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Fekete-Szegö Problem for Certain Analytic Functions of Complex Order Associated With Cardioid Domain

Aijaz Ahmed[1], **Abdul Wasim Shaikh**[1], **Shujaat Ali Shah**[2,*]

[1]*Institute of Mathematics and Computer Science, University of Sindh, Allama I.I. Kazi Campus, Jamshoro, Sindh, Pakistan*

[2]*Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, Nawabshah, Pakistan*

[*]*Corresponding author:* shahglike@yahoo.com, shujaatali@quest.edu.pk

Abstract. In this present investigation, using a linear multiplier differential operator $\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)$, we introduced a novel subclass of starlike and convex functions of complex order γ defined on the open unit disk $\mathfrak{E}$, associated with cardioid domain. This subclass provides a solution to the Fekete–Szegö problem, along with the consideration of numerous new special cases.

1. Introduction

Let $\mathfrak{A}$ denote the family of functions f and has the following series representation

$$f(\psi) = \psi + \sum_{n=2}^{\infty} a_n \psi^n \tag{1.1}$$

analyticity in the disk $\mathfrak{E} = \left\{\psi : \left|\psi\right| < 1\right\}$. And let $\mathcal{S}$ be the set of functions that preserve univalence in $\mathfrak{E}$.

Consider the function f and g to be analytic in the unit disk $\mathfrak{E}$, It is stated that f is subordinate to g, symbolically $f \prec g$, If an analytic function $u : \mathfrak{E} \to \mathfrak{E}$ with $u(0) = 0$ exists, such that

$f(\psi) = g(u(\psi))$. if g is univalent, the condition $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathfrak{E}) \subseteq g(\mathfrak{E})$.

It is a familiar result that for $f \in \mathcal{S}$, $\left|a_3 - a_2^2\right| \leq 1$. A well-known theorem by Fekete–Szegö (see [1]) expresses that for $f \in \mathcal{S}$ represented by (1.1).

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$$\left|a_3-\rho a_2^2\right|\leq\begin{cases}3-4\rho & \text{if } \rho\leq 0,\\ 1+2\exp\left(\frac{-2\rho}{1-\rho}\right) & \text{if } 0<\rho<1,\\ 4\rho-3 & \text{if } \rho\geq 1.\end{cases}$$

This inequality holds sharp as, for every ρ, there is a function in $\mathcal{S}$ satisfying the equality. Pfluger later examined the complex values of ρ, (see [2]) and proposed the inequality:

$$\left|a_3-\rho a_2^2\right|\leq 1+2\left|\exp\left(\frac{-2\rho}{1-\rho}\right)\right|.$$

So far, several researchers have focused on extending the given inequality above generalized for more extensive classes of analytic functions.

Among the subclasses of the class $\mathcal{S}$, the subclasses S^* and C of starlike functions and of convex functions respectively, are the most essential and noteworthy. These subclasses can be expressed using subordination as follows:

$$S^*=\left\{f\in\mathfrak{A}:\frac{\psi f'(\psi)}{f(\psi)}\prec\frac{1+\psi}{1-\psi}\right\}$$

and

$$C=\left\{f\in\mathfrak{A}:1+\frac{\psi f''(\psi)}{f'(\psi)}\prec\frac{1+\psi}{1-\psi}\right\}.$$

The following generalization of S^* and C was proposed by Ma and Minda [3].

$$S^*(\varphi)=\left\{f\in\mathfrak{A}:\frac{\psi f'(\psi)}{f(\psi)}\prec\varphi(\psi)\right\}$$

and

$$C(\varphi)=\left\{f\in\mathfrak{A}:1+\frac{\psi f''(\psi)}{f'(\psi)}\prec\varphi(\psi)\right\},$$

where $\varphi(\psi)$ denotes a real part function, which is positive and satisfies the normalization condition $\varphi(0)=1$, and $\varphi'(0)>0$. Moreover, φ transforms $\mathfrak{E}$ into a starlike region with respect to 1, which is symmetric about the real axis. Various subclasses of the class $\mathfrak{A}$ of normalized analytic functions have been studied as special cases. For more details, see ([4], [5], [6], [7], [8], [9]).

