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FIXED POINT AND TRIPLED FIXED POINT THEOREMS UNDER PATA-TYPE CONDITIONS IN ORDERED METRIC SPACES

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Abstract. In this paper, we first prove a version of the fixed point theorem obtained in [V. Pata, A fixed point theorem in metric spaces, J. Fixed Point Theory Appl. 10 (2011) 299–305], adjusted for monotone mappings in ordered metric spaces, as well as some generalizations. Then we apply them to obtain results of this type for tripled fixed points in two cases—for monotone and mixed-monotone mappings with three variables. An example is given to show the difference between some of these results.

1. Introduction

A very interesting extension of the Banach Contraction Principle was recently obtained by V. Pata in [1]. Some researchers followed this approach and already several other fixed point results in the spirit of Pata have appeared, see, e.g., [3, 4, 2, 5].

On the other hand, fixed points of monotone mappings in ordered metric spaces have been a matter of investigation ever since the first results given by Ran and Reurings in [6]. This includes so-called coupled and tripled fixed points. Generally speaking, fixed point results in ordered spaces use weaker contractive conditions (restricted to comparable pairs of points), but at the expense of an additional assumption that the given mapping is monotone.

Some coupled fixed point results with Pata-type conditions have been recently obtained in [3, 4].

In this paper, we first prove "ordered versions" of the basic Pata's result, as well as some generalizations. Then we apply them to obtain results of this type for tripled fixed points in two cases—for monotone and mixed-monotone mappings with three variables. An example is given to show the difference between some of these results.

2. Preliminaries

We begin with some notation and preliminaries. Throughout the paper, $(\mathcal{X}, d, \preceq)$ always denotes a partially ordered metric space, i.e., a triple where $(\mathcal{X}, \preceq)$ is a partially ordered set and $(\mathcal{X}, d)$ is a metric space.

For $x, y \in \mathcal{X}$, $x \asymp y$ will denote that x and y are comparable, i.e., either $x \preceq y$ or $y \preceq x$ holds.

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Recall that the space $(\mathcal{X}, d, \preceq)$ is said to be regular if it has the following properties:

(i) if for a non-decreasing sequence $\{x_n\}$, $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all n;

(ii) if for a non-increasing sequence $\{x_n\}$, $x_n \to x$ as $n \to \infty$, then $x_n \succeq x$ for all n.

Throughout the paper, $\psi : [0,1] \to [0,\infty)$ will be a fixed increasing function, continuous at zero, satisfying $\psi(0) = 0$.

3. Pata-type fixed point results in ordered metric spaces

In this section, x_0 will be an arbitrary stable point in the given ordered metric space $(\mathcal{X}, d, \preceq)$, and $\|x\|$ will be defined by

$$\|x\| = d(x, x_0).$$

It will be clear that the results do not depend on the particular choice of point x_0.

Theorem 3.1. *Let the space $(\mathcal{X}, d, \preceq)$ be complete and let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants. Let $f : \mathcal{X} \to \mathcal{X}$ be a non-decreasing map such that there exists x_0 satisfying $x_0 \asymp fx_0$ and suppose that the inequality*

$$d(fx, fy) \leq (1-\varepsilon)d(x,y) + \Lambda\varepsilon^\alpha\psi(\varepsilon)[1 + \|x\| + \|y\|]^\beta \tag{3.1}$$

is satisfied for every $\varepsilon \in [0,1]$ and all $x, y \in \mathcal{X}$ with $x \asymp y$. If f is continuous or $(\mathcal{X}, d, \preceq)$ is regular, then f has a fixed point $z \in \mathcal{X}$. Moreover,

(i) the set of fixed points of f is a singleton if and only if it is totally ordered;

(ii) the set of fixed points of f is a singleton if for every two points $u, v \in \mathcal{X}$ there exists $w \in \mathcal{X}$, comparable with u, v and fw.

Proof. As remarked before the formulation of theorem, we can take that the point for which $x_0 \asymp fx_0$ is the same as the one for which $\|x\|$ is defined to be equal to $d(x, x_0)$. Also, without loss of generality, we can assume that $x_0 \preceq fx_0$. Then the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n$, $n = 0, 1, \dots,$ is non-decreasing. Suppose further that $x_n \neq fx_n$ for each n (otherwise there is nothing to prove).

