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APPROXIMATING FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

BAPURAO C. DHAGE

Abstract. In this paper, we prove a fixed point theorem for the selfmaps of a closed convex and bounded subset of the Banach space satisfying a generalized nonexpansive type condition. Some results concerning the approximations of fixed points with Krasnoselskii and Mann type iterations are also proved under suitable conditions. Our results include the well-known result of Kannan (1968) and Bose and Mukherjee (1981) as the special cases with a different and constructive method.

1. Introduction

Let (X, d) be a metric space. Then Banach contraction principle states that if X is complete and $f : X \to X$ satisfies the condition

$$d(fx, fy) \leq \alpha d(x, y) \tag{1.1}$$

for all $x, y \in X$ and $0 \leq \alpha < 1$, then f has a unique fixed point. The mapping f satisfying the condition (1.1) is called contraction and when $\alpha = 1$, f is called nonexpansive. The nonexansive mappings have been studied by Kirk and Goebel [6] for fixed points. Bogin [1] considered a class of generalized nonexpansive mappings characterized by the inequality

$$d(fx, fy) \leq ad(x, y) + b[d(x, fx) + d(y, fy)] + b[d(x, fy) + d(y, fx)] \tag{1.2}$$

for all $x, y \in X$, where a, b, c are nonnegative real numbers satisfying

$$a + 2b + 2c = 1 \tag{1.3}$$

for the study of fixed points. Recently Ciric [3] generalized the above class of mappings (1.2)-(1.3) to a wider class mappings characterized by the inequality

$$\begin{aligned} d(fx, fy) \leq{}& a \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\right\} \\ &+ b \max\{d(x, fx), d(y, fy)\} \\ &+ c[d(x, fy) + d(y, fx)] \end{aligned} \tag{1.4}$$

for all $x, y \in X$, where the real numbers $a, b, c \geq 0$ satisfy the condition

$$a + b + 2c = 1. \tag{1.5}$$

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Similarly, the study of nonexansive mappings in Banach spaces has been made extensively by several authors. Bose and Mukherjee [2] studied the class of generalized nonexansive mappings for the study of fixed points characterized by the inequality

$$\|fx - fy\| \leq a\,\|x - y\| + b\,[\|x - fx\| + \|y - fy\|] + c\,[\|x - fx\| + \|y - fy\|] \tag{1.6}$$

for all $x, y \in X$, where a, b, c are nonnegative real numbers, $a > 0$ satisfying the condition

$$3a + 2b + 4c = 1. \tag{1.7}$$

The aim of the present note is to generalize the above class of mappings (1.6)-(1.7) and prove a couple of fixed point theorems under a generalized contraction condition with a different method which in turn generalize fixed point theorems of Bose and Mukherjee [2] as the special cases.

2. Generalized Nonexpansive Mappings

Given a non-empty, closed, convex and bounded subset C of the Banach space X, consider the class of nonexpansive type mappings $f : C \to C$ characterized by the inequality

$$\begin{aligned}\|fx - fy\| \leq{}& a \max\Big\{\|x-y\|, \|x - fx\|, \|y - fy\|, \frac{1}{2}[\|x - fy\| + \|y - fx\|]\Big\}\\ &+ b\,\big[\|x - fx\| + \|y - fy\|\big]\\ &+ c \max\big\{\|x - fy\|, \|y - fx\|\big\}\end{aligned} \tag{2.1}$$

for all $x, y \in X$, where the real numbers $a, b, c \geq 0$ satisfy the inequality

$$a + b + c \leq \frac{1}{2}. \tag{2.2}$$

The generalized nonexpansive mappings characterized by the inequalities (2.1) and (2.2) have been considered in Dhage [4] in the setting of a metric space for fixed points and are different from the class of Ciric's mappings characterized by the inequalities (1.6) and (1.7). In this section we prove a couple of results concerning the existence of fixed point for the class of generalized nonexpansive mappings (2.1) and (2.2) in a Banach space via a scheme of Krasnoselskii type iterations.

