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ESTIMATION OF COMPARATIVE GROWTH PROPERTIES OF ENTIRE AND MEROMORPHIC FUNCTIONS IN TERMS OF THEIR RELATIVE ORDER

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Abstract. In this paper we discuss some comparative growth properties of entire and meromorphic functions on the basis of their relative order which improve some earlier results.

1. Introduction, Definitions and Notations.

Let f be a non constant entire function in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$.Then $M_f(r)$ is strictly increasing,its inverse

$$M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$$

exists and is such that

$$\lim_{r\to\infty} M_f^{-1}(r) = \infty.$$

Two entire functions f and g are said to be asymptotically equivalent if there exists l $(0 < l < \infty)$ such that

$$\frac{M_f(r)}{M_g(r)} \to l \text{ as } r \to \infty$$

and in that case we write $f \sim g$.Clearly if $f \sim g$ then $g \sim f$.

The order and lower order of an entire function are defined in the following way:

Definition 1. *The order ρ_f and lower order λ_f of an entire function f are defined as follows:*

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

The function f is said to be of regular growth if $\rho_f = \lambda_f$.

The notion of order of an entire function was much improved by the introduction of the relative order of two entire functions.In this connection Bernal [1] gave the following definition.

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Definition 2. [1]*Let f and g be two entire functions.The relative order $\rho_g(f)$ of f with respect to g is defined as follows:*

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \textit{ for all sufficiently large values of } r\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

If we take $g(z)$ as $\exp z$ then we see that $\rho_g(f) = \rho_f$ and this shows that the relative order generalised the concept of the order of an entire function.

Similarly the relative lower order $\lambda_g(f)$ is defined as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

For the case of a meromorphic function this generalisation was due to Lahiri and Banerjee [5].They introduced the notion of relative order $\rho_g(f)$ of f with respect to g where f is meromorphic as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^{\mu}) \text{ for all sufficiently large values of } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

Similarly the relative lower order $\lambda_g(f)$ of f with respect to g is defined by

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In a recent paper Datta and Biswas [2] studied some growth properties of entire functions using relative order.In this paper we discuss some comparative growth properties of entire and meromorphic functions in terms of their relative order which improves some results of Datta and Biswas [2].

2. Lemmas.

In this section we present two lemmas which will be needed in the sequel.

Lemma 1. *If g_1 and g_2 be two entire functions with property (A) such that $g_1 \sim g_2$.If f be meromorphic then $\rho_{g_1}(f) = \rho_{g_2}(f)$.*

Lemma 1 follows from Theorem 5 {cf.[3]} on putting $L(r) \equiv 1$.

Lemma 2 ([4])**.** *If f, g be two meromorphic function and g is of regular growth.Then $\rho_g(f) = \frac{\rho_f}{\rho_g}$.*

3. Theorems.

In this section we present the main results of our paper.

Theorem 1. *Let f be meromorphic and g, h be two entire functions with non zero finite orders.Then*

$$\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_h^{-1} T_f(r)} \le \frac{\rho_g(f)}{\rho_h(f)} \le \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_h^{-1} T_f(r)}.$$

Proof. From the definition of relative order we get for arbitrary $\varepsilon\,(>0)$ and for all sufficiently large values of r that

$$\log T_g^{-1}T_f(r) < (\rho_g(f)+\varepsilon)\log r. \tag{1}$$

Also for a sequence of values of r tending to infinity we get that

$$\log T_g^{-1}T_f(r) > (\rho_g(f)-\varepsilon)\log r. \tag{2}$$

Again for arbitrary $\varepsilon\,(>0)$ and for all sufficiently large values of r we obtain that

$$\log T_h^{-1}T_f(r) < (\rho_h(f)+\varepsilon)\log r \tag{3}$$

and for a sequence of values of r tending to infinity we get that

$$\log T_h^{-1}T_f(r) > (\rho_h(f)-\varepsilon)\log r. \tag{4}$$

Now from (1) and (4) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1}T_f(r)}{\log T_h^{-1}T_f(r)} < \frac{(\rho_g(f)+\varepsilon)}{(\rho_h(f)-\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_h^{-1}T_f(r)} \le \frac{\rho_g(f)}{\rho_h(f)}. \tag{5}$$

Also from (2) and (3) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1}T_f(r)}{\log T_h^{-1}T_f(r)} > \frac{(\rho_g(f)-\varepsilon)}{(\rho_h(f)+\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_h^{-1}T_f(r)} \ge \frac{\rho_g(f)}{\rho_h(f)}. \tag{6}$$

