ , ( $C$  is a positive constant) such that

$$Kx = \sum_{n \in \mathbb{N}} a_n x_n.$$

**Proposition 2.9.** [16] Let  $(H, (\cdot, \cdot))$  be a Hilbert space,  $K \in B(H)$ , and a sequence  $F = \{x_n\}_{n \in \mathbb{N}} \subseteq H$ , then the following conditions are equivalent:

- (i)  $F = \{x_n\}_{n \in \mathbb{N}}$  is an atomic system for  $K$ ;
- (ii)  $F = \{x_n\}_{n \in \mathbb{N}}$  is a  $K$ -frame for  $H$ ;
- (iii) For any  $x \in H$  there exists  $\{y_n\}_{n \in \mathbb{N}} \subset H$  a Bessel sequence for  $H$  such that

$$Kx = \sum_{n \in \mathbb{N}} (x, y_n)x_n, \quad \forall x \in H.$$

Now we are ready to define 2- $\mathcal{K}$ -frames in 2-Hilbert space  $\mathcal{H}$  associated to  $\theta$ , where  $\theta$  as element fixed in  $\mathcal{H}$ , and  $\mathcal{K} \in B_\theta(\mathcal{H})$ . Then describe some fundamental properties of them.

### 3. MAIN RESULTS

In the first entry, we will introduce 2- $\mathcal{K}$ -frames and some basic facts about this concept. Then try to define their analysis operators and frame operators.

**3.1. 2- $\mathcal{K}$ -frames in 2-Hilbert space.** First we need to define  $B_\theta(\mathcal{H})$  the collection of all  $\theta$ -bounded linear operators on  $\mathcal{H}$ , where  $\theta$  is a fixed element in  $\mathcal{H}$  and state a representation of Riesz theorem in 2-Hilbert space.

**Defintion 3.1.1.** [11] Let  $\mathcal{H}$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ . A linear operator  $\mathcal{K}$  on  $\mathcal{H}$  is said a  $\theta$ -bounded operator if there exists  $M > 0$  such that

$$\|\mathcal{K}\|_\theta < M, \quad \text{where} \quad \|\mathcal{K}\|_\theta = \sup_{\|f, \theta\|=1} \{\|\mathcal{K}f, \theta\|\}, f \in \mathcal{H}.$$

We denote by  $B_\theta(\mathcal{H})$  the set of all  $\theta$ -bounded linear operators on  $\mathcal{H}$ .

To state the representation of Riesz theorem in 2-Hilbert space, we remind the definition of  $\theta$ -congruence as it was defined in [11].

**Defintion 3.1.2.** [11] Let  $\mathcal{X}$  be a vector space of dimension greater than 1 over  $\mathbb{K}$ ,  $\theta$  is a fixed element in  $\mathcal{X}$ , and  $x, y \in \mathcal{X}$ .  $x$  is said to be  $\theta$ -congruent to  $y$  if  $x - y \in \mathcal{L}_\theta$ , where  $\mathcal{L}_\theta$  is the subspace generated with  $\theta$ .

For the rest of this paper, We denote by  $\mathcal{I}$  the identity operator for  $\mathcal{H}$ , by  $R(\mathcal{K})$  and  $N(\mathcal{K})$  the range and the kernel of  $\mathcal{K}$  respectively, and by  $\mathcal{K}^+$  the pseudo-inverse of  $\mathcal{K}$  when  $R(\mathcal{K})$  is closed.

**Theoreme 3.1.3.** Let  $(\mathcal{H}, (., .|\theta))$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ . If  $\Gamma$  is a linear functional bounded on  $\mathcal{H}$ , then there exists a unique  $\rho \in \mathcal{H}$  up to  $\theta$ -congruence satisfying:

$$\Gamma(f) = (f, \rho|\theta) \quad \text{with} \quad \|\Gamma\|_\theta = \sup_{\|f, \theta\|=1} \{|\Gamma(f)|\}, f \in \mathcal{H}.$$

**Proof.** We pose that  $\mathcal{N} = N(\Gamma)$  the kernel of  $\Gamma$ ,  $\mathcal{N}^{\perp_\theta} = \{f \in \mathcal{H}; (f, g|\theta) = 0, \forall g \in \mathcal{N}\}$ .

