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FIXED POINT OF ORDER 2 ON G-METRIC SPACE

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Abstract. In this article we introduce a new concept of fixed point that is fixed point of order 2 on G-metric space and some results are achieved.

1. Introduction and preliminaries

In 2003, Mustafa and Sims [4] introduced a more appropriate and robust notion of a generalized metric space as follows.

Definition 1.1. [4] Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:

(1) $G(x, y, z) = 0$ if and only if $x = y = z$;
(2) $G(x, x, y) > 0$, for all $x \neq y$;
(3) $G(x, y, z) \geq G(x, x, y)$, for all $x, y, z \in X$;
(4) $G(x, y, z) = G(x, z, y) = G(z, y, x) = \cdots$ (symmetric in all three variables);
(5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$, for all $x, y, z, w \in X$.

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2. Suppose that (X,G) is a G-metric space, $T : X \to X$ is a function and $x_0 \in X$ is fixed point of T. We call x_0 is a fixed pointof order 2 if it is not alone point and the following satisfies:

$$\lim_{x \to x_0} \frac{G(Tx, Tx, x_0)}{G(x, x, x_0)} = 1 \tag{1.1}$$

We remember the following definitions. We will show that for the case (a) there is not fixed point of order 2 but in two other cases there is fixed point of order 2.

Definition 1.3. Suppose that (X,G) is a G-metric space, $T : X \to X$ is a function.

(a) T is a contraction, if there exist $k \in [0, 1)$ such that $G(Tx, Ty, Tz) \leq kG(x, y, z)$ for all $x, y, z \in X$.
(b) T is a contractive mapping, if $G(Tx, Ty, Tz) < G(x, y, z)$ for all $x, y, z \in X$ which $x \neq y \neq z$.
(c) T is non-expansive mapping, if $G(Tx, Ty, Tz) \leq G(x, y, z)$ for all $x, y, z \in X$.

In the following we consider first some properties for fixed point of order 2.

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2. Main Results

Proposition 2.1. *If $x_0 \in X$ is a fixed point of order 2 for T on X. Then T is continuous at x_0.*

Proof. $\lim_{n\to\infty} G(Tx,Tx,x_0) = \lim_{x\to x_0} \frac{G(Tx,Tx,x_0)}{G(x,x,x_0)} G(x,x,x_0)$
$\lim_{x\to x_0} \frac{G(Tx,Tx,x_0)}{G(x,x,x_0)} \lim_{x\to x_0} G(x,x,x_0) = 0.$ □

Proposition 2.2. *Let (X,G) be a metric space and $T : X \to X$ be a function such that $x_0 \in X$ is a fixed point for T, not alone point for X and alone point for $T(X)$. Then x_0 is not fixed point of order 2 for T.*

Proof. According to assumption x_0 is alone point for $T(X)$. There is a neighborhood of x_0, like $N(x_0)$ such that $N(x_0) \cap T(X)$ and each $x \in N(x_0)$ implies that $G(Tx,Tx,x_0) = 0$. Therefore, $\lim_{x\to x_0} \frac{G(Tx,Tx,x_0)}{G(x,x,x_0)} = 0$, i.e; x_0 is not a fixed point of order 2 for T. □

Proposition 2.3. *Suppose that $x_0 \in X$ be a fixed point for $T_i : X \to X$ which $i = 1,2,...,n$ where $(n \in N)$ and also $\lim_{x\to x_0} \frac{G(T_ix,T_ix,x_0)}{G(x,x,x_0)} = \lambda_i$. Then x_0 is a fixed point of order 2 for $T_1T_2...T_n$ if and only if $\lambda_1\lambda_2...\lambda_n = 1$.*

