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Generalization of Kodaira's Embedding Theorem for Compact Kähler Manifolds with Semi-Positive Chern Class

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Abstract. Kodaira embedding theorem states that a compact complex manifold can only be embedded into a complex projective space P^N for some N if it admits a positive line bundle. Based on Chow's theorem, projective algebraic manifolds are algebraic. The purpose of this study is to generalize Kodaira's known theorem. We show that if X is a compact Kähler manifold of complex dimension n and its first Chern class is semi-positive and of rank $n-1$ at one point of X, then X can be embedded into P^N for some integer N. Furthermore, we prove that this embedding is unique up to biholomorphism. We also show that X admits a Kähler metric whose Ricci curvature is bounded from below.

1. Introduction

The Kodaira embedding theorem is a fundamental result in complex geometry, providing a criterion for determining whether a compact complex manifold can be embedded into a complex projective space. The theorem states that a compact complex manifold admits such an embedding if it possesses a positive holomorphic line bundle [1]. Alongside this, Chow's theorem establishes that projective algebraic manifolds are inherently algebraic [2], enabling the translation of complex analytic problems into algebraic geometry. These classical results form the basis for understanding the interplay between differential geometry and algebraic geometry.

According to the Kodaira embedding theorem, a compact complex manifold can only be embedded into a complex projective space P^N for some N if it admits a positive line bundle. According to Chow's theorem, projective algebraic manifolds are algebraic, that is, they are defined by the zeros of homogeneous polynomials. Thus, using the embedding theorem along with Chow's theorem,

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one can convert analysis problems into algebraic ones [2]. Theorems by Kodaira and Chow allow for the translation of a complex analysis question into the language of algebraic geometry. For line bundles that are not strictly positive, Siu, Demailly, Riemanschneider, and Abdelkader have studied vanishing and embedding theorems. To be more specific, Riemenschneider [3] extended the Kodaira embedding theorem to semi-positive line bundles for Kähler manifolds in his 1965 paper, Riemenschneider [3]. The Grauert-Riemenschneider conjecture was solved by Siu [4] in 1984 who generalized the Kodaira embedding theorem to semi-positive line bundles for non-Kähler manifolds. In 1985 Demailly [5], generalized the theorem to a line bundle whose curvature might be slightly negative. Abdelkader [6] in 1998 proved that the assumption on the line bundle may be weakened if it is replaced by the canonical line bundle K Additionally, the Kodaira vanishing and embedding theorem was generalized by Zhu and Stefano Mammola in their works (Zhu [7] and Mammola [8]). For the results on the closed range for $\overline{\partial}$ on non-pseudoconvex domains, see [9]- [19]. While Kodaira's theorem depends on the fact that the first Chern class $c_1(E)$ is positive, Stefano Mammola [8] was concerned on line bundles with $c_1(E) = 0$. Recently X. Yang [20] proved that any compact Kähler manifold with positive holomorphic sectional curvature must be projective. This gives a metric criterion of the projectivity in terms of its curvature. While Stefano Mammola [8] was interested in line bundles with $c_1(E) = 0$, Kodaira's theorem depends on the first Chern class $c_1(E)$ being positive. Similar results for this problem were considered in [21]- [37].

Our main aim is to show the embedding of some compact Kähler manifolds into a complex projective space P^N for some integer N. That is, we provide a new extension of the Kodaira vanishing and embedding theorem using a different approach of Abdelkader [6]. We demonstrate that if X is a compact Kähler manifold of complex dimension n and if its first Chern class is semi-positive at any point of X and of rank $n-1$ at one point of X, then X is a Hodge manifold. To do this, we use the results of Girbau [38], [39] and Theorem 8.3 in [40]. As a result, X is projective algebraic and can be embedded into P^N for some integer N using the Kodaira [1] technique. Theorem 4.1 contains the paper's original contribution (and Corollary 1,2).

2. Methods

2.1. **First Chern class of Kähler manifolds.** The purpose of this subsection is to show that the curvature form of the dual of the canonical line bundle K^* is the first Chern class of a compact Kähler manifold X. Assuming X is a complex manifold with complex dimension n, $\{U_j\}_{j\in J}$ is an open covering of X consisting of coordinate neighborhoods U_j with holomorphic coordinates $(z_j^1, z_j^2, \ldots, z_j^n)$.