A function $f(\psi)\in\mathfrak{A}$ is said to be in he class of starlike functions of complex order γ, denoted $S^*(\gamma)$, if and only if

$$R\left\{1+\frac{1}{\gamma}\left(\frac{\psi f'(\psi)}{f(\psi)}-1\right)\right\}>0,\left(\psi\in\mathfrak{E};\ \gamma\in\mathbb{C}\setminus\{0\}\right).$$

Moreover, a function $f\in\mathfrak{A}$ belongs to the clas $C(\gamma)$ of convex functions of complex order γ, provided it satisfies the inequality

$$R\left\{1+\frac{1}{\gamma}\left(\frac{\psi f''(\psi)}{f'(\psi)}-1\right)\right\}>0,\left(\psi\in\mathfrak{E};\ \gamma\in\mathbb{C}\setminus\{0\}\right).$$

The well-known classes $S^*(\gamma)$ and $C(\gamma)$ were investigted by Nasr and Aouf [10] and Wiatrowski [11]. It should be noted that $S^*(1)$ and $C(1)$ represent the familiar classes of starlike functions and convex functions, respectively.

The extension of the above two classes was given by Ravichandran et al. [12] in the following way:

$$S^*(\varphi,\gamma)=\left\{f\in\mathfrak{A}:1+\frac{1}{\gamma}\left(\frac{\psi f'(\psi)}{f(\psi)}\right)\prec\varphi(z),\ \ (\psi\in\mathfrak{E};\ \gamma\in\mathbb{C}\setminus\{0\})\right\}$$

and

$$C(\varphi,\gamma)=\left\{f\in\mathfrak{A}:1+\frac{1}{\gamma}\left(\frac{\psi f''(\psi)}{f'(\psi)}\right)\prec\varphi(z),\ \ (\psi\in\mathfrak{E};\ \gamma\in\mathbb{C}\setminus\{0\})\right\}.$$

Such type of functions are known as Ma–Minda starlike and convex functions of order γ, $(\gamma\in\mathbb{C}\setminus\{0\})$ respectively.

The linear multiplier differential operator $\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}f(\psi)$ was defined by Deniz and Orhan in [13] as follows:

$$\mathfrak{D}^{0}_{\lambda,\mu}f(\psi)=f(\psi)$$

$$\mathfrak{D}^{1}_{\lambda,\mu}f(\psi)=\mathfrak{D}_{\lambda,\mu}f(\psi)=\lambda\mu\psi^2(f(\psi))''+(\lambda-\mu)\psi(f(\psi))'+(1-\lambda+\mu)f(\psi)$$

$$\mathfrak{D}^{2}_{\lambda,\mu}f(\psi)=\mathfrak{D}_{\lambda,\mu}\left(\mathfrak{D}^{1}_{\lambda,\mu}f(\psi)\right)$$

$$\vdots$$

$$\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}f(\psi)=\mathfrak{D}_{\lambda,\mu}\left(\mathfrak{D}^{\mathfrak{M}-g}_{\lambda,\mu}f(\psi)\right),$$

where $\lambda\geq\mu\geq0$ and $\mathfrak{M}\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

If f is of the form (1.1), based on the definition of the operator $\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}f(\psi)$ it is simple to observe

that

$$\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}f(\psi)=\psi+\sum_{n=2}^{\infty}\left[1+(\lambda\mu n+\lambda-\mu)(n-1)\right]^{\mathfrak{M}}a_n\psi^n \tag{1.2}$$

It is important to mention that $\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}$ is a more generalized version of the linear operators reviewed earlier. In special cases, for $f\in\mathfrak{A}$ we observe the following

- $\mathfrak{D}^{\mathfrak{M}}_{1,0}f(\psi)\equiv\mathfrak{D}^{\mathfrak{M}}f(\psi)$, studied by Sãlãgean [14].
- $\mathfrak{D}^{\mathfrak{M}}_{\lambda,0}f(\psi)\equiv\mathfrak{D}^{\mathfrak{M}}_{\lambda}f(\psi)$ is studied by Al-Oboudi [15].
- $\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}f(\psi)$ is firstly considered for $0\leq\mu\leq\lambda\leq1$, by Rãducanu and Orhan [16].