Since any two terms of $\{x_n\}$ are comparable, the inequality (3.1) can be used in the same way as in the proof of [1, Theorem 1] to obtain that:

(1) The sequence $\{d(x_n, x_{n+1})\}$ is strictly decreasing and tends to some $d^* \geq 0$ as $n \to \infty$;

(2) The sequence $\{c_n\}$ is bounded, where $c_n = \|x_n\|$;

(3) $d^* = 0$.

(4) $\{x_n\}$ is a Cauchy sequence, thus converging to some $z \in X$.

If the mapping f is continuous, then $fx_n \to fz = z$ since $fx_n = x_{n+1}$.

If the space $(X, d, \preceq)$ is regular, then for the non-decreasing sequence $\{x_n\}$ we have that $x_n \preceq z$ for each n. Now, for $\varepsilon = 0$, we get from (3.1) that

$$d(fx_n, fz) \leq d(x_n, z) \to 0$$

wherefrom $fx_n \to fz$, i.e., $fz = z$.

Uniqueness of the fixed point.

(i) If the set of fixed points $Fix(f)$ is a singleton, then it is totally ordered. Conversely, assume that $Fix(f)$ is totally ordered and that u, v are two (comparable) fixed points of f. Applying (3.1), we get

$$d(u,v) = d(fu, fv) \leq (1-\varepsilon)d(u,v) + K\varepsilon^\alpha\psi(\varepsilon),$$

where $[1+\|u\|+\|v\|]^{\beta}=K>0$, i.e.,

$$\varepsilon d(u,v)\le K\varepsilon^{\alpha}\psi(\varepsilon),$$

for each $\varepsilon\in[0,1]$, and it follows that $u=v$.

(ii) Suppose now that for every two points $u,v\in\mathcal{X}$ there exists $w\in\mathcal{X}$, comparable with u, v and fw. Assume that u and v are distinct fixed points of f. If they are comparable, we get a contradiction as in (i). If not, choose w as stated. Then, $u=f^nu\asymp f^nw$ and $v=f^nv\asymp f^nw$; we will prove that $d(u,f^nw)\downarrow u^*=0$ and $d(v,f^nw)\downarrow v^*=0$.

Indeed, for $\varepsilon=0$, we get from (3.1) that

$$d(u,f^nw)=d(ff^{n-1}u,ff^{n-1}w)\le d(f^{n-1}u,f^{n-1}w)=d(u,f^{n-1}w)$$

and, similarly,

$$d(v,f^nw)=d(ff^{n-1}v,ff^{n-1}w)\le d(f^{n-1}v,f^{n-1}w)=d(v,f^{n-1}w),$$

i.e., $d(u,f^nw)\downarrow u^*$ and $d(v,f^nw)\downarrow v^*$. It remains to prove that $u^*=v^*=0$.

We will prove first that the sequence $c_n=\|f^nw\|$ is bounded. We have

$$c_n=d(f^nw,x_0)\le d(f^nw,f^{n+1}w)+d(f^{n+1}w,fw)+d(fw,x_0). \tag{3.2}$$

Since, by assumption, $w\asymp fw$, we have that the sequence $d(f^nw,f^{n+1}w)$ decreases. Indeed, taking again $\varepsilon=0$ in (3.1), we get that

$$d(f^nw,f^{n+1}w)=d(ff^{n-1}w,ff^nw)\le d(f^{n-1}w,f^nw)\le\cdots\le d(w,fw).$$

Then it follows from (3.2) that

$$\begin{aligned}c_n&\le d(w,fw)+d(ff^nw,fw)+d(fw,w)+d(w,x_0)\\&=2d(w,fw)+d(w,x_0)+(1-\varepsilon)d(w,f^nw)\\&\qquad+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|f^nw\|+\|w\|]^{\beta}\\&\le 2d(w,fw)+d(w,x_0)+(1-\varepsilon)d(w,x_0)\\&\qquad+(1-\varepsilon)d(x_0,f^nw)+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+c_n+d(w,x_0)]^{\beta}.\end{aligned}$$

Hence, we get that

$$\varepsilon c_n\le a\varepsilon^{\alpha}\psi(\varepsilon)c_n^{\alpha}+b,$$

for some constants $a,b>0$. Now, in the same way as in [1, Lemma 2.1], it follows that $\{c_n\}$ is a bounded sequence.