Thus from (5) and (6) Theorem 1 follows.This completes the proof. □

Corollary 1. *If g and h are of regular growths then using Lemma 2 we get from Theorem 1 that*

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_h^{-1}T_f(r)} \le \frac{\rho_h}{\rho_g} \le \limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_h^{-1}T_f(r)}.$$

Corollary 2. *If g and h are of regular growths and $g\sim h$ then using Lemma 1 we get from Theorem 1 that*

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_h^{-1}T_f(r)} \le 1 \le \limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_h^{-1}T_f(r)}.$$

Remark 1. *The converse of Corollary 2 is not always true which is evident from the following example.*

Example 1. *Let $g\,(z)=\exp z$ and $h\,(z)=\exp(2z)$ so that $M_g(r)=e^r$ and $M_h(r)=e^{2r}$ .Now*

$$\frac{M_g(r)}{M_h(r)} \to 0 \text{ as } r\to\infty$$

and so $g_1 \nsim g_2$ .Also

$$T_g(r)=\frac{r}{\pi} \text{ and } T_h(r)=\frac{r}{\pi}$$

and therefore

$$T_g^{-1}(r) = \pi r \text{ and } T_h^{-1}(r) = \frac{\pi}{2} r.$$

But

$$\begin{aligned}\lim_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_h^{-1} T_f(r)} &= \lim_{r\to\infty} \frac{\log \pi T_f(r)}{\log \frac{\pi}{2} T_f(r)} \\ &= 1.\end{aligned}$$

Theorem 2. *Let f,h be meromorphic and g be entire functions with non zero finite order. Then*

$$\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_g^{-1} T_h(r)} \leq \frac{\rho_g(f)}{\rho_g(h)} \leq \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_g^{-1} T_h(r)}.$$

Proof. From the definition of relative order we get for arbitrary $\varepsilon\,(>0)$ and for all sufficiently large values of r that

$$\log T_g^{-1} T_h(r) < (\rho_g(h) + \varepsilon) \log r \tag{7}$$

and for a sequence of values of r tending to infinity we get that

$$\log T_g^{-1} T_h(r) > (\rho_g(h) - \varepsilon) \log r. \tag{8}$$

Now from (1) and (8) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1} T_f(r)}{\log T_g^{-1} T_h(r)} < \frac{(\rho_g(f) + \varepsilon)}{(\rho_g(h) - \varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_g^{-1} T_h(r)} \leq \frac{\rho_g(f)}{\rho_g(h)}. \tag{9}$$

Also from (2) and (7) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1} T_f(r)}{\log T_g^{-1} T_h(r)} > \frac{(\rho_g(f) - \varepsilon)}{(\rho_g(h) + \varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_g^{-1} T_h(r)} \geq \frac{\rho_g(f)}{\rho_g(h)}. \tag{10}$$

From (9) and (10) we obtain Theorem 2.This completes the proof. □

Corollary 3. *If g and h are of regular growths then using Lemma 2 we get from Theorem 2 that*

$$\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_g^{-1} T_h(r)} \leq \frac{\rho_f}{\rho_h} \leq \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_g^{-1} T_h(r)}.$$

Theorem 3. *Let f, h be meromorphic and g, k be entire functions with non zero finite orders. Then*

$$\liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_k^{-1} T_h(r)} \leq \frac{\rho_g(f)}{\rho_k(h)} \leq \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log T_k^{-1} T_h(r)}.$$

Proof. From the definition of relative order we get for arbitrary $\varepsilon\,(>0)$ and for all sufficiently large values of r that

$$\log T_k^{-1}T_h(r) < (\rho_k(h)+\varepsilon)\log r. \tag{11}$$

Also for a sequence of values of r tending to infinity we get that

$$\log T_k^{-1}T_h(r) > (\rho_k(h)-\varepsilon)\log r. \tag{12}$$

Now from (1) and (12) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1}T_f(r)}{\log T_k^{-1}T_h(r)} < \frac{(\rho_g(f)+\varepsilon)}{(\rho_k(h)-\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_k^{-1}T_h(r)} \le \frac{\rho_g(f)}{\rho_k(h)}. \tag{13}$$

Also from (2) and (11) we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1}T_f(r)}{\log T_k^{-1}T_h(r)} > \frac{(\rho_g(f)-\varepsilon)}{(\rho_k(h)+\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_k^{-1}T_h(r)} \ge \frac{\rho_g(f)}{\rho_k(h)}. \tag{14}$$

From (13) and (14) we obtain Theorem 3.This completes the proof. □

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