We now define a new subclass of analytic functions with the aid of the differential operator.$\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}f(\psi)$ as follow:

Definition 1.1. *Let γ be a complex number, not equal to zero, and let $f \in \mathfrak{A}$,such that $\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi) \neq 0$ for $\psi \in \mathfrak{E} - \{0\}$. We say that f belongs to $\mathcal{S}^{(\alpha,\gamma)}_{car}(\lambda, \mu; \varphi)_{\mathfrak{M}}$ if*

$$1+\frac{1}{\gamma}\left(\frac{\psi\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)' + \alpha\psi^2\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)''}{(1-\alpha)\ \mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\ +\ \alpha\psi\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)'} - 1\right) \prec \varphi(\psi),$$

for $0 \leq \mu \leq \lambda$, $\mathfrak{M} \in \mathbb{N}_0$, $0 \leq \alpha \leq 1$. Where

$$\varphi(\psi) = 1 + \frac{4}{3}\psi + \frac{2}{3}\psi^2;\ \ (\psi \in \mathfrak{E}) \tag{1.3}$$

Alternatively, $f \in \mathcal{S}^{(\alpha,\gamma)}_{car}(\lambda, \mu; \varphi)_{\mathfrak{M}}$ when the function

$$1+\frac{1}{\gamma}\left(\frac{\psi\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)' + \alpha\psi^2\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)''}{(1-\alpha)\ \mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\ +\ \alpha\psi\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)'} - 1\right)$$

takes its values from the cardioid domain given by (1.3).

Definition 1.2. *A function $f \in \mathcal{S}^{(\gamma)}_{car}(\lambda, \mu; \varphi)_{\mathfrak{M}}$, and has the form (1.1), if*

$$1+\frac{1}{\gamma}\left(\frac{\psi\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)'}{\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)} - 1\right) \prec \varphi(\psi),\ \left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\ \ \neq 0 \text{ for } \psi \in \mathfrak{E} - \{0\},\ \gamma \in \mathbb{C}\setminus\{0\}\right),$$

where $\varphi(z)$ is given by (1.3) and $0 \leq \mu \leq \lambda$, $\mathfrak{M} \in \mathbb{N}_0$.

Definition 1.3. *A function $f \in C^{(\gamma)}_{car}(\lambda, \mu; \varphi)_{\mathfrak{M}}$, and has the form (1.1), if the following conditions are satisfied*

$$1+\frac{1}{\gamma}\left(\frac{\psi\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)''}{\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu} f(\psi)\right)'}\right) \prec \varphi(\psi),$$

where $\gamma \in \mathbb{C}\setminus\{0\}$, $0 \leq \mu \leq \lambda$, $\mathfrak{M} \in \mathbb{N}_0$ and $\varphi(\psi)$ is given by (1.3).

By fixing the parameters $\alpha, \gamma, \mathfrak{M}$ and λ, μ to specific values, we obtain the well-known classes $\mathcal{S}^{(0,1)}_{car}(0{,}0; \varphi)_0 \equiv \mathcal{S}_{car}$ and $C^{(1)}_{car}(0{,}0; \varphi)_0 \equiv C_{car}$ introduced and examined by Kanika Sharma et al [17].

2. Main results

Let P denote a class of analytic function in $\mathfrak{E}$ with $p(0) = 1$ and $R(p(0)) > 0$. For the derivation of our main results, we need to refer back to the following Lemma (see, [18]).