\* If  $\Gamma \neq 0$ , there exists an element  $\rho \in \mathcal{N}^{\perp_\theta}$  such that  $\Gamma(\rho) = 1, \|\rho, \theta\| = 1$ .

For any  $f \in \mathcal{H}$ , we can express as  $f = (\Gamma(f)\rho + U)$ , where  $U = (f - \Gamma(f)\rho) \in \mathcal{N}$ .

Thus, we have

$$\Gamma(f) = (f, \rho|\theta) \quad \forall f \in \mathcal{H}.$$

If we assume that there is another  $\rho'$  such that  $\Gamma(f) = (f, \rho|\theta) = (f, \rho'|\theta) \quad \forall f \in \mathcal{H}$ .

Then,  $(\rho - \rho', \rho - \rho'|\theta) = 0$  Hence,  $\rho - \rho' \in \mathcal{L}_\theta$ .

\* If  $\Gamma = 0$  then, it follows that

$$\Gamma(f) = (f, 0|\theta) \quad \forall f \in \mathcal{H}.$$

**Corollary 3.1.4.** Let  $(\mathcal{H}, (\cdot, \cdot)_\theta)$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ . For every  $\theta$ -bounded linear operator  $\mathcal{K}$  as an element in  $B_\theta(\mathcal{H})$ , there exists  $\mathcal{K}^*$  the adjoint operator of  $\mathcal{K}$  such that:

$$(f, \mathcal{K}^*g)_\theta = (\mathcal{K}f, g)_\theta \quad \forall f \in \mathcal{H} \quad \forall g \in \mathcal{H}, \text{ and } \|\mathcal{K}^*\|_\theta = \|\mathcal{K}\|_\theta.$$

*Proof.* For each element  $g$  fixed in  $\mathcal{H}$ , just tack a linear functional bounded defined by

$$\varphi_g : \mathcal{H} \rightarrow \mathbb{C}, \quad \varphi_g(f) = (\mathcal{K}f, g)_\theta \quad \forall f \in \mathcal{H}.$$

According (Theorem 3.1.3.), there exists a unique element  $h_g \in \mathcal{H}$  up to  $\theta$ -congruence such that

$$(\mathcal{K}f, g)_\theta = (f, h_g)_\theta \quad \forall f \in \mathcal{H}.$$

Consider the application  $\mathcal{K}^*$  defined as follows:

$$\mathcal{K}^* : \mathcal{H} \rightarrow \mathcal{H}, \quad \mathcal{K}^*(g) = h_g.$$

$\mathcal{K}^*$  is a linear operator due to the uniqueness of  $h_g$ .

Given that  $\mathcal{K}$  is  $\theta$ -bounded with,

$$\|\mathcal{K}\|_\theta = \sup\{\|(\mathcal{K}f, \theta)\|, f \in \mathcal{H}; \|f, \theta\| = 1\}, \quad \|\mathcal{K}^*\|_\theta = \sup\{\|(\mathcal{K}^*(g), \theta)\|, g \in \mathcal{H}; \|g, \theta\| = 1\}.$$

Since  $\forall f, g \in \mathcal{H}$ ,  $(\mathcal{K}f, g)_\theta = (f, \mathcal{K}^*g)_\theta$  by applying (2,3), we find that  $\|\mathcal{K}^*\|_\theta = \|\mathcal{K}\|_\theta$ . Then,  $\mathcal{K}^*$  is a  $\theta$ -bounded linear operator satisfying:

$$(\mathcal{K}f, g)_\theta = (f, \mathcal{K}^*g)_\theta \quad \forall f \in \mathcal{H}, \quad \forall g \in \mathcal{H}, \quad \|\mathcal{K}^*\|_\theta = \|\mathcal{K}\|_\theta.$$

$\mathcal{K}^*$  is a  $\theta$ -bounded linear operator known as the adjoint of  $\mathcal{K}$ .