A holomorphic tangent vector at $z \in U_j$ is an element of the form $v = \sum_{\alpha=1}^{n} \eta_j^\alpha \frac{\partial}{\partial z_j^\alpha}$, where η_j^α are C^∞ functions on U_j. The set T_zX of all complex tangent vectors at z is a complex vector space $T_zX \cong \mathbb{C}^n$. $TX = \cup_{z\in X} T_zX$ is called the complex tangent bundle of X and $\overline{TX}$ the conjugate tangent bundle of X. A holomorphic cotangent vector at $z \in U_j$ is an element of the form $\tau = \sum_{\alpha=1}^{n} \eta_{j\alpha}^* \, dz_j^\alpha$, where $dz_j^\alpha \frac{\partial}{\partial z_j^\beta} = \delta_\beta^\alpha$. The set T_z^*X of all complex cotangent vectors at z is a complex vector space $T_z^*X \cong \mathbb{C}^n$ and $T^*X = \cup_{z\in X} T_z^*X$ the dual bundle of TX.

Definition 2.1 ([40]). *A Hermitian metric G on a complex manifold X of complex dimension n is given by $G(z) = \sum_{\alpha,\beta=1}^n g_{j\alpha\overline{\beta}}(z)\, dz_j^\alpha\, d\overline{z}_j^\beta$, where $g_{j\alpha\overline{\beta}}(z)$ is a C^∞-section of $T^*X \otimes \overline{T}^*X$ such that:*
(i) $\overline{g_{j\alpha\overline{\beta}}(z)} = g_{j\beta\overline{\alpha}}(z)$ (Hermitian symmetric),
(ii) for any $\mu = (\mu^1, ..., \mu^n) \in \mathbb{C}^n$,

$$\sum_{\alpha,\beta=1}^{n} g_{j\alpha\overline{\beta}}(z)\, \mu^\alpha\, \overline{\mu}^\beta \geq 0,$$

the equality holds if and only if $\mu = 0$.

Associate to G a real (1,1)-differential form $\omega = \frac{i}{2}\sum_{\alpha,\beta=1}^n g_{j\alpha\overline{\beta}}(z)\, dz_j^\alpha \wedge dz_j^{\overline{\beta}}$. If $d\omega = 0$, then the metric G is called Kähler metric and ω is called the Kähler form associated to the metric G. If Kähler metric can be defined on a complex manifold X, then we can say that it is a Kähler manifold. Assume that $\{f_{ij}\}$ is a system of transition function for E. A Hermitian metric along the fibers of E is a system of functions $h = \{h_j\}$, $j \in J$, satisfies h_j is C^∞ positive function in U_j and $h_j =\mid f_{ij} \mid^2 h_i$ in $U_i \cap U_j$. A holomorphic line bundle E is positive if $\partial\overline{\partial}\log h_j > 0$. The connection of E, with respect to a local holomorphic frame, is given by $\theta = \partial h.h^{-1}$. The curvature matrix of a hermitian connection is easily seen to be given by $\Theta = i\overline{\partial}\theta = i\overline{\partial}\partial \log h$.

Denote by $H^k(X,\mathbb{C})$ (resp. $H^k(X,\mathbb{R})$ the complex (real) De Rham cohomology group of degree k, and by $H^{r,s}(X,\mathbb{Z})$ the subspace of classes which can be represented as closed (r,s)-forms, $r+s=k$.

Example 2.1 ([40]). *The complex projective space P^N is Kähler.*

Definition 2.2 ([40]). *The Fubini-Study metric, a natural Kähler metric ω_{FS} on P^N, is defined by*

$$\Pi^*\omega_{FS} = \frac{i}{2\pi} d'd''log(|\xi_0|^2 + |\xi_1|^2 + ... + |\xi_n|^2),$$

where $\xi_0, \xi_1, ..., \xi_n$ are coordinates of $\mathbb{C}^{n+1}$ and $\Pi : \mathbb{C}^{n+1}\{0\} \longrightarrow P^N$ is the projection.

Let $z = (\xi_1/\xi_0, ..., \xi_n/\xi_0)$ be non homogeneous coordinates on $\mathbb{C}^n \subset P^N$. Then a calculation shows that

$$\omega_{FS} = \frac{i}{2\pi} d'd''\log(1+|z|^2), \quad \int_{P^N} \omega_{FS}^n = 1.$$

It is also well-known from topology that $\omega_{FS} \in H^2(P^N,\mathbb{Z})$ is a generator of the cohomology algebra $H^2(P^N,\mathbb{Z})$.