Lemma 2.1. *Let $p \in P$ with $p(\psi) = 1 + c_1\psi + c_2\psi^2 + \ldots$. Then*

$$|c_n| \leq 2, \text{ for } n \geq 1.$$

If $|c_1| = 2$, then $p(\psi) \equiv p_1(\psi) = \frac{1+\delta_1\psi}{1-\delta_1\psi}$ with $\delta_1 = \frac{c_1}{2}$. Conversely, if $p(\psi) \equiv p_1(\psi)$ in the case of some $|\delta_1| = 1$. Then $c_1 = 2\delta_1$ and $|c_1| = 2$. Additionally, we have

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|^2}{2}$, then $p(\psi) \equiv p_2(\psi)$, where

$$p_2(\psi) = \frac{1 + \psi\frac{\delta_2\psi+\delta_1}{1+\overline{\delta_1}\delta_2\psi}}{1 - \psi\frac{\delta_2\psi+\delta_1}{1+\overline{\delta_1}\delta_2\psi}},$$

and $\delta_1 = \frac{c_1}{2}$, $\delta_2 = \frac{2c_2-c_1^2}{4-|c_1|^2}$. Conversely, if $p(\psi) \equiv p_2(\psi)$ for some $|\delta_1| < 1$ and $|\delta_2| = 1$ then $\delta_1 = \frac{c_1}{2}$, $\delta_2 = \frac{2c_2-c_1^2}{4-|c_1|^2}$ and $\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}$.

Lemma 2.2. *[19]. Let $p \in P$ with $p(\psi) - 1 = c_1\psi + c_2\psi^2 + \ldots$. Then, for any $\rho \in \mathbb{C}$*

$$\left|c_2 - \rho c_1^2\right| \leq 2 \max\left\{1, \left|2\rho - 1\right|\right\}.$$

Theorem 2.1. *Let $\gamma \in \mathbb{C}\setminus\{0\}$, and $0 \leq \mu \leq \lambda$, $0 \leq \alpha \leq 1$, $\mathfrak{M} \in \mathbb{N}_0$, $\rho \in \mathbb{C}$. If f, represented in the form (1.1), is in $\mathcal{S}_{car}^{(\alpha,\gamma)}(\lambda, \mu; \varphi)_{\mathfrak{M}}$, then*

$$|a_2| \leq \frac{4|\gamma|}{3(\alpha+1)A^{\mathfrak{M}}}, \tag{2.1}$$

$$|a_3| \leq \frac{|\gamma(8\gamma+3)|}{9(2\alpha+1)B^{\mathfrak{M}}} \tag{2.2}$$

and

$$\left|a_3 - \rho a_2^2\right| \leq \frac{2|\gamma|}{3(2\alpha+1)B^{\mathfrak{M}}} \max\left\{1, \left|\frac{16\rho\gamma(2\alpha+1)B^{\mathfrak{M}} - (\alpha+1)^2(8\gamma-3)A^{2\mathfrak{M}}}{6(\alpha+1)^2 A^{2\mathfrak{M}}}\right|\right\}, \tag{2.3}$$

where $A = [1 + (2\lambda\mu + \lambda - \mu)]$ and $B = [1 + 2(3\lambda\mu + \lambda - \mu)]$. Consider the functions

$$\frac{\psi\left(\Delta_{\lambda,\mu}^{\mathfrak{B},\mathfrak{M}} f(\psi)\right)'}{\Delta_{\lambda,\mu}^{\mathfrak{B},\mathfrak{M}} f(\psi)} = 1 + \gamma[p_1(\psi) - 1] \tag{2.4}$$

and

$$\frac{\psi\left(\Delta_{\lambda,\mu}^{\mathfrak{B},\mathfrak{M}} f(\psi)\right)'}{\Delta_{\lambda,\mu}^{\mathfrak{B},\mathfrak{M}} f(\psi)} = 1 + \gamma[p_2(\psi) - 1], \tag{2.5}$$

where $p_1(\psi)$ and $p_2(\psi)$ are described in Lemma 2.1. Equality in (2.1) is valid provided that (2.4) is true. The equality in (2.2) holds when both (2.4) and (2.5) are satisfied. In the same way, the equality in (2.3) is satisfied for each ρ provided that (2.4) and (2.5) are valid.