Theorem 2.1. *Let C be a non-empty, closed, convex and bounded subset of the normed linear space X and let $f : C \to C$ be a mapping satisfying the inequality* (2.1) *and* (2.2) *with $a > 0$. If the sequence $\{x_n\}$ defined by*

$$x_{n+1} = (1-t)x_n + t f x_n, \quad n = 0, 1, 2, ...; \tag{2.3}$$

for some $t \in (0, 1)$ and for some $x = x_0 \in C$ converges to u, then u is a unique fixed point of f.

Proof. By (2.1), one gets

$$\begin{aligned}\|x_{n+1} - fu\| &\le (1-t)\|x_n - fu\| + t\|fx_n - fu\| \\ &\le (1-t)\|x_n - fu\| + a\Big\{\|x_n - u\|, \|x_n - fx\|, \|u - fu\|, \\ &\qquad \frac{1}{2}[\|x_n - fu\| + \|u - fx_n\|]\Big\} \\ &\quad + b\,[\|x_n - fx_n + \|u - fu\| + \|] \\ &\quad + c\ \max\{\|x_n - fu\|, \|u - fx_n\|\}. \end{aligned} \tag{2.4}$$

Now,

$$x_{n+1} = (1-t)x_n + tfx_n,$$

and so we have

$$(x_{n+1} - x_n) = -t(x_n - fx_n).$$

This further implies that

$$\|x_{n+1} - x_n\| = t\|x_n - fx_n\| \longrightarrow 0 \quad \text{as } n \to \infty.$$

Taking the limit as $n \to \infty$ in (2.4), we obtain

$$\begin{aligned}\|u - fu\| &\le (1-t)\|u - fu\| \\ &\quad + t\,a\ \max\Big\{0, 0, \|u - fu\|, \frac{1}{2}\|u - fu\|\Big\} \\ &\quad + t\,b\,[0 + \|u - fu\|] + t\,c\ \max\{\|u - fu\|, 0\} \\ &\le [(1-t) + ta + tb + tc]\|u - fu\| \\ &\le (1 - t + a + b + c)\|u - fu\|. \end{aligned}$$

Since $a + b + c < 1$, we may choose $t \in (0,1)$ such that $t > a + b + c$. Then from the above inequality, we obtain so $u = fu$.

To prove uniqueness, let $v(\neq u)$ be another fixed point of f. Then by (2.1),

$$\begin{aligned}\|u - v\| &= \|fu - fv\| \\ &\le a\ \max\Big\{\|u - v\|, \|u - fu\|, \|v - fv\|, \frac{1}{2}[\|u - fv\| + \|v - fu\|]\Big\} \\ &\quad + b\,[\|u - fu\| + \|v - fv\|] + c\ \max\{\|u - fv\|, \|v - fu\|\} \\ &= (a + c)\,\|u - v)\| \end{aligned}$$

which is a contradiction. Hence $u = v$ and the proof of the theorem is complete. □

Theorem 2.2. *Let C be a non-empty, closed, convex and bounded subset of a Banach space X. If $f : C \to C$ satisfies the inequalities* (2.1) *and* (2.2) *with $a > 0$, $b > 0$, then f has a unique fixed point.*

Proof. Let $x = x_0 \in C$ be arbitrary and consider the sequence $\{x_n\}$ defined by (2.3). Then, we have

$$x_1 - x_2 = (1-t)(x_0 - x_1) + t(fx_0 - fx_2).$$

Then, by (2.1), we obtain

$$\begin{aligned}\|x_1 - x_2\| &\le (1-t)\|x_0 - x_1\| + t\|fx_0 - fx_1\| \\ &\le (1-t)\|x_0 - x_1\| \\ &\quad + ta \max\Big\{\|x_0 - x_1\|, \|x_0 - fx_0\|, \|x_1 - fx_1\|, \\ &\qquad\qquad \frac{1}{2}[\|x_0 - x_1\| + \|x_1 - x_2\|]\Big\} \\ &\quad + tb\,[\|x_0 - fx_0\| + \|x_1 - fx_1\|] \\ &\quad + tc \max\{\|x_0 - fx_1\|, \|x_1 - fx_0\|\}. \qquad (2.5)\end{aligned}$$