Definition 2.3 ([40]). *$c_1(E) = \delta^*(E)$ is the first Chern class of the holomorhic line bundle E. That is, the first Chern class $c_1(E)$ of E is an element of the de Rham cohomology group $H^2(X,\mathbb{Z})$.*

Following De Rham theorem (see [40]), one obtains

$$H^r(X,\mathbb{C}) \cong \frac{\{\varphi \in A^r(X,\mathbb{C}), d\varphi = 0\}}{\{\varphi \in A^r(X,\mathbb{C}), \varphi = d\psi, \ \text{ for some } \psi \in A^{r-1}(X,\mathbb{C})\}}.$$

Since $Z \subset \mathbb{C}$, then there is a map $H^2(X,Z) \xrightarrow{i} H^2(X,\mathbb{C})$, which imply to, $c(E) \longrightarrow c(E)_{\mathbb{C}}$, where $c(E)_{\mathbb{C}}$ is an element in $H^2(X,\mathbb{C})$. The de Rham cohomology class of $c(E)_{\mathbb{C}}$ is represented by $\frac{i}{2\pi}\sum_{\alpha,\beta=1}^{n}\Omega_{j\alpha\overline{\beta}}(z)\,dz_j^{\alpha}\wedge d\overline{z}_j^{\beta}$. (see [40]). Hence, a Kähler form on X determines an element of $H^2(X,\mathbb{C})$. Assume that $\mathcal{O}$ is the sheaf of holomorphic functions on X, $\mathcal{O}^*$ is the sheaf of non vanishing holomorphic functions, Z is the constant sheaf of integers, i is the inclusion map, and $exp. : \mathcal{O} \longrightarrow \mathcal{O}^*$ is defined by $exp.(f)(z) = exp.(2\pi i f(z))$. Following [41], since the short sequence

$$0 \longrightarrow Z \xrightarrow{\text{i}} \mathcal{O} \xrightarrow{\text{exp.}} \mathcal{O}^* \longrightarrow 0$$

is exact, then the exact cohomology sequence

$$.... \longrightarrow H^1(X,Z) \longrightarrow H^1(X,\mathcal{O}) \xrightarrow{\delta} H^1(X,\mathcal{O}^*) \xrightarrow{\delta^*} H^2(X,Z) \longrightarrow H^2(X,\mathcal{O}) \longrightarrow ..... \tag{2.1}$$

is obtained.

Definition 2.4 ([40]). *A differential form of type (1,1) on a complex manifold X is said to be positive if, in local coordinates at any point a,*

$$\varphi = i\sum_{\mu,\nu=1}^{n}\varphi_{\mu\nu}(z)\,dz^{\mu}\,d\overline{z}^{\nu},$$

and the matrix $(\varphi_{\mu\nu})$ is a positive definite Hermitian symmetric matrix for each fixed point z near a.

Definition 2.5 ([40]). *The Kähler metric $G = \sum_{\alpha,\beta=1}^{n} g_{j\alpha\overline{\beta}}(z)\,dz_j^{\alpha}\,d\overline{z}_j^{\beta}$ is a Hodge metric on X if the equivalence class $[\omega]$ of ω is equal to $c_{\mathbb{C}}$ for some $c \in H^2(X,Z)$, where*

$$\omega = \frac{i}{2}\sum_{\alpha,\beta=1}^{n} g_{j\alpha\overline{\beta}}(z)\,dz_j^{\alpha}\wedge d\overline{z}_j^{\beta}.$$

If X has a Hodge metric, then X is said to a Hodge manifold.

Example 2.2 ([41], Example V.4.5). *Let X be a compact projective algebraic manifold, then X is a submanifold of P^N for some N.*

Indeed, let ω be the fundamental form associated with the Fubini-Study metric on P^N. Since ω is the negative of the Chern form for the universal bundle $U_{1,N+1} \longrightarrow P^N$, it follows that ω is a Hodge form on P^N (see [41], Propositions III.4.3 and III.4.6). The restriction of ω (as a differential form) to X will also be a Hodge form, and hence X is a Hodge manifold. In general, a complex submanifold of a Hodge manifold is again a Hodge manifold.