Proof. Denote $(1-\alpha)\,\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}f(\psi)+\alpha\psi\left(\mathfrak{D}^{\mathfrak{M}}_{\lambda,\mu}f(\psi)\right)'=\Delta^{\mathfrak{B},\mathfrak{M}}_{\lambda,\mu}f(\psi)=\psi+\beta_2\psi^2+\beta_3\psi^3+....$ Then □

$$\beta_2=(\alpha+1)\,A^{\mathfrak{m}}a_2,\beta_3=(2\alpha+1)\,B^{\mathfrak{m}}a_3 \tag{2.6}$$

By definition of the class $\mathcal{S}^{(\alpha,\gamma)}_{car}(\lambda,\mu\,;\varphi)$, there exists a function $u(\psi)$ that is analytic such that $\left|u(\psi)\right|\leq\left|\psi\right|$ in $\mathfrak{E}$ and $p(\psi)=\varphi(u(\psi))$, therefore

$$\frac{\psi\left(\Delta^{\mathfrak{B},\mathfrak{M}}_{\lambda,\mu}f(\psi)\right)'}{\Delta^{\mathfrak{B},\mathfrak{M}}_{\lambda,\mu}f(\psi)}=1+\gamma\left[p(\psi)-1\right],$$

so that

$$\frac{\psi\left(1+2\beta_2\psi+3\beta_3\psi^2+..\right)}{\psi+\beta_2\psi^2+\beta_3\psi^3+...}=1-\gamma+\gamma\left(1+\frac{2c_1}{3}z+\left(\frac{2c_2}{3}-\frac{c_1^2}{6}\right)z^2+\left(\frac{2c_2}{3}-\frac{c_1c_2}{3}\right)z^3+...\right)-$$

which results in the equality

$$\psi+2\beta_2\psi^2+3\beta_3\psi^3+...=\psi+\left(\beta_2+\frac{2\gamma c_1}{3}\right)\psi^2+\left(\beta_3+\frac{2\gamma c_1}{3}\beta_2+\gamma\left(\frac{2c_2}{3}-\frac{c_1^2}{6}\right)\right)\psi^3+...$$

Equating the coefficients to get

$$\beta_2=\frac{2\gamma c_1}{3},\ \ \beta_3=\frac{\gamma c_2}{3}+\frac{\gamma(8\gamma-3)\,c_1^2}{36} \tag{2.7}$$

so that, from (2.6) and (2.7),

$$a_2=\frac{2\gamma c_1}{3(\alpha+1)\,A^{\mathfrak{m}}},\ \ a_3=\frac{\gamma}{3(2\alpha+1)\,B^{\mathfrak{m}}}\left\{\frac{(8\gamma-3)}{12}c_1^2+c_2\right\} \tag{2.8}$$

Using (2.8) and Lemma 2.1, we obtain

$$|a_2|=\left|\frac{2\gamma c_1}{3(\alpha+1)\,A^{\mathfrak{m}}}\right|\leq\frac{4\left|\gamma\right|}{3(\alpha+1)\,A^{\mathfrak{m}}},$$

and

$$|a_3|=\left|\frac{\gamma}{3(2\alpha+1)\,B^{\mathfrak{m}}}\left\{\frac{(8\gamma-3)}{12}c_1^2+c_2\right\}\right|=\frac{\left|\gamma\right|}{3(2\alpha+1)\,B^{\mathfrak{m}}}\left|\frac{(8\gamma-3)}{12}c_1^2+c_2\right|$$

resulting in

$$|a_3| \leq \frac{\left|\gamma\left(8\gamma-3\right)\right|}{9\left(2\alpha+1\right)B^{\mathfrak{M}}}.$$