Now,

$$x_1 = (1-t)x_0 + tfx_0,$$

and so we have

$$\Rightarrow x_1 - x_0 = -t(x_0 - fx_0).$$

This further implies that

$$t\|x_0 - fx_0\| = \|x_0 - x_1\|.$$

Again,

$$x_2 = (1-t)x_1 + tfx_1,$$

and so we have

$$x_2 - x_1 = -t(x_1 - fx_1)$$

which again implies that

$$t\|x_1 - fx_1\| = \|x_1 - x_2\|.$$

Similarly,

$$(x_0 - fx_1) = (x_0 - x_1) + (x_1 - fx_1),$$

implies

$$t(x_0 - fx_1) = t(x_0 - x_1) + t(x_1 - x_2)),$$

and

$$t\|x_0 - fx_1\| \le t\|x_0 - x_1\| + t\|x_1 - x_2\|.$$

Again,

$$x_1 - fx_0 = x_1 - x_0 + x_0 - fx_0 = (x_1 - x_2) + (x_0 - fx_0),$$

which gives

$$t(x_1 - fx_0) = t(x_1 - x_0) + t(x_0 - f(x_0) = (1-t)(x_0 - x_1),$$

or,

$$t\|x_1 - fx_0\| = (1-t)\|x_0 - x_1\|.$$

Substituting the above values in (2.5),

$$
\begin{aligned}
\|x_1-x_2\| \le{}& (1-t)\|x_0-x_1\| \\
&+ a\,\max\Big\{\|x_0-x_1\|,\|x_0-x_1\|,\|x_1-x_2\|, \\
&\qquad \frac{1}{2}\big[\big[(1-t)\|x_0-x_1\|+t\|x_0-x_1\|+\|x_1-x_2\|\big]\Big\} \\
&+ b\,\big[\|x_0-x_1\|+\|x_1-x_2\|\big] \\
&+ c\,\max\big\{(1-t)\|x_0-x_1\|, t\|x_0-x_1\|+\|x_1-x_2\|\big\} \\
={}& (1-t)\|x_0-x_1\| \\
&+ a\,\max\Big\{\|x_0-x_1\|,\|x_1-x_2\|,\frac{1}{2}\big[\big[\|x_0-x_1\|+\|x_1-x_2\|\big]\Big\} \\
&+ b\,\big[\|x_0-x_1\|+\|x_1-x_2\|\big] \\
&+ c\,\max\big\{(1-t)\|x_0-x_1\|, t\|x_0-x_1\|+\|x_1-x_2\|\big\}. \qquad (2.6)
\end{aligned}
$$

Now there are three cases:

Case I: Suppose that

$$\max\Big\{\|x_0-x_1\|,\|x_1-x_2\|,\frac{1}{2}\big[\|x_0-x_1\|+\|x_1-x_2\|\big]\Big\} = \|x_0-x_1\|$$

and

$$\max\big\{(1-t)\|x_0-x_1\|, t\|x_0-x_1\|+\|x_1-x_2\|\big\} = \|x_0-x_1\|$$

for $t > \frac{1}{2}$. Then from (2.6),

$$
\begin{aligned}
(1-b)\|x_1-x_2\| \le{}& (1-t)\|x_0-x_1\| + (a+b)\|x_0-x_1\| \\
&+ ct\,\|x_0-x_1\| + c\|x_1-x_2\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x_1-x_2\| &\le \left(\frac{(1-t)+a+b+ct}{1-b-c}\right)\|x_0-x_1\| \\
&\le \left(\frac{(1-t)+a+b+c}{1-b-c}\right)\|x_0-x_1\| \\
&= \alpha_1\|x_0-x_1\|
\end{aligned}
$$

Case II: Suppose that

$$\max\Big\{\|x_0-x_1\|,\|x_1-x_2\|,\frac{1}{2}\big[\|x_0-x_1\|+\|x_1-x_2\|\big]\Big\} = \|x_1-x_2\|.$$