Example 2.3 ([41]). *A complex submanifold of a Hodge manifold is again a Hodge manifold.*

Example 2.4 ([41]). *Let X be a compact complex manifold that is an unramified covering of a Hodge manifold Y; i.e., there's a holomorphic mapping $X \xrightarrow{\pi} Y$ specified $\pi^{-1}(p)$ is distinct and π could be a local biholomorphism at every point $x \in X$. Then X could be a Hodge manifold.*

Indeed, if ω is a Hodge form on Y and then $\pi^*\omega$ will be a Hodge form on X. Similarly, if $X \xrightarrow{f} Y$ is an immersion, then $f^*\omega$ will give a Hodge manifold structure to X.

Example 2.5 ([41]). *Let X be a connected compact Riemann surface. As a result, X is a Hodge manifold. Due to Poincare duality,*

$$\mathbb{C} \cong H^0(X,\mathbb{C}) \cong H^2(X,\mathbb{C}),$$

and, additionally,

$$H^2(X,\mathbb{C}) = H^{1,1}(X).$$

Let $\tilde{\omega}$ be the fundamental form on X connected to a Hermitian metric. If so, $\tilde{\omega}$ is a closed form of type $(1,1)$ and a basis element for the one-dimensional de Rham group $H^2(X,\mathbb{C})$. If $c = \int_M \tilde{\omega}$, then $\omega = c^{-1}\tilde{\omega}$ will be an integral positive form on X of type $(1,1)$. As a result, Hodge is X. This illustration generalizes to the claim that any Kahler manifold X with the property that $dim H^{1,1}(X) = 1$ is definitely Hodge.

Example 2.6 ([41], Proposition V.5.3). *If D is a bounded domain in $\mathbb{C}^n$ and Γ is a fixed point free correctly discontinuous subgroup of the group of biholomorphisms of D onto itself with the characteristic that $X = D/\Gamma$ is compact. As a result, X is a Hodge manifold.*

Definition 2.6 ([40]). *The transition function system $\{K_{ij}\}$ defines the canonical line bundle $K = K_X = \bigwedge^n T^*$ of X, where*

$$K_{ij} = \frac{\partial(z_j^1, ..., z_j^n)}{\partial(z_i^1, ..., z_i^n)}, \text{ on } U_i \cap U_j.$$

Since $dz_j^\alpha = \sum_{\beta=1}^n \frac{\partial z_j^\alpha}{\partial z_i^\beta} dz_i^\beta$ on $U_i \cap U_j$. Then

$$(g_{i\alpha\overline{\beta}}) = {}^t\left(\frac{\partial z_j^\alpha}{\partial z_i^\beta}\right)(g_{j\alpha\overline{\beta}})\overline{\left(\frac{\partial z_j^\tau}{\partial z_i^\gamma}\right)}, \text{ on } U_i \cap U_j.$$

As a result, the system $g = \{g_j\}$, $g_i = det(g_{i\alpha\overline{\beta}})$ determines a Hermitian metric along the fibers of $K = \{K_{jk}\}$ with regard to the covering $\{U_j\}_{j\in J}$ and satisfies $g_i = |K_{ij}|^2 g_j$ on $U_i \cap U_j$. Let $c_1(K)$ be the first Chern class of K and K^* be the dual line bundle of K. Then, following [41] the sheaf identities is given by

$$\Omega^n(E) = K \otimes E, \tag{2.2}$$

and

$$c_1(K^*) = -c_1(K). \tag{2.3}$$

In terms of the metric $\{g_i\}$, the curvature form $\Theta = \{\Theta_j\}$ of K is given by

$$\Theta_j = \sum_{\alpha,\beta=1}^n \Theta_{j\alpha\overline{\beta}}(z)\, dz_j^\alpha \wedge d\overline{z}_j^\beta, \quad \Theta_{j\alpha\overline{\beta}} = \frac{\partial^2 \log g_j}{\partial z_j^\alpha \partial \overline{z}_j^\beta}.$$

Moreover, following Theorem 4.5 of Chapter III in [41] the first Chern class $c_1(K)$ of K is given by

$$c_1(K) = -\frac{1}{2\pi i}\partial\overline{\partial} \log g_j.$$

Then

$$c_1(K) = \frac{i}{2\pi}\partial\overline{\partial} \log g_j = \frac{i}{2\pi}\Theta. \tag{2.4}$$

On the other hand, following [40] the first Chern class $c_1(X)$ of X is given by

$$c_1(X) = -c_1(K). \tag{2.5}$$

Then, from (2.2) and (2.4), we obtain

$$c_1(K^*) = -c_1(K) = c_1(X).$$

Thus, the curvature form of K^* is the first Chern class of X.