We are now able to prove that, e.g., $u^*=0$. Indeed,

$$\begin{aligned}d(u,f^nw)=d(ff^{n-1}u,ff^{n-1}w)&\le(1-\varepsilon)d(f^{n-1}u,f^{n-1}w)\\&\quad+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|u\|+\|f^{n-1}w\|]^{\beta},\end{aligned}$$

i.e.,

$$d(u,f^nw)\le(1-\varepsilon)d(u,f^{n-1}w)+K\varepsilon^{\alpha}\psi(\varepsilon),$$

for some $K>0$. Passing to the limit as $n\to\infty$, we get that

$$u^*\le(1-\varepsilon)u^*+K\varepsilon^{\alpha}\psi(\varepsilon),$$

i.e., $u^*=0$. In the same way, $v^*=0$ is proved.

It follows that

$$d(u,v)\le d(u,f^nw)+d(f^nw,v)\to 0+0=0,$$

i.e., $u=v$. □

Remark 3.1. Theorem 3.1 is strictly stronger than [6, Theorem 2.1]. On the one side, the hypotheses of [6, Theorem 2.1] imply those of Theorem 3.1, which follows in the same way as it was proved in [1, §3] that the classical Banach's contractive condition implies Pata's condition (3.1).

On the other side, the example of function

$$f : [1, +\infty) \to [1, +\infty), \quad f(x) = -2 + x - 2\sqrt{x} + 4\sqrt[4]{x}$$

(see [1, Example, p. 303]) shows that condition (3.1) can be satisfied when Banach's condition is not. It is also an example of the situation when condition (ii) for the uniqueness of fixed point (in the previous theorem) is fulfilled (since the given space is totally ordered).

It is well known that there are a lot of generalizations of Banach Contraction Principle, obtained by modifying the basic contractive conditions (see, e.g., [7]). Some of them already have their Pata-type versions (see [4, 2]). We shall present here a result of Pata-type for so-called generalized contractions, in the "ordered" version.

Theorem 3.2. *Let the space* $(\mathcal{X}, d, \preceq)$ *be complete and let* $\Lambda \geq 0$, $\alpha \geq 1$ *and* $\beta \in [0, \alpha]$ *be fixed constants. Let* $f : \mathcal{X} \to \mathcal{X}$ *be a non-decreasing map such that there exists* x_0 *satisfying* $x_0 \asymp fx_0$ *and suppose that the inequality*

$$(3.3) \quad d(fx, fy) \leq (1 - \varepsilon) \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2} \right\} + \Lambda \varepsilon^\alpha \psi(\varepsilon)[1 + \|x\| + \|y\| + \|fx\| + \|fy\|]^\beta$$

is satisfied for every $\varepsilon \in [0, 1]$ *and all* $x, y \in \mathcal{X}$ *with* $x \asymp y$. *If* f *is continuous or* $(\mathcal{X}, d, \preceq)$ *is regular, then* f *has a fixed point* $z \in \mathcal{X}$. *Moreover,*

(i) the set of fixed points of f *is a singleton if and only if it is totally ordered;*

(ii) the set of fixed points of f *is a singleton if for every two points* $u, v \in \mathcal{X}$ *there exists* $w \in \mathcal{X}$, *comparable with* u, v *and* fw.

Proof. **1.** As usual, starting with the given point x_0, construct the sequence $\{x_n\}$ by $x_{n+1} = fx_n$, $n = 0, 1, \ldots$ Similarly as in the proof of Theorem 3.1, this sequence is monotone, hence condition (3.3) can be used for its elements. Suppose that $x_n \neq x_{n+1}$ for each n.