Then,

$$
\begin{aligned}
\|x_1-x_2\| \le{}& (1-t)\|x_0-x_1\| + a\,\|x_1-x_2\| \\
&+ b\,\|x_0-x_1\| + b\|x_1-x_2\| \\
&+ ct\,\|x_0-x_1\| + c\|x_1-x_2\| \\
\le{}& \left(\frac{1-t+b+c}{1-a-b-c}\right)\|x_0-x_1\| \\
\le{}& \alpha_2\|x_0-x_1\| \qquad [t > a+2b+2c]
\end{aligned}
$$

Case III: Suppose that

$$\max\left\{\|x_0-x_1\|,\|x_1-x_2\|,\frac{1}{2}\left[\|x_0-x_1\|+\|x_1-x_2\|\right]\right\}=\frac{1}{2}[\|x_1-x_2\|+\|x_0-x_1\|].$$

Then,

$$\begin{aligned}\|x_1-x_2\| &\le (1-t)\|x_0-x_1\|+\frac{a}{2}\|x_0-x_1\|+\frac{a}{2}\|x_1-x_2\|\\ &\quad + b\,\|x_0-x_1\|+b\|x_1-x_2\|\\ &\quad + c\,\|x_0-x_1\|+c\|x_1-x_2\|\\ &\le \left(\frac{1-t+\frac{a}{2}+b+c}{1-\frac{a}{2}-b-c}\right)\|x_0-x_1\|\\ &\le \alpha_3\|x_0-x_1\| \qquad [t>a+2b+2c].\end{aligned}$$

Let $\alpha=\max\{\alpha_1,\alpha_2,\alpha_3\}$, then in all above three cases we obtain

$$\|x_1-x_2\|\le\alpha\|x_0-x_1\|.$$

Therefore,

$$\begin{aligned}\|x_n-x_{n+1}\| &\le \sum_{i=n}^{n+p}\|x_i-x_{i+1}\|\\ &\le \frac{\alpha^n}{1-\alpha}\|x_0-x_1\|\\ &\longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy is a sequence in C. Since C is a closed subset of a complete space, it is complete. Hence $\{x_n\}$ is convergent and converse to a point $u\in C$. The rest of the proof is similar to Theorem 2.1 and so we omit the details. □

Corollary 2.1. *Let C be a non-empty, closed, convex and bounded subset of the normed linear space X and let $f:C\to C$. Suppose that there exists a positive integer r such that f satisfies the contraction condition*

$$\begin{aligned}\|f^rx-f^ry\| &\le a\,\max\left\{\|x-y\|,\|x-f^rx\|,\|y-f^ry\|,\frac{1}{2}[\|x-f^ry\|+\|y-f^rx\|]\right\}\\ &\quad + b\left[\|x-f^rx\|+\|y-f^ry\|\right]\\ &\quad + c\,\max\left\{\|x-f^ry\|,\|y-f^rx\|\right\}\end{aligned} \tag{2.7}$$

for all $x,y\in C$, where the real numbers $a,b,c\ge 0$, $a>0$, satisfy the inequality

$$a+b+c\le\frac{1}{2}. \tag{2.8}$$

If the sequence $\{x_n\}$ defined by

$$x_{n+1}=(1-t)x_n+tf^rx_n,\quad n=0,1,2,...; \tag{2.9}$$

for some $t\in(0,1)$ and for some $x=x_0\in C$ converges to u, then u is a unique fixed point of f.

Proof. By Theorem 2.1 above, the mapping f^r has a unique fixed point, say $p\in C$. Then we have $f^r(p)=p$. Therefore, $f^r(fp)=f^{r+1}(p)=f(f^r(p))=fp$ showing that fp is again a fixed point of f^r. By uniqueness of p, we get $fp=p$. Thus, f has a unique fixed point p in C and the sequence of iterations given by (2.9) converges to p. The proof of the theorem is complete. □

In the following section we prove that the Mann iterations of the mapping f in a uniformly convex Banach space satisfying (2.1) and (2.2).

3. Convergence of Mann Iterations

The following definitions is well-known in the literature.

Definition 3.1. *A self mapping f of a convex subset C of a Banach space X is said to be quasi-nonexpansive provided f has a fixed point and if p is a fixed point of f, then*

$$\|fx - p\| \leq \|x - p\|$$

for all $x \in C$.