From (2.8), we obtain

$$a_3-\rho a_2^2=\frac{\gamma}{3\left(2\alpha+1\right)B^{\mathfrak{M}}}\left\{c_2-c_1^2\left(\frac{16\rho\gamma\left(2\alpha+1\right)B^{\mathfrak{M}}-\left(8\gamma-3\right)\left(\alpha+1\right)^2A^{2\mathfrak{M}}}{12\left(\alpha+1\right)^2A^{2\mathfrak{M}}}\right)\right\}.$$

Let $\rho=\left(\frac{16\rho\gamma(2\alpha+1)B^{\mathfrak{M}}-(8\gamma-3)(\alpha+1)^2A^{2\mathfrak{M}}}{12(\alpha+1)^2A^{2\mathfrak{M}}}\right)$, thus, we have

$$\left|a_3-\rho a_2^2\right|=\frac{\left|\gamma\right|}{3\left(2\alpha+1\right)B^{\mathfrak{M}}}\left|c_2-c_1^2\rho\right|,$$

Applying Lemma 2.2 for ρ, we obtain the required result. Thus, we have completed the proof of Theorem 2.1.

Let us now obtain the accuracies based on the estimates in (2.1)−(2.3).

The equality stated in (2.1) is valid, if $c_1=2$. Equivalently, we have $p\left(\psi\right)\equiv p_1\left(\psi\right)=\frac{1+\psi}{1-\psi}$. It follows that the extremal function in $\mathcal{S}_{car}^{(\alpha,\gamma)}\left(\lambda,\mu\,;\varphi\right)_{\mathfrak{M}}$, takes the form of

$$\frac{\psi\left(\Delta_{\lambda,\mu}^{\mathfrak{B},\mathfrak{M}}f\left(\psi\right)\right)'}{\Delta_{\lambda,\mu}^{\mathfrak{B},\mathfrak{M}}f\left(\psi\right)}=\frac{1+\left(2\gamma-1\right)\psi}{1-\psi} \tag{2.9}$$

Next, in (2.2), the equality is valid if $c_1=c_2=2$. So, the extremal functions in $\mathcal{S}_{car}^{(\alpha,\gamma)}\left(\lambda,\mu\,;\varphi\right)_{\mathfrak{M}}$ is given by (2.9). Finally, the equality holds in (2.3). The extremal function obtained for (2.1)−(2.2) is also valid for (2.3).

Thus, the proof of Theorem 2.1 is is now complete.

In addition,

Theorem 2.2. *Let $f\in\mathcal{S}_{car}^{(\alpha,\gamma)}\left(\lambda,\mu\,;\varphi\right)_{\mathfrak{M}}$ and has the form (1.1). Then for a complex number ρ*

$$\left|a_3-\rho a_2^2\right|\leq\frac{2\left|\rho\gamma\right|}{3\left(2\alpha+1\right)B^{\mathfrak{M}}}\max\left\{1,\left|\frac{16\gamma\left(2\alpha+1\right)B^{\mathfrak{M}}-\rho\left(8\gamma+3\right)\left(\alpha+1\right)^2A^{2\mathfrak{M}}}{6\rho\left(\alpha+1\right)^2A^{2\mathfrak{M}}}\right|\right\}.$$

Remark 2.1. *For $\alpha=0$ in Theorem 2.1, we obtain the new result for the subclass $\mathcal{S}_{car}^{(\gamma)}\left(\lambda,\mu\,;\varphi\right)_{\mathfrak{M}}$ as follows:*