Proof. T_i is continuous at x_0 for all $i = 1,2,...,n$ by a simple change of variable that

$$\lim_{x\to x_0} \frac{G(T_k(T_{k+1}...T_nx),T_k(T_{k+1}...T_nx),x_0)}{G(T_{k+1}...T_nx,T_{k+1}...T_nx,x_0)} = \lim_{t\to x_0} \frac{G(T_kt,T_kt,x_0)}{t,t,x_0}$$

and the last limit is equal with λ_k for $k = 1,2,...,n$. Hence,

$$\lim_{x\to x_0} \frac{G(T_1T_2...T_nx,T_1T_2...T_nx,x_0)}{G(x,x,x_0)} =$$

$$\lim_{x\to x_0} \frac{G(T_1(T_2...T_n)x,T_1(T_2...T_n)x,x_0)}{G(T_2...Tn,T_2...Tn,x_0)} \frac{G(T_2(T_3...T_n)x,T_2(T_3...T_n)x,x_0)}{G(T_3...Tn,T_3...Tn,x_0)} \cdots \frac{G(T_nx,T_nx,x_0)}{G(x,x,x_0)}$$

$$\lambda_1\lambda_2...\lambda_n$$

□

Proposition 2.4. *Let$x_0 \in X$ be a fixed point for $T_i : X \to X$ for $i = 1,2,...,n$ and $n \in N$.*

(a) *If x_0 is fixed point of order 2 for all T_i, then x_0 is fixed point for $T_1T_2...T_n$.*
(b) *If x_0 is fixed point order 2 for T_1T_2 and T_2 , then x_0 is fixed point of order 2 for T_1.*

Proof. (a) By proposition 2.1.

(b) x_0 is fixed point of order 2 for T_1T_2 and T_2. Thus, $\lim_{x\to x_0} \frac{G(T_1T_2x,T_1T_2x,x_0)}{G(x,x,x_0)} = 1$, $\lim_{x\to x_0} \frac{G(T_2x,T_2x,x_0)}{G(x,x,x_0)} = 1$. Since T is continuous at x_0 for $t = T_2x$.

$$1 = \frac{\lim_{x\to x_0} \frac{G(T_1T_2x,T_1T_2x,x_0)}{G(x,x,x_0)}}{\lim_{x\to x_0} \frac{G(T_2x,T_2x,x_0)}{G(x,x,x_0)}} = \lim_{x\to x_0} \frac{G(T_1T_2x,T_1T_2x,x_0)}{G(T_2x,T_2x,x_0)} = \lim_{t\to x_0} \frac{G(T_1t,T_1t,x_0)}{G(t,t,x_0)}$$

□

Proposition 2.5. *Suppose that x_0 is not alone point and is a fixed point for $T_i : X \to X$ for $i = 1, 2, ..., n$ and $n \in N$.*

(a) *If T_i be a contractive mapping or non expansive mapping for $i = 1, 2, ..., n$ and $n \in N$ and $\lim_{x\to x_0} \frac{G(T_i x, T_i x, x_0)}{G(x,x,x_0)} = \lambda_i$. Then $x_0 \in X$ is a fixed point of order 2 for $T_1T_2...T_n$ if and only if x_0 is a fixed point of order 2 for all T_i.*

(b) *If $\lim_{x\to x_0} \frac{G(T_1 x, T_1 x, x_0)}{G(x,x,x_0)} = \lambda$ then x_0 is a fixed point of order 2 for T_1 if and only if x_0 be a fixed point of order 2 for T_1^n where n is arbitrary positive integer.*

(c) *If T_1 be a contractive mapping or non-expansive mapping, then x_0 is a fixed point of order 2 for T_1 if and only if there exist $n \in N$ such that x_0 be a fixed point of order 2 for T_1^n.*

Proof. (a) Let T_i be a contractive mapping for all $i = 1, 2, ..., n$. If x_0 is a fixed point of order 2 for all T_i then by proposition 2.3, x_0 is a fixed point of order 2 for $T_1T_2...T_n$. Now assume that x_0 is a fixed point of order 2 for $T_1T_2...T_n$ then by proposition 2.2, $1 = \lim_{x\to x_0} \frac{G(T_1T_2...T_n x, T_1T_2...T_n x, x_0)}{G(x,x,x_0)} = \lambda_1\lambda_2...\lambda_n$. But all T_i are contractive mappings so $\frac{G(T_1 x, T_1 x, x_0)}{G(x,x,x_0)} < 1$ which implies that $\lambda_i \leq 1$ for all $i = 1, 2, ...n$. Hence, $\lambda_1 = \lambda_2 = ... = \lambda_n = 1$. Proof for non expansive is similar.