2.2. **Vanishing theorems.** It would be very useful if there were vanishing theorems for some specific sheaves. It will be the same hypotheses that we will use for both theorems: Let X be an n-dimensional compact Kähler manifold and let L be a holomorphic line bundle on X with a curvature satisfying

$$\frac{1}{2\pi i}\Theta_L = \omega.$$

It is important to note that in this set-up we have

Theorem 2.1. *(Kodaira Vanishing). For $p+q>n$, we have*

$$H^q\left(X, \mathcal{H}^p \otimes L\right) = 0.$$

Theorem 2.2. *(Serre Vanishing). For any holomorphic vector bundle E and any $q>0$, we have*

$$H^1\left(E \otimes L^{\otimes N}\right) = 0,$$

for $N \gg 0$.

Remark 2.1. *It follows from the Kodaira vanishing theorem and Serre duality that*

$$H^q\left(X, \mathcal{H}^p \otimes L^{-1}\right) = 0,$$

for $p+q<n$.

Remark 2.2. *The question of how large N needs to be to make the theorem hold is an active research question. The proof we'll give is theoretically constructive, but the general consensus is that the techniques of the proof have been pushed as far as they can for bounding N.*

This section aims to show several compact Kähler manifold vanishing theorems. Lemma 4.4 of Chapter III of [41] proves the following lemma.

To prove both of these theorems, we use a result that we proved in the course of proving the Kähler identities on March 17th: if E is a holomorphic vector bundle with connection Θ_E, then $[\Lambda, \Theta_E] = i\left(\Delta_D - \Delta_{\bar{D}}\right)$.

Let $p+q>n$. We're interested in $H^q\left(X, \mathcal{H}^p \otimes L\right)$. The Dolbeault sequence identifies this cohomology group with the space of $\bar{\partial}$-closed (p,q)-forms (tensored with sections of L) mod $\bar{\partial}$ exact ones, but as we are in the compact Kähler case, we can think of these as harmonic forms, i.e. as

$$\ker\left(\Delta_{\bar{D}} : \Omega^{p,q} \otimes L \to \Omega^{p,q} \otimes L\right).$$

Let $\eta \in \ker \Delta_{\bar{D}}$. On the one hand, using the inner product (,) defined in the March 17th notes, we see that

$$(\eta, \Delta_D \eta) = (D\eta, D\eta) + (D^*\eta, D^*\eta) \geq 0,$$

and of course

$$(\eta, \Delta_{\bar{D}} \eta) = (\eta, 0) = 0.$$

Thus $(\eta, (\Delta_D \eta - \Delta_{\bar{D}})\, \eta) \geq 0$.

On the other hand, this same expression, after dividing by 2π, is

$$\begin{aligned}\left(\eta, \frac{1}{2\pi i}\left[\Lambda, \Theta_L\right] \eta\right) &= (\eta, [\Lambda, \omega\wedge]\eta) \\ &= (\eta, [\Lambda, L]\eta) \\ &= (n-p-q)(\eta, \eta) \\ &\leq 0.\end{aligned}$$

This is only possible if all inequalities are equalities, and hence (η, η), and so η, must be 0.

Lemma 2.1 ([41]). *There is a one-to-one correspondence between the elements of the cohomology group $H^1(X, \mathcal{O}^*)$ and the equivalence classes of holomorphic line bundles on X.*

Theorem 2.3 ([40]). *If X is a complex manifold and if $\omega \in C^\infty(X, \Omega^{1,1}(X))$ is a d-closed real $(1,1)$-form satisfies $[\omega] \in H^2(X, \mathbb{Z})$, then there exists a holomorphic Hermitian line bundle (E, b) on X satisfies $\frac{i}{2\pi}\Theta_{E,b} = \omega$.*

Theorem 2.4. *(Kodaira's embedding theorem) The following statements are equivalent if X is a compact complex manifold:*

(1) X is projective, i.e., a holomorphic embedding with the form $\varphi : X \longrightarrow P^N$ exists.

(2) There is a Kähler form on X called ω that belongs to the integer class and is the image of the morphism $H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R})$.

(3) There is a positive holomorphic line bundle $E \longrightarrow X$.