Theorem 2.3. *If A function f of the form (1.1), is in the class $\mathcal{S}_{car}^{(\gamma)}\left(\lambda,\mu\,;\varphi\right)_{\mathfrak{M}}$, with $\gamma\in\mathbb{C}\setminus\{0\}$, $\rho\in\mathbb{C}$ and $0\leq\mu\leq\lambda$, $\mathfrak{M}\in\mathbb{N}_0$, then*

$$|a_2|\leq\frac{4\left|\gamma\right|}{3A^{\mathfrak{M}}},$$

$$|a_3|\leq\frac{\left|\gamma\left(8\gamma+3\right)\right|}{B^{\mathfrak{M}}}$$

and

$$\left|a_3 - \rho a_2^2\right| \leq \frac{2\left|\gamma\right|}{3B^{\mathfrak{M}}} \max\left\{1, \left|\frac{16\rho\gamma B^{\mathfrak{M}} - (8\gamma - 3) A^{2\mathfrak{M}}}{6A^{2\mathfrak{M}}}\right|\right\}.$$

Remark 2.2. *For $\alpha = 1$ in Theorem 2.1, following new result is obtained for the subclass $C_{car}^{(\gamma)}(\lambda, \mu ;\varphi)_{\mathfrak{M}}$.*

Theorem 2.4. *Let γ be a nonzero complex number and $0 \leq \mu \leq \lambda$, $\mathfrak{M} \in \mathbb{N}_0$, $\rho \in \mathbb{C}$.If f, represented in the form (1.1), is in $C_{car}^{(\gamma)}(\lambda, \mu ;\varphi)_{\mathfrak{M}}$, then*

$$|a_2| \leq \frac{4\left|\gamma\right|}{6A^{\mathfrak{M}}},$$

$$|a_3| \leq \frac{\left|\gamma (8\gamma + 3)\right|}{27B^{\mathfrak{M}}}$$

and

$$\left|a_3 - \rho a_2^2\right| \leq \frac{2\left|\gamma\right|}{3(2\alpha + 1) B^{\mathfrak{M}}} \max\left\{1, \left|\frac{48\rho\gamma B^{\mathfrak{M}} - 4(8\gamma - 3) A^{2\mathfrak{M}}}{24A^{2\mathfrak{M}}}\right|\right\}.$$

Remark 2.3. *For $\gamma = 1$ and particular values for the parameters $\mathfrak{M}$, λ and μ in Theorem 2.3 and Theorem 2.4, we obtain the following known results for the subclasses $\mathcal{S}_{car}^{*}$ of starlike functions and C_{car} of convex functions associated with cardioid domain studied by Lei Shi et al. in [19].*

Theorem 2.5. *If $f(\psi)$ of the form (1.1) belongs to $\mathcal{S}_{car}^{*}$, then*

$$|a_2| \leq \frac{4}{3},$$

and

$$|a_3| \leq \frac{11}{9}.$$

Theorem 2.6. *If $f \in C_{car}$ and has the series form (1.1), then*

$$|a_2| \leq \frac{2}{3},$$

and

$$|a_3| \leq \frac{11}{27}.$$

Remark 2.4. *For $\gamma = 1$, $\alpha = 0$ and specific values to the parameters $\mathfrak{M}$, λ and μ in Theorem 2.2, we obtain the following known result investiagtd by Kanika Sharma et al. in [17]*

Theorem 2.7. *If $f(\psi) = \psi + \sum_{n=2}^{\infty} a_n \psi^n \in \mathcal{S}_{car}^{*}$, then*

$$\left|a_3 - \rho a_2^2\right| \leq \frac{2\left|\rho\right|}{3} \max\left\{1, \left|\frac{16 - 11\rho}{6\rho}\right|\right\}.$$

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References

[1] M. Fekete, G. Szegö, Eine Bemerkung Über Ungerade Schlichte Funktionen, J. Lond. Math. Soc. s1-8 (1933), 85–89. https://doi.org/10.1112/jlms/s1-8.2.85.