**Definition 3.1.5.** Let  $(\mathcal{H}, (\cdot, \cdot)_\theta)$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ , and  $\mathcal{K} \in B_\theta(\mathcal{H})$ . A sequence  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$  is called 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$ , if there exist two constants  $0 < E \leq F < \infty$  satisfying:

$$E\|\mathcal{K}^*f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(f, k_n)_\theta|^2 \leq F\|f, \theta\|^2, \quad \forall f \in \mathcal{H}. \quad (3.1)$$

$E$  and  $F$  are the lower and upper 2- $\mathcal{K}$ -frame bounds respectively.

$\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is called a 2-frame for  $\mathcal{H}$  associated to  $\theta$  if  $\mathcal{K} = \mathcal{I}$ .

$\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is called a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$  if only the right inequality of (3.1) holds.

**Proposition 3.1.6.** Let  $(\mathcal{H}, (\cdot, \cdot)_\theta)$  be a 2-Hilbert space,  $\theta$  a fixed element in  $\mathcal{H}$ ,  $\mathcal{K} \in B_\theta(\mathcal{H})$ , and  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$ . For all  $T \in B_\theta(\mathcal{H})$ , the sequence  $\{Tk_n\}_{n \in \mathbb{N}}$  is also 2-T $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$ .

**Proof.** Since  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$ . Applying (3,1) to  $T^*f$  as an element of  $\mathcal{H}$  for any  $f$  in  $\mathcal{H}$ , then there exist  $0 < E \leq F < \infty$  such that

$$E\|\mathcal{K}^*(T^*f), \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(T^*f, k_n|\theta)|^2 \leq F\|T^*f, \theta\|^2, \quad \forall f \in \mathcal{H}.$$

Hence,

$$E\|(T\mathcal{K})^*f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(f, Tk_n|\theta)|^2 \leq F\|T\|_\theta^2\|f, \theta\|^2, \quad \forall f \in \mathcal{H}.$$

Which shows that  $\{Tk_n\}_{n \in \mathbb{N}}$  is a 2- $T\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$ , with lower and upper 2- $T\mathcal{K}$ -frame bounds  $E, F\|T\|_\theta^2$  respectively.

As a consequence of above proposition we have the following proposition.

**Proposition 3.1.7.** Let  $(\mathcal{H}, (\cdot, \cdot)_\theta)$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ . Every 2-frame  $\{f_n\}_{n \in \mathbb{N}}$  for  $\mathcal{H}$  associated to  $\theta$ ,  $\{\mathcal{K}f_n\}_{n \in \mathbb{N}}$  is a 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$  for all  $\mathcal{K} \in B_\theta(\mathcal{H})$ .

**Example 2.**

Suppose that the Hilbert space  $H = \mathbb{C}^3$ , and suppose that  $\{e_n\}_{n=1}^3$  be the orthonormal basis for  $H$ ,  $\theta_0 = (1, 0, 0) \in H$  where,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let  $F = \{e_n\}_{n=1}^3 = \{e_1, e_2, e_3\}$ . Obviously the sequence  $F$  is 2-frame for the standad 2-inner product space  $H$  associated to  $\theta_0$ . Indeed  $\forall f \in \mathbb{C}^3$ ,  $f = xe_1 + ye_2 + ze_3$ , by a simple calculation we have

$$\|f, \theta_0\|^2 = y^2 + z^2, \quad \sum_{n=1}^3 |(f, f_n|\theta_0)|^2 = y^2 + z^2$$

Now define  $\mathcal{K}$  a  $\theta_0$ -bounded linear operator as follows:

$$\mathcal{K} : H \rightarrow H, \quad \mathcal{K}(e_1) = e_1, \mathcal{K}(e_2) = e_1, \mathcal{K}(e_3) = e_2, \quad \|\mathcal{K}\|_{\theta_0} \leq 1.$$

Indeed  $\forall f \in \mathbb{C}^3$ ,  $\mathcal{K}f = (x + y)e_1 + ze_2$ , by a simple calculation we have  $\|\mathcal{K}f, \theta_0\|^2 = z^2$ .

According to the (Proposition 3.1.7.) a sequence  $\mathcal{F} = \{\mathcal{K}e_n\}_{n=1}^3 = \{k_1, k_2, k_3\}$  is a 2- $\mathcal{K}$ -frame for the standard 2-inner product  $H$  associated to  $\theta_0$ .