Proof. 1) Given that the restriction of the Fubini-Study formω_{FS} is an integer Kähler form, 1) and 2) are evident. A direct result of the Lefschetz Theorem on $(1,1)$-classes is 2) $\Longrightarrow$ 3) (see Theorem 3.2). 3) $\Longrightarrow$ 1) For the proof see [42] or [43]; Section 5.3. □

Let us give an immediate application.

Theorem 2.5 ([40]). *If $H^2(X, \mathcal{O}) = 0$, then X is a compact Kähler manifold and is a projective manifold.*

Indeed, by the Hodge decomposition theorem, the assumption implies that $H^2(X, \mathbb{R})$ can be represented by real closed forms of type $(1,1)$. Now start from a Kähler class α represented by a Kähler form $\tilde{\alpha}$. Then as $H^2(X, \mathbb{Q})$ is dense inside $H^2(X, \mathbb{R})$, α can approximated by rational cohomology classes α_n. Choosing representative $\tilde{\alpha}_n$ of α_n, which are real closed forms of type $(1,1)$ converging to α, we conclude that $\tilde{\alpha}_n$ must be positive non-degenerate for n large enough, as it is an open property of real $(1,1)$-forms on compact complex manifolds. Thus α_n is Kähler for large n, and X is projective.

Theorem 2.6 ([40], Theorem 8.3)**.** *If $H^2(X,\mathcal{O})=0$, then X is a compact Kähler manifold and is a Hodge manifold.*

Theorem 2.7 ([38,39])**.** *Given that E is a holomorphic line bundle over X and X is a compact Kähler manifold of complex dimension n. Assume that the curvature form of E at every point of X is semi-positive and of rank k, $1 \leqslant k \leqslant n$, at any point of X (respectively, at one point of X). Then*

$$H^s(X,\Omega^r(E))=0, \text{ for } r+s \geqslant 2n-k+1,$$
$$(\text{resp. } H^s(X,\Omega^n(E))=0, \text{ for } s \geqslant n-k+1),$$

where $H^s(X,\Omega^r(E))$ denotes the s-th cohomology group of X with coefficients in the sheaf $\Omega^r(E)$.

3. Results

This section aims to generalize the known theorem of Kodaira, which characterizes that compact complex manifolds admit an embedding into complex projective space. When X is a compact Kähler manifold of complex dimension n and its first Chern class is semi-positive and of rank $n-1$ at one point of X, we prove that X may be embedded into P^N for some integer N.

Theorem 3.1. *Let $c_1(X)$ be the first Chern class of a compact Kähler manifold of complex dimension n. Assume that $c_1(X)$ is of rank $n-1$ at one point of X and semi-positive at each points of X. X is then Hodge manifold.*

Proof. In order to prove this theorem, we must first show that $H^2(X,\mathcal{O})=0$, and as a result, the theorem follows from Theorem 2.5. According to Definition 2 and Equation (2.3), any condition on the positivity (or negativity) of the K's first Chern class is identical to the same condition on the K's curvature form. Equations (2.3) and (2.5) and the theorem's hypothesis lead to the conclusion that the curvature form of K^* is semi-positive at each point of X and of rank $n-1$ at one point of X. Following Theorem 2.6, one obtains

$$H^s(X,\Omega^n(K^*))=0, \text{ for } s \geqslant 2.$$

However, we have

$$H^s(X,\mathcal{O})=H^s(X,\Omega^0(K\otimes K^*))=H^s(X,\Omega^n(K^*)).$$

Then, we obtain

$$H^s(X,\mathcal{O})=0, \quad \text{for } s \geqslant 2.$$

When we apply this result to the exact sequence of cohomology (2.1), we obtain the exact sequence:

$$\dots . \longrightarrow H^1(X,\mathcal{O}^*) \xrightarrow{c} H^2(X,\mathbb{Z}) \longrightarrow 0.$$