(b) By proposition 2.2, $\lim_{x\to x_0} \frac{G(T_1^n x,, x_0)}{G(x,x,x_0)} = \lambda^n$. Then $\lambda^n = 1$ if and only if $\lambda = 1$ because $\lambda \geq 0$.

(c) Let T_1 be a contractive mapping and there exists $n \in N$ such that x_0 is a fixed point of order 2 for T_1^n.T_1 is a contractive mapping. So

$$G(T_1^n x, T_1^n x, x_0) < ... < G(T_1 x, T_1 x, x_0) < G(x, x, x_0)$$

$$1 = \lim_{x\to x_0} \frac{G(T_1^n x, T_1^n x, x_0)}{G(x, x, x_0} \leq \frac{G(T_1 x, T_1 x, x_0)}{G(x, x, x_0} \leq 1.$$

Therefore, $\lim_{x\to x_0} \frac{G(T_1 x, T_1 x, x_0)}{G(x,x,x_0} = 1.$ □

Proposition 2.6. *Suppose that (X, G) is a metric space, $T : X \to X$ is a function and x_0 is a fixed point of T. If T is contraction then x_0 is not a fixed point of order 2 for T.*

Proof. Since T is a contractive mapping so there exists $\alpha \in [0, 1)$ such that $G(Tx, Ty, Tz) \leq \alpha G(x, y, z)$ for all $x, y, z \in X$. Therefore $\frac{G(Tx, Tx, x_0)}{G(x,x,x_0)} \leq \alpha < 1$ and x_0 can not be a fixed point of order 2 for T. □

Proposition 2.7. *Suppose that $x_0 \in X$ be a fixed point of order 2 for $T : X \to X$ where T is one to one and g is left inverse of T. Then x_0 is also a fixed point of order 2 for g.*

Proof. It is clear that x_0 is a fixed point for g. On the other hand, since T is continuous at x_0 for $t = Tx$ so

$$1=\lim_{x\to x_0}\frac{G(Tx,Tx,x_0)}{G(x,x,x_0)} = \lim_{x\to x_0}\frac{G(g(T(Tx)),g(T(Tx)),x_0)}{G(gTx,gTx,x_0)} = \lim_{t\to x_0}\frac{G(g(Tt),g(Tt),x_0)}{G(gt,gt,x_0)} = \lim_{t\to x_0}\frac{G(t,t,x_0)}{G(gt,gt,x_0)} = \lim_{t\to x_0}\frac{1}{\frac{G(gt,gt,x_0)}{G(t,t,x_0)}}$$

Therefore, $\lim_{t\to x_0}\frac{G(gt,gt,x_0)}{G(t,t,x_0)}=1$. □

In the following we give another condition for the fixed point of order 2.

Proposition 2.8. *Suppose that x_0 is not alone point and is a fixed point for $T : x \to X$.*

(a) *If $\lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(x,x,x_0)}=0$ then x_0 is a fixed point of order 2 for T.*
(b) *If $\lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(Tx,Tx,x_0)}=0$ then x_0 is a fixed point of order 2 for T.*

Proof. (a) From the definition of G-metric space we have

$$\begin{aligned} \mid G(x,x,x_0)-G(Tx,Tx,x_0)\mid &\le G(Tx,Tx,x) \\ 1-\frac{G(Tx,Tx,x_0)}{G(x,x,x_0)} &\le \frac{G(Tx,Tx,x)}{G(x,x,x_0)} \\ &\le 1+\frac{G(Tx,Tx,x_0)}{G(x,x,x_0)} \end{aligned}$$

$\lim_{x\to x_0}\frac{G(Tx,Tx,x_0)}{G(x,x,x_0)}=1$.