The converse of the (Proposition 3.1.7.) is not true. In fact, by the following proposition, for all  $\mathcal{K} \in B_\theta(\mathcal{H})$  with a closed range  $R(\mathcal{K})$ , every 2- $\mathcal{K}$ -frame is a 2-frame for  $R(\mathcal{K})$ .

**Proposition 3.1.8.** Let  $(\mathcal{H}, (\cdot, \cdot)_\theta)$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ , and  $\mathcal{K} \in B_\theta(\mathcal{H})$  with a closed range  $R(\mathcal{K})$ . Every 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$  is a 2-frame for  $R(\mathcal{K})$  associated to  $\theta$ .

*Proof.* Suppose that  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$  with frame bounds E and F, by (3,1) we have

$$E\|\mathcal{K}^*f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(f, k_n)_\theta|^2 \leq F\|f, \theta\|^2, \quad \forall f \in \mathcal{H}.$$

Since  $R(\mathcal{K})$  is closed, there exists a pseudo-inverse  $\mathcal{K}^+$  of  $\mathcal{K}$  such that  $\mathcal{K}\mathcal{K}^+f = f \quad \forall f \in R(\mathcal{K})$ , namely  $\mathcal{K}(\mathcal{K}^+|R(\mathcal{K})) = I_{R(\mathcal{K})}$  so we have  $(I_{R(\mathcal{K})})^* = (\mathcal{K}^+|R(\mathcal{K}))^*\mathcal{K}^*$ .

Hence, we obtain

$$\|f, \theta\| = \|(\mathcal{K}^+|R(\mathcal{K}))^*\mathcal{K}^*f, \theta\| \leq \|\mathcal{K}^+\|_\theta \|\mathcal{K}^*f, \theta\|, \quad \forall f \in R(\mathcal{K}).$$

That is

$$\|\mathcal{K}^+\|_\theta^{-2} \|f, \theta\|^2 \leq \|\mathcal{K}^*f, \theta\|^2, \quad \forall f \in R(\mathcal{K}).$$

By combining the two inequalities above we obtain that the following relationship:

$$E\|\mathcal{K}^+\|_\theta^{-2} \|f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(f, k_n)_\theta|^2 \leq F\|f, \theta\|^2, \quad \forall f \in R(\mathcal{K}).$$

Which shows that  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2-frame for the inner-product space  $R(\mathcal{K})$  with the frame bounds  $E\|\mathcal{K}^+\|_\theta^{-2}$ , F.

In the following we will assume that  $R(\mathcal{K})$  is closed, since this can assure that the pseudo-inverse  $\mathcal{K}^+$  of  $\mathcal{K}$  exists.

**Definition 3.1.9.** Let  $(\mathcal{H}, (\cdot, \cdot)_\theta)$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ , and  $\mathcal{K} \in B_\theta(\mathcal{H})$ . A sequence  $\mathcal{F} = \{k_n\} \subseteq \mathcal{H}$  is called a  $\theta$ -atomic system for  $\mathcal{K}$  if the following conditions are satisfied.

- 1)  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$ ;
- 2) For any  $f \in \mathcal{H}_\theta$  there exists  $a_f = \{a_n\}_{n \in \mathbb{N}} \in l_2(\mathbb{N})$ ; Where  $\|a_f\|_{l_2(\mathbb{N})} \leq C\|f, \theta\|$ , (C is a positive constant), satisfying:

$$\mathcal{K}f = \sum_{n \in \mathbb{N}} a_n k_n.$$

**3.2. Operators for 2- $\mathcal{K}$ -frames.** This section is devoted to establishing 2 pre-frame, 2-frame operator for a 2- $\mathcal{K}$ -frame, where  $\mathcal{K} \in B_\theta(\mathcal{H})$ , and to determine a representation of each element f in a closed subspace  $R(\mathcal{K})$ .