In a uniformly Banach space, Senter and Dotson, Jr., have conditions under which the sequence of Mann types of iterates of a quasi-nonexpansive mapping converges to a fixed point of the mapping in question. We denote by $\mathcal{F}(f)$ the set of all fixed points of f in C.

Condition I: Let C be a convex subset of a uniformly convex Banach space X. A mapping $f : C \to C$ is said to satisfy Condition I if there is a nondecreasing function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ satisfying $\|x - fx\| > \beta(d(x, \mathcal{F}(f)))$ for all $x \in C$, where $\beta(d(x, \mathcal{F}(f))) = \inf_{\{} \|x - p\| : p \in \mathcal{F}(f)\}$.

Condition I: Let C be a convex subset of a uniformly convex Banach space X. A mapping $f : C \to C$ is said to satisfy Condition I if there is a real number $\alpha > 0$ such that $\|x - fx\| \geq \alpha\, d(x, \mathcal{F}(f))$ for all $x \in C$.

It can be easily shown that a mapping which satisfies Condition II also satisfies Condition I. Now, we state a key theorem of Senter and Dotson [9] which is used in what follows. Before going to the theorem we define the Mann iterations of the mapping f on a subset C of the Banach space X. Let $x_1 \in C$ be arbitrary and let $M(x_1, t_n, f)$ be a sequence $\{x_n\}$ defined by $x_{n+1} = (1 - t_n)x_n + t_n f(x_n)$, where $t_n \in [\beta, \gamma]$, $0 < \beta < \gamma < 1$ and $n \in \mathbb{N}$.

Theorem 3.1 (Senter and Dotson [9])**.** *Let X be a uniformly convex Banach space, C a closed, convex and bounded subset of X and let f be a nonexpansive mapping of C into itself. If f satisfies Condition I, then for arbitrary $x_1 \in C$, the sequence $M(x_1, t_n, f)$ converges to a member of $\mathcal{F}(f)$.*

Below we prove a result concerning the convergence of the sequence of Mann iterations to the fixed point of generalized nonexpansive mappings in a uniformly Banach space.

Theorem 3.2. *let C be a closed, convex and bounded subset of a uniformly Banach space X and let $f : C \to C$ be a generalized nonexpansive mapping satisfying the inequalities* (2.1) *and* (2.2)*. Then f has a unique fixed point p and for arbitrary $x_1 \in C$, the sequence $M(x_1, t_n, f)$ of Mann iterations converges to p.*

Proof. By Theorem 2.1, f has a unique fixed point p in C. We show that the sequence $M(x_1, t_n, f)$ of Mann iterations converges to p for arbitrary $x_1 \in C$. This will be achieved in the following two steps:

Step I: f is quasi-nonexpansive on C.

We first show that f is a quasi-nonexpansive mapping on C into itself. Assume the contrary, that is, $\|fx - p\| > \|x - p\|$ for some $x \in C$. Then by (2.1), we have

$$
\begin{aligned}
\|fx - p\| &= \|fx - fp\| \\
&\leq a \max\{\|x - p\|, \|x - fx\|, \|p - fp\|, \frac{1}{2}\left[\|x - fp\| + \|p - fx\|\right]\} \\
&\quad + b\left[\|x - fx\| + \|p - fp\|\right] \\
&\quad + c \max\{\|x - fp\|, \|p - fx\|\} \\
&= a \max\{\|x - p\|, \|x - fx\|, \frac{1}{2}\left[\|x - p\| + \|fx - p\|\right]\} \\
&\quad + b\,\|x - fx\| + c \max\{\|x - p\|, \|fx - p\|\} \\
&\leq a \max\{\|x - p\|, \|x - fx\|, \|fx - p\|\} \\
&\quad + b\,\|x - fx\| + c\,\|fx - p\| \\
&\leq a \max\{\|x - fx\|, \|fx - p\|\} \\
&\quad + b\,\|x - fx\| + c\,\|fx - p\|. \qquad (3.1)
\end{aligned}
$$

Now there are two cases:

Case I: Suppose that

$$\max\{\|x - fx\|, \|fx - p\|\} = \|x - p\|.$$

Then from (3.1), we obtain

$$\|fx - p\| \leq (a + b + c)\|fx - p\|$$

which is a contradiction, since $a + b = c \leq \frac{1}{2}$.