Thus, every element in $H^2(X,\mathbb{Z})$ is the Chern class of some line bundle. Let $\{b_1,...,b_m\}$ be a basis for the free part of $H^2(X,\mathbb{Z})$ so that

$$H^2(X,\mathbb{C}) = \mathbb{C}b_1 + ... + \mathbb{C}b_m.$$

Each $b_\lambda = c(F_\lambda)$, $F_\lambda \in H^2(X,\mathcal{O}^*)$ and hence is cohomologous to a real 2-form of type $(1,1)$. We wish to modify the Kähler form on X to get a Hodge metric on X. Since $\omega \in H^2(X,\mathbb{C})$, then ω is cohomologous to a form $\sum \rho_\lambda b_\lambda$ (i.e., $\omega \sim \sum \rho_\lambda b_\lambda$), where $\rho_\lambda \in \mathbb{R}$ (ω is real and the b_λ are real). Given ε we can always find integers $k_\lambda, r \in \mathbb{Z}$ such that

$$\left|\rho_\lambda - \frac{k_\lambda}{r}\right| < \varepsilon, \quad \lambda = 1,2,...,m.$$

But then for small enough ε, one obtains

$$\omega' = \omega - \sum \left(\rho_\lambda - \frac{k_\lambda}{r}\right)\gamma_\lambda,$$

defines a Kähler form on X where $\gamma_\lambda \sim b_\lambda$ is a real $(1,1)$ form. Hence $\tilde{\gamma} = r\omega'$ is a Kähler form. But

$$\tilde{\omega} \sim r\sum \rho_\lambda b_\lambda - r\sum\left(\rho_\lambda - \frac{k_\lambda}{r}\right)b_\lambda = \sum k_\lambda b_\lambda \in H^2(X,\mathbb{Z}).$$

Thus $\tilde{\omega}$ defines a Hodge metric on X, and hence X is Hodge manifold. Then the theorem is proved. □

Remark 3.1. *The proof of Theorem 3.1 can be obtained also, as follows: By (the second line of) Theorem 2.6 with $E = K^*$, $H^2(X,\mathcal{O}) = 0$. Then by Theorem 2.4, X is projective-algebraic.*

Theorem 3.2 ([1]). *Each Hodge manifold has algebraic properties.*

Theorems 3.1 and 3.2 yield the following immediate result:

Corollary 3.1. *Let $c_1(X)$ be the first Chern of the compact Kähler manifold X of complex dimension n. Assume that $c_1(X)$ is of rank $n-1$ at one point of X and semi-positive at each point of X. If X is projective algebraic, it can be embedded into P^N for some integer N.*

Theorem 3.3 ([44]). *Let E be a holomorphic line bundle over X and X be a compact Kähler manifold of complex dimension n. At a point $x_0 \in X$, if E is semi-positive and Θ has k positive eigenvalues, then*

$$H^s(X, E\otimes K_X) = 0, \quad \textit{for} \quad s > n-k.$$

Corollary 3.2. *Let X be a compact Kähler manifold of complex dimension n and $c_1(X)$ be its first Chern. Suppose that $c_1(X)$ is semi-positive at each point of X and has k positive eigenvalues at a point $x_0 \in X$. Then X is projective algebraic, i.e., X can be embedded into P^N for some integer N.*

4. Discussion

Kodaira embedding theorem provides an effective characterization of projectivity of a Kähler manifold in terms the second cohomology, which characterizes the compact complex manifolds that can be embedded into a projective space, i.e. that are projective varieties. So, thanks to Kodaira and Chow's theorems, a question of Complex Analysis can be translated in the language of Algebraic Geometry. We used the theory of harmonic forms to prove the extension of Kodaira vanishing theorem, which is necessary in the proof of the Kodaira embedding theorem. The vanishing theorem is proved using methods of complex analytic differential geometry. If X is supposed to be a smooth, projective variety one can expect to and an algebraic proof of the vanishing theorem but at the present there is no purely algebraic proof.

5. Conclusions

In this work we proved an extension of a well known theorem due to Kodaira, which characterizes the compact complex manifolds that can be embedded into a projective space i.e. that are projective varieties. We presented some formulae related to the Chern-Ricci curvatures and scalar curvatures of special Hermitian metrics. We proved that if X is a compact Kähler manifold of complex dimension n and its first Chern class is semi-positive and of rank $n-1$ at one point of X, then X can be embedded into P^N for some integer N. So, thanks to Kodaira and Chow's theorems, a question of Complex Analysis can be translated in the language of Algebraic Geometry. We will use the theory of harmonic forms to prove the extension of Kodaira vanishing theorem, which is necessary in the proof of the Kodaira embedding theorem. The vanishing theorem is proved using methods of complex analytic differential geometry. If X is supposed to be a smooth, projective variety one can expect to and an algebraic proof of the vanishing theorem but at the present there is no purely algebraic proof.

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