**Theorem 3.2.1.** Let  $(\mathcal{H}, (\cdot, \cdot))$  be a 2-Hilbert space,  $\theta$  is a fixed element in  $\mathcal{H}$ , and  $\mathcal{K} \in B_\theta(\mathcal{H})$ . A sequence  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2- $\mathcal{K}$ -frame for  $(\mathcal{H}, (\cdot, \cdot))$  associated to  $\theta$  if and only if  $\mathcal{F}$  is a  $\mathcal{K}$ -frame for the Hilbert space  $(\mathcal{H}_\theta, (\cdot, \cdot)_\theta)$  with the same bounds.

*Proof.*  $\mathcal{H}_\theta$  is the completion of the inner product space  $(\mathcal{M}_\theta, (\cdot, \cdot)_\theta)$ . See more Lemma 5.1.2 in [4].

$\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$ . Applying (3,1) we have

$$E\|\mathcal{K}^*f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(f, k_n \theta)|^2 \leq F\|f, \theta\|^2, \quad \forall f \in \mathcal{H}.$$

Since  $\forall f \in \mathcal{H} \quad \|\mathcal{K}^*(f + \mathcal{L}_\theta), \theta\| = \|\mathcal{K}^*f, \theta\|$ . The above inequality is equivalent to

$$E\|\mathcal{K}^*f\|_\theta^2 \leq \sum_{n \in \mathbb{N}} |(f, k_n \theta)|^2 \leq F\|f\|_\theta^2, \quad \forall f \in \mathcal{H}_\theta.$$

By the above theorem ensures that, every question about 2- $\mathcal{K}$ -frames for 2-Hilbert space  $(\mathcal{H}, (\cdot, \cdot))$  associated to  $\theta$  can be solved as a question about  $\mathcal{K}$ -frames for the Hilbert space  $(\mathcal{H}_\theta, (\cdot, \cdot)_\theta)$ .

**Theorem 3.2.2.** Let  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  be a 2- $\mathcal{K}$ -frame for a 2-Hilbert space  $\mathcal{H}$  associated to  $\theta$  as an element fixed in  $\mathcal{H}$  with frame bounds  $E$  and  $F$ , where  $\mathcal{K} \in B_\theta(\mathcal{H})$ . Then for  $\mathcal{F}$  the 2-pre- $\mathcal{K}$ -frame operator or the 2- $\mathcal{K}$ -synthesis operator.

$\mathcal{T}_\theta : l_2(\mathbb{N}) \rightarrow \mathcal{H}_\theta$ , where

$$\mathcal{T}_\theta(\{a_n\}_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} a_n k_n$$

is well defined and bounded.

*Proof.* Let  $\{a_n\}_{n \in \mathbb{N}} \in l_2(\mathbb{N})$ , by (2,3) for  $m < n$ ;

$$\begin{aligned} \left\| \sum_{i=0}^n a_i k_i - \sum_{i=0}^m a_i k_i \right\|_\theta &= \left\| \sum_{i=m+1}^n a_i k_i \right\|_\theta \\ &= \sup_{\|h, \theta\|=1} \left\{ \left| \sum_{i=m+1}^n a_i k_i, h \theta \right| \right\}, \quad h \in \mathcal{H} \\ &= \left( \sum_{i=m+1}^n |a_i|^2 \right)^{\frac{1}{2}} \sup_{\|h, \theta\|=1} \left\{ \left( \sum_{i=m+1}^n |(k_i, h \theta)|^2 \right)^{\frac{1}{2}} \right\}, \quad h \in \mathcal{H} \\ &\leq \sqrt{F} \left( \sum_{i=m+1}^n |a_i|^2 \right)^{\frac{1}{2}} \end{aligned}$$

this implies that  $\sum_{n \in \mathbb{N}} a_n k_n$  is well defined as an element of  $\mathcal{H}_\theta$ . Moreover, for each  $\{a_n\}_{n \in \mathbb{N}}$  a sequence in  $l_2(\mathbb{N})$ , the argument as above shows that

$\|\mathcal{T}_\theta\{a_n\}_{n \in \mathbb{N}}\|_\theta \leq \sqrt{F}\|\{a_n\}_{n \in \mathbb{N}}\|_{l_2}$ . In particular,  $\|\mathcal{T}_\theta\| \leq \sqrt{F}$  i.e.  $\mathcal{T}_\theta$  is bounded. The adjoint operator of  $\mathcal{T}_\theta$  is given by

$$\mathcal{T}_\theta^* : \mathcal{H}_\theta \rightarrow l_2(\mathbb{N}),$$

$$\mathcal{T}_\theta^*(f) = \{(f, k_n|\theta)\}_{n \in \mathbb{N}}, \quad \forall f \in \mathcal{H}_\theta.$$

$\mathcal{T}_\theta^*$  is called the 2- $\mathcal{K}$ -analysis operator for  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$ .