Case II: Suppose that

$$\max\{\|x - fx\|, \|fx - p\|\} = \|x - p\|.$$

Then from (3.1), we obtain

$$
\begin{aligned}
\|fx - p\| &\leq (a + b + c)\|x - fx\| \\
&\leq (a + b + c)[\|x - p\| + \|fx - p\|] \\
&= (a + b + c)\|x - p\| + (a + b + c)\|fx - p\|
\end{aligned}
$$

which further implies that

$$\|fx - p\| \leq \left[\frac{a + b + c}{1 - (a + b + c)}\right] \|x - p\|$$

which is a contradiction, since $\dfrac{a + b + c}{1 - (a + b + c)} \leq 1$.

Thus, in both the cases, we obtain a contradiction. Therefore, we conclude that $\|fx - p\| \leq \|x - p\|$ for all $x \in C$ and consequently f is quasi-nonexpansive on C.

Step I: f satisfies Condition II on C.

let $x \in C$ be arbitrary. Then,

$$\|x - p\| \leq \|x - fx\| + \|fx - p\|. \qquad (3.2)$$

Now, by (2.1),

$$\begin{aligned}\|fx-p\| &= \|fx-fp\| \\ &\le a\,\max\{\|x-p\|,\|x-fx\|,\|p-fp\|,\frac{1}{2}\left[\|x-fp\|+\|p-fx\|\right]\} \\ &\quad + b\left[\|x-fx\|+\|p-fp\|\right] \\ &\quad + c\,\max\{\|x-fp\|,\|p-fx\|\} \\ &= a\,\max\{\|x-p\|,\|x-fx\|\}+b\,\|x-fx\|+c\,\|x-p\|. \qquad (3.3)\end{aligned}$$

Now there are two cases:

Case I: Suppose that

$$\max\{\|x-fx\|,\|fx-p\|\}=\|x-p\|.$$

Then from (3.1), we obtain

$$\|fx-p\|\le (a+c)\|x-p\|+b\|x-fx\|.$$

Substituting above value in (3.2), we obtain

$$\|x-fx\|\ge \frac{1}{3}\|x-p\|=\frac{1}{3}d(x,\mathcal{F}(f)).$$

Case II: Suppose that

$$\max\{\|x-fx\|,\|x-p\|\}=\|x-fx\|.$$

Then from (3.1), we obtain

$$\|fx-p\|\le (a+b)\|x-fx\|+c\|x-p\|.$$

Substituting above value in (3.2), we obtain

$$\|x-fx\|\ge \frac{1}{3}\|x-p\|=\frac{1}{3}d(x,\mathcal{F}(f)).$$

Thus, f satisfies Condition II with $\alpha=\frac{1}{3}$. Consequently f satisfies Condition I and by an application of Theorem 3.1, for arbitrary $x_1\in C$, the sequence $M(x_1,t_n,f)$ of Mann iterations of f converges to p. This completes the proof. □

Corollary 3.1. *let C be a closed, convex and bounded subset of a uniformly Banach space X and let $f:C\to C$. Suppose that there exists a positive integer r such that f satisfies the generalized contraction condition* (2.7) *and* (2.8). *Then f has a unique fixed point p and for arbitrary $x_1\in C$, the sequence $M(x_1,t_n,f^r)$ of Mann iterations converges to p.*

Proof. By Theorem 2.1, the mapping f has a unique fixed point p in C which is also a unique fixed point of f^r. Now the desired conclusion follows by a direct application of Theorem 3.2. □

As a consequence of Theorem 3.2, we obtain the following fixed point theorem of Bose and Mukherjee [2] as a corollary.

Corollary 3.2. *let C be a closed, convex and bounded subset of a uniformly Banach space X and let $f:C\to C$ be a generalized nonexpansive mapping satisfying the inequalities* (1.6) *and* (1.7). *Then f has a unique fixed point p and for arbitrary $x_1\in C$, the sequence $M(x_1,t_n,f)$ of Mann iterations converges to p.*

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Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India