In order to prove that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing, suppose, to the contrary, that

$$d(x_k, x_{k+1}) = \max\{d(x_{k-1}, x_k), d(x_k, x_{k+1})\}$$

for some $k \in \mathbb{N}$. Then, applying (3.3) with $x = x_{k-1}$, $y = x_k$, we get that

$$\begin{aligned} d(x_k, x_{k+1}) &= d(fx_{k-1}, fx_k) \\ &\leq (1 - \varepsilon) \max \left\{ d(x_{k-1}, x_k), d(x_k, x_{k+1}), \frac{1}{2} d(x_{k-1}, x_{k+1}) \right\} \\ &\quad + \Lambda \varepsilon^\alpha \psi(\varepsilon)[1 + \|x_{k-1}\| + 2\|x_k\| + \|x_{k+1}\|]^\beta \\ &= (1 - \varepsilon) d(x_k, x_{k+1}) + K \varepsilon^\alpha \psi(\varepsilon), \end{aligned}$$

for some $K > 0$. It follows that $d(x_k, x_{k+1}) = 0$, a contradiction! Hence, $\{d(x_n, x_{n+1})\}$ is a (strictly) decreasing sequence, thus tending to some $d^* \geq 0$.

2. Denote $c_n = \|x_n\|$. We will prove that the sequence $\{c_n\}$ is bounded.

We have that

$$
\begin{aligned}
c_n &= d(x_n, x_0) \le d(x_n, x_{n+1}) + d(fx_n, fx_0) + c_1 \\
&\le 2c_1 + (1-\varepsilon)\max\left\{d(x_n, x_0), d(x_n, x_{n+1}), d(x_0, x_1), \frac{1}{2}(d(x_n, x_1) + d(x_{n+1}, x_0))\right\} \\
&\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x_n\| + \|x_{n+1}\| + \|x_1\|]^{\beta} \\
&\le 2c_1 + (1-\varepsilon)\max\{c_n, c_1, c_n + c_1\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + c_n + c_1 + c_n + c_1]^{\beta} \\
&\le 2c_1 + (1-\varepsilon)(c_n + c_1) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + 2c_1 + 2c_n]^{\alpha}
\end{aligned}
$$

(it was used that $d(x_n, x_{n+1}) \le c_1$, $d(x_n, x_1) + d(x_{n+1}, x_0) \le d(x_n, x_0) + d(x_1, x_0) + d(x_{n+1}, x_n) + d(x_n, x_0) \le 2(c_n + c_1)$ and $\|x_{n+1}\| \le d(x_{n+1}, x_n) + d(x_n, x_0) \le c_1 + c_n$). Finally, we get that

$$\varepsilon c_n \le a\varepsilon^{\alpha}\psi(\varepsilon)c_n^{\alpha} + b$$

for some constants $a, b > 0$. In the same way as in the proof of [1, Lemma 3], it follows that the sequence $\{c_n\}$ is bounded.

3. Now we use the boundedness of $\{c_n\}$ to prove that $d^* = 0$.

Indeed, we have that

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(fx_n, fx_{n-1}) \\
&\le (1-\varepsilon)d(x_n, x_{n-1}) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x_n\| + 2\|x_{n-1}\| + \|x_{n+1}\|]^{\beta} \\
&\le (1-\varepsilon)d(x_n, x_{n-1}) + K\varepsilon^{\alpha}\psi(\varepsilon),
\end{aligned}
$$

for some $K > 0$. Passing to the limit as $n \to \infty$, it follows that $d^* = 0$.

4. In order to prove that $\{x_n\}$ is a Cauchy sequence, suppose the contrary. Then, using the standard procedure (see, e.g., [8, Lemma 2.1]), we get that there exist $\delta > 0$ and two increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$, such that $n_k > m_k > k$ and the sequences $d(x_{n(k)+1}, x_{m(k)})$ and $d(x_{n(k)}, x_{m(k)-1})$ tend to δ as $n \to \infty$. Putting $x = x_{n(k)}$, $y = x_{m(k)-1}$ in (3.3), and using the boundedness of $\{c_n\}$, we get that

$$d(x_{n(k)+1}, x_{m(k)}) \le (1-\varepsilon)d(x_{n(k)}, x_{m(k)-1}) + K\varepsilon^{\alpha}\psi(\varepsilon).$$

Passing to the limit as $k \to \infty$, we get that $\delta = 0$, a contradiction!