Clearly,  $\mathcal{T}_\theta^*$  is well defined and bounded operator. Theremore, it follows by (3.1)  $\|\mathcal{T}_\theta^*\| \leq \sqrt{F}$ .

**Definition 3.2.3.** Let  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  be a 2- $\mathcal{K}$ -frame for a 2-Hilbert space  $\mathcal{H}$  associated to  $\theta$  as an element fixed in  $\mathcal{H}$  with frame bounds E and F, where  $\mathcal{K} \in B_\theta(\mathcal{H})$ . Then by composing  $\mathcal{T}_\theta$  with its adjoint  $\mathcal{T}_\theta^*$ , we obtain the 2- $\mathcal{K}$ -frame operator  $\mathcal{S}_\theta = \mathcal{T}_\theta\mathcal{T}_\theta^*$  for  $\mathcal{F}$  defined as:

$$\mathcal{S}_\theta : \mathcal{H}_\theta \rightarrow \mathcal{H}_\theta, \quad \text{where}$$

$$\mathcal{S}_\theta f = \sum_{n \in \mathbb{N}} (f, k_n|\theta)k_n \quad \forall f \in \mathcal{H}_\theta.$$

Clearly,  $\mathcal{S}_\theta = \mathcal{T}_\theta\mathcal{T}_\theta^*$  is bounded. Moreover  $\|\mathcal{S}_\theta\| \leq F$ .

Furthermore, by  $\mathcal{S}_\theta = \mathcal{T}_\theta\mathcal{T}_\theta^*$  we can easily show that,

$$(\mathcal{S}_\theta f, f|\theta) = \sum_{n \in \mathbb{N}} |(f, k_n|\theta)|^2. \quad (3.2)$$

By a simple calculation we have

$$\forall f \in \mathcal{H}_\theta \quad E\|\mathcal{K}^*f\|_\theta^2 = (E\mathcal{K}\mathcal{K}^*f, f|\theta), \quad F\|f\|_\theta^2 = (Ff, f|\theta).$$

Combining (3.1) and (3,2) we have

$$(E\mathcal{K}\mathcal{K}^*f, f|\theta) \leq (\mathcal{S}_\theta f, f|\theta) \leq (Ff, f|\theta), \quad \forall f \in \mathcal{H}_\theta.$$

It follows that

$$E\mathcal{K}\mathcal{K}^* \leq \mathcal{S}_\theta \leq F\mathcal{I}_{\mathcal{H}_\theta}.$$

In general the frame operator of 2- $\mathcal{K}$ -frame is note inversible on  $\mathcal{H}_\theta$ .

**Proposition 3.2.4.** Let  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  be a 2- $\mathcal{K}$ -frame for a 2-Hilbert space  $\mathcal{H}$  associated to  $\theta$  as an element fixed in  $\mathcal{H}$  with frame bounds E and F, where  $\mathcal{K} \in B_\theta(\mathcal{H})$ . If the range  $R(\mathcal{K})$  of  $\mathcal{K}$  is closed, then the 2-frame operator,

$$\mathcal{S}_\theta : \mathcal{K}(\mathcal{H}_\theta) \rightarrow \mathcal{S}_\theta(\mathcal{K}(\mathcal{H}_\theta))$$

is a homeomorphism.