(b) Prove of this part is similarly as prove of (a). □

Proposition 2.9. *Suppose that x_0 is a fixed point for $T : X \to X$ and $\psi : X \to R^+$ is a real valued function.*

(a) *If x_0 be a fixed point of order 2 for T then $\lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(x,x,x_0)}\le 2$.*
(b) *If $G(Tx,Tx,x)\le 2\psi(x)-\psi(Tx)\le G(x,x,x_0)$ for all $x\in X$ then x_0 is a fixed point of order 2 for T if and only if $\lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(x,x,x_0)}=0$.*

Proof. (a) From the inequality

$$\begin{aligned} G(Tx,Tx,x) &\le G(Tx,x_0,x_0)+G(x_0,Tx,x) \\ &\le G(Tx,Tx,x_0)+G(x,x,x_0) \\ \frac{G(Tx,Tx,x)}{G(x,x,x_0)} &\le \frac{G(Tx,Tx,x_0)}{G(x,x,x_0)}+1. \end{aligned}$$

Therefore, $\lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(x,x,x_0)}\le 2$.

(b) From inequality $G(Tx,Tx,x)\le 2\psi(x)-\psi(Tx)\le G(x,x,x_0)$,

$$\begin{aligned} G(x,x,Tx)+G(Tx,Tx,T^2x)+...+G(T^{n-1}x,T^{n-1}x,T^nx) &\le \Sigma_{i=1}^n 2\psi(T^{i-1}x)-\psi(T^ix) \\ &= 2\psi(x)-\psi(T^nx) \end{aligned}$$

and

$$\begin{aligned}\frac{G(T^{n-1}x, T^{n-1}x, T^n x}{G(x,x,x_0)} &= \frac{G(T^{n-1}x, T^{n-1}x, T^n x)}{G(T^{n-1}x, T^{n-1}x, T^{n-2}x)}\frac{G(T^{n-1}x, T^{n-1}x, T^{n-2}x)}{G(T^{n-2}x, T^{n-2}x, T^{n-3}x)}\cdots \\ &= \cdots\frac{G(T^2x, T^2x, x_0)}{G(Tx,Tx,x_0)}\frac{G(Tx,Tx,x_0)}{G(x,x,x_0},\end{aligned}$$

since $\lim_{x\to x_0}\frac{G(T^{n-1}x,T^{n-1}x,T^n x)}{G(x,x,x_0)} = \lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(x,x,x_0)}$ and $\lim_{x\to x_0}\frac{G(T^{n-k}x,T^{n-k}x,T^n x)}{G(x,x,x_0)} = 1$ which $k = 1, 2, ...n-1$, so $\lim_{x\to x_0}\frac{G(T^{n-1}x,T^{n-1}x,T^n x)}{G(x,x,x_0)} = \lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(x,x,x_0)}$. From inequality $G(Tx,Tx,x) \le 2\psi(x) - \psi(Tx) \le G(x,x,x_0)$. It is clear that $\psi(T^n x)$ is strict decreasing.

$$\begin{aligned}n\frac{G(Tx,Tx,x)}{G(x,x,x_0)} &\le \lim_{x\to x_0}\frac{2\psi(x)-\psi(T^n x)}{G(x,x,x_0)} \\ &\le \lim_{x\to x_0}\frac{2\psi(x)-\psi(T^n x)}{2\psi(x)-\psi(Tx)} \\ &\le \lim_{x\to x_0}\frac{2\psi(x)-\psi(T^n x)}{2\psi(x)-\psi(T^n x)} \\ &= 1.\end{aligned}$$

Hence, $\lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(x,x,x_0)} = \frac{1}{n}$. Since n is arbitrary positive integer, $\lim_{x\to x_0}\frac{G(Tx,Tx,x)}{G(x,x,x_0)} = 0$. □

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