[2] A. Pfluger, The Fekete-Szegö Inequality by a Variational Method, Ann. Acad. Sci. Fenn., Ser. A I Math. 10 (1985), 447–454. https://doi.org/10.5186/aasfm.1985.1049.

[3] W.C. Ma, D. Minda, A Unified Treatment of Some Special Classes of Univalent Functions, in: Proceedings of the Conference on Complex Analysis, Tianjin, China, (1992), 157-169.

[4] W. Janowski, Extremal Problems for a Family of Functions with Positive Real Part and for Some Related Families, Ann. Polon. Math. 23 (1970), 159–177. https://doi.org/10.4064/ap-23-2-159-177.

[5] J. Sokół, J. Stankiewicz, Radius of Convexity of Some Subclasses of Strongly Starlike Functions, Zesz. Nauk. Politech. Rzesz. Mat. Fiz. 19 (1996), 101-105.

[6] N.E. Cho, V. Kumar, S.S. Kumar, V. Ravichandran, Radius Problems for Starlike Functions Associated with the Sine Function, Bull. Iran. Math. Soc. 45 (2019), 213–232. https://doi.org/10.1007/s41980-018-0127-5.

[7] R. Mendiratta, S. Nagpal, V. Ravichandran, On a Subclass of Strongly Starlike Functions Associated with Exponential Function, Bull. Malays. Math. Sci. Soc. 38 (2015), 365–386. https://doi.org/10.1007/s40840-014-0026-8.

[8] K. Ullah, H.M. Srivastava, A. Rafiq, M. Arif, S. Arjika, A Study of Sharp Coefficient Bounds for a New Subfamily of Starlike Functions, J. Inequal. Appl. 2021 (2021), 194. https://doi.org/10.1186/s13660-021-02729-1.

[9] Op. Nave, Modification of Semi-Analytical Method Applied System of ODE, Mod. Appl. Sci. 14 (2020), 75-81. https://doi.org/10.5539/mas.v14n6p75.

[10] M.A. Nasr, M.K. Aouf, Starlike Function of Complex Order, J. Nat. Sci. Math. 25 (1985), 1-12.

[11] P. Wiatrowski, The Coefficients of a Certain Family of Holomorphic Functions, Univ. Lodzk. Nauk. Math. Przyrod. Ser. II, Zeszyt 39 (1971), 75-85.

[12] V. Ravichandran, Y. Polatoglu, M. Bolcal, A. Sen, Certain Subclasses of Starlike and Convex Functions of Complex Order, Hacettepe J. Math. Stat. 34 (2005), 9-15.

[13] E. Deniz, H. Orhan, The Fekete-Szegö Problem for a Generalized Subclass of Analytic Functions, Kyungpook Math. J. 50 (2010), 37–47. https://doi.org/10.5666/KMJ.2010.50.1.037.

[14] G.Ş. Šl'gean, Subclasses of Univalent Functions, Complex Analysis, Proc. 5th Rom.-Finn. Semin, Bucharest (1981), 362-372.

[15] F.M. Al-Oboudi, On Univalent Functions Defined by a Generalized Sălăgean Operator, Int. J. Math. Math. Sci. 2004 (2004), 1429–1436. https://doi.org/10.1155/S0161171204108090.

[16] D. Raducanu, H. Orhan, Subclasses of Analytic Functions Defined by a Generalized Differential Operator, Int. J. Math. Anal. 4 (2010), 1-15.

[17] K. Sharma, N.K. Jain, V. Ravichandran, Starlike Functions Associated with a Cardioid, Afr. Mat. 27 (2016), 923–939. https://doi.org/10.1007/s13370-015-0387-7.

[18] C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, (1975).

[19] L. Shi, I. Ali, M. Arif, N.E. Cho, S. Hussain, H. Khan, A Study of Third Hankel Determinant Problem for Certain Subfamilies of Analytic Functions Involving Cardioid Domain, Mathematics 7 (2019), 418. https://doi.org/10.3390/math7050418.