**Proof.** Since  $R(\mathcal{K})$  is closed, there exists a pseudo-inverse  $\mathcal{K}^+$  of  $\mathcal{K}$  such that

$\mathcal{K}\mathcal{K}^+f = f \quad \forall f \in R(\mathcal{K})$ , namely  $\mathcal{K}(\mathcal{K}^+|R(\mathcal{K})) = I_{R(\mathcal{K})}$  so we have  $(I_{R(\mathcal{K})})^* = (\mathcal{K}^+|R(\mathcal{K}))^*\mathcal{K}^*$ . Hence, we obtain

$$\|f, \theta\| = \|(\mathcal{K}^+|\mathcal{K}(\mathcal{H}_\theta))^*\mathcal{K}^*f, \theta\| \leq \|\mathcal{K}^+\|_\theta \|\mathcal{K}^*f\|_\theta, \quad \forall f \in K(\mathcal{H}_\theta)$$

that is

$$\|\mathcal{K}^+\|_\theta^{-2} \|f, \theta\|^2 \leq \|\mathcal{K}^*f, \theta\|^2, \quad \forall f \in \mathcal{K}(\mathcal{H}_\theta)$$

combined with (3,1), we get

$$E\|\mathcal{K}^+\|_\theta^{-2} \|f, \theta\|^2 \leq (\mathcal{S}_\theta f, f, \theta) \leq F\|f, \theta\|^2, \quad \forall f \in \mathcal{K}(\mathcal{H}_\theta).$$

Given a definition of 2- $\mathcal{K}$ -frame we have

$$E\|\mathcal{K}^+\|_\theta^{-2} \|f\|_\theta \leq \|\mathcal{S}_\theta f, \theta\| \leq F\|f\|_\theta, \quad \forall f \in \mathcal{K}(\mathcal{H}_\theta).$$

Which implies that  $S : \mathcal{K}(\mathcal{H}_\theta) \rightarrow S(\mathcal{K}(\mathcal{H}_\theta))$  is a homomorphism, furthermore, we have

$$\frac{1}{F}\|f\|_\theta \leq \|\mathcal{S}_\theta^{-1}(f), \theta\| \leq \frac{1}{E}\|\mathcal{K}^+\|_\theta^2 \|f\|_\theta, \quad \forall f \in S(\mathcal{K}(\mathcal{H}_\theta)).$$

**Proposition 3.2.5.** Let a sequence  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}$ ,  $\theta$  is a fixed element in  $\mathcal{H}$ , and  $\mathcal{K} \in B_\theta(\mathcal{H})$ . Then the following statements are equivalent

- i)  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a  $\theta$ -atomic system for  $\mathcal{K}$ ;
- ii)  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$ ;
- iii)  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$  and there exists a 2-Bessel sequence  $\{g_n\}_{n \in \mathbb{N}}$  for  $\mathcal{H}$  associated to  $\theta$  satisfying:

$$\forall f \in \mathcal{H}_\theta \quad \mathcal{K}f = \sum_{n \in \mathbb{N}} (f, g_n|\theta) k_n.$$

*Proof.* (i)  $\implies$  (ii)

Since  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a  $\theta$ -atomic system for  $\mathcal{K}$ ,

$$\forall f \in \mathcal{H}_\theta \quad \|\mathcal{K}^*f, \theta\| = \sup_{\|g, \theta\|=1} \{|(\mathcal{K}^*f, g|\theta)|, g \in \mathcal{H}_\theta\} = \sup_{\|g, \theta\|=1} \{|(f, \mathcal{K}g|\theta)|, g \in \mathcal{H}_\theta\}.$$

And by taking  $g$  instead of  $f$  in the condition (ii) of (Definition 3.1.9.) we obtain that

$$\mathcal{K}g = \sum_{n \in \mathbb{N}} a_n k_n.$$

Then

$$\begin{aligned} \|\mathcal{K}^*f, \theta\| &= \sup_{\|g, \theta\|=1} \{|(f, \sum_{n \in \mathbb{N}} a_n k_n|\theta)|, g \in \mathcal{H}_\theta\} \\ &= \sup_{\|g, \theta\|=1} \{| \sum_{n \in \mathbb{N}} \bar{a}_n (f, k_n|\theta)|, g \in \mathcal{H}_\theta\} \\ &\leq \left( \sum_{n \in \mathbb{N}} |a_n|^2 \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{N}} |(f, k_n|\theta)|^2 \right)^{\frac{1}{2}} \\ &\leq C \sup_{\|g, \theta\|=1} \|g, \theta\| \left( \sum_{n \in \mathbb{N}} |(f, k_n|\theta)|^2 \right)^{\frac{1}{2}} \end{aligned}$$

which proves that there exists a positive constant  $E = \frac{1}{C^2}$  such that

$$\frac{1}{C^2} \|\mathcal{K}^* f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(f, k_n | \theta)|^2, \quad \forall f \in \mathcal{H}_\theta.$$