Hence, $\{x_n\}$ is a Cauchy sequence, and it converges to some $z \in X$.

5. The proof that $fz = z$ in either of the given cases is the same as for Theorem 3.1.

6. The uniqueness of the the fixed point under one of the assumptions (i) or (ii) can be proved similarly as in Theorem 3.1.

□

Remark 3.2. Similarly as in the classical situation, treated in [7], it can be proved that Theorem 3.2 contains as special cases several other Pata-type results in their order versions. In particular, this includes Kannan, Chatterjea, Reich, Zamfirescu and Hardy-Rogers results. Since the exact formulations and proofs are obvious, we omit the details.

4. Tripled fixed point results for monotone and mixed-monotone mappings

We will use the following terminology.

Definition 4.1. *Let $F : \mathcal{X}^3 \to \mathcal{X}$ be a mapping.*

(1) *F is called non-decreasing if it is non-decreasing in all three variables.*
(2) *F is called mixed-monotone if it is non-decreasing in the first and third variables, and non-increasing in the second variable.*
(3) *A point $Y = (x, y, z) \in \mathcal{X}^3$ is called a tripled fixed point of the first kind (or Borcut kind* [9]*) if*

$$F(x,y,z) = z, \quad F(y,x,z) = y, \quad F(z,y,x) = z. \tag{4.1}$$

(4) *A point $Y = (x, y, z) \in \mathcal{X}^3$ is called a tripled fixed point of the second kind (or Berinde-Borcut kind* [10]*) if*

$$F(x,y,z) = z, \quad F(y,x,y) = y, \quad F(z,y,x) = z. \tag{4.2}$$

Remark 4.1. In what follows, tripled fixed point results of the first kind will be proved for monotone mappings, while those of the second type will be connected with mixed-monotone mappings. It will be clear in the sequel that part (3) of the previous definition can be modified in several ways. In fact, any three combinations of elements x, y, z can be taken instead of (x, y, z), (y, x, z) and (z, y, x) in (4.1), with the only condition that the first entry of each triple matches the right-hand side. In particular, the "cyclic" case, i.e., the condition

$$F(x,y,z) = x, \quad F(y,z,x) = \text{ and } F(z,x,y) = z$$

can be considered. It will also be clear which modifications should be made to the results that follows, so we will not state them explicitly. Moreover, the same treatment can be applied in the case of arbitrary number of variables.

It is important to notice that this considerably differs from the case of "mixed-monotone situation". Namely, as was shown in [11], in this case only some particular combinations are possible (in particular, the cyclic case cannot be treated in this way).

The following lemma is easy to prove.

Lemma 4.1. *(i) If relations $\sqsubseteq_1$ and $\sqsubseteq_2$ are defined on $\mathcal{X}^3$ by*

$$Y \sqsubseteq_1 V \Leftrightarrow x \preceq u \wedge y \preceq v \wedge z \preceq w, \quad Y = (x,y,z),\ V = (u,v,w) \in \mathcal{X}^3$$

and

$$Y \sqsubseteq_2 V \Leftrightarrow x \preceq u \wedge y \succeq v \wedge z \preceq w, \quad Y = (x,y,z),\ V = (u,v,w) \in \mathcal{X}^3,$$

and $D : \mathcal{X}^3 \times \mathcal{X}^3 \to \mathbb{R}^+$ is given by

$$D(Y,V) = d(x,u) + d(y,v) + d(z,w), \quad Y = (x,y,z),\ V = (u,v,w) \in \mathcal{X}^3,$$

then $(\mathcal{X}^3, D, \sqsubseteq_i)$, $i = 1, 2$ are ordered metric space. The space $(\mathcal{X}^3, D)$ is complete if and only if $(\mathcal{X}, d)$ is complete. Moreover, the spaces $(\mathcal{X}^3, D, \sqsubseteq_i)$