Since  $\forall f \in \mathcal{H} \quad \|\mathcal{K}^*(f + \mathcal{L}\theta), \theta\| = \|\mathcal{K}^* f, \theta\|$ . The above inequality is equivalent to

$$\frac{1}{C^2} \|\mathcal{K}^* f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |(f, k_n | \theta)|^2, \quad \forall f \in \mathcal{H}.$$

Moreover,  $\mathcal{F}$  is a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$ . Then  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$ .

(ii)  $\implies$  (iii)

If  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2- $\mathcal{K}$ -frame for  $\mathcal{H}$  associated to  $\theta$  for  $\mathcal{H}$ , then  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a  $\mathcal{K}$ -frame for  $\mathcal{H}_\theta$  associated to  $\theta$ .

According to the (proposition 3.1.) [16], there exists  $\{g_n\}_{n \in \mathbb{N}}$  a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$  such that:

$$Kf = \sum_{n \in \mathbb{N}} (f, g_n | \theta) k_n, \quad \forall f \in \mathcal{H}_\theta.$$

(iii)  $\implies$  (i)

Let  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$ .

According to (iii) there exists  $\{g_n\}_{n \in \mathbb{N}}$  a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$  such that

$$\mathcal{K}f = \sum_{n \in \mathbb{N}} (f, g_n | \theta) k_n, \quad \forall f \in \mathcal{H}_\theta$$

since  $\{g_n\}_{n \in \mathbb{N}}$  be a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$  there exist a constant  $F > 0$  satisfying:

$$\sum_{n \in \mathbb{N}} |(f, g_n | \theta)|^2 \leq F \|f, \theta\|^2 \quad \forall f \in \mathcal{H}.$$

We pose

$$a_n = (f, g_n | \theta) \quad \forall n \in \mathbb{N}.$$

Hence, there exist  $a_f = \{a_n\}_{n \in \mathbb{N}} \in l^2(\mathbb{N})$  Where  $\|a_f\|_{l^2(\mathbb{N})} \leq \sqrt{F} \|f, \theta\|$ , ( $\sqrt{F}$  is a positive constant), satisfying:

$$\mathcal{K}f = \sum_{n \in \mathbb{N}} a_n k_n.$$

We conclude that  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  is a  $\theta$ -atomic system for  $\mathcal{K}$ .

**Proposition 3.2.6.** Let  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  be a 2- $\mathcal{K}$ -frame for a 2-Hilbert space  $\mathcal{H}$  associated to  $\theta$  as an element fixed in  $\mathcal{H}$  with frame bounds E and F, where  $\mathcal{K} \in B_\theta(\mathcal{H})$ . There exists  $\{g_n\}_{n \in \mathbb{N}}$  a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$  such that

$$\forall f \in \mathcal{H}_\theta, \quad Kf = \sum_{n \in \mathbb{N}} (f, g_n | \theta) k_n.$$

**Remark.**

Let  $\mathcal{F} = \{k_n\}_{n \in \mathbb{N}}$  be a 2- $\mathcal{K}$ -frame for a 2-Hilbert space  $\mathcal{H}$  associated to  $\theta$  as an element fixed in  $\mathcal{H}$  with frame bounds E and F, where  $\mathcal{K} \in B_\theta(\mathcal{H})$ . There exist  $\{g_n\}_{n \in \mathbb{N}}$  a 2-Bessel sequence for  $\mathcal{H}$  associated to  $\theta$  such that

$$\forall f \in \mathcal{H}, \quad \mathcal{K}f = \alpha \mathcal{K}\theta + \sum_{n \in \mathbb{N}} (f, g_n | \theta) k_n;$$

for some  $\alpha \in \mathbb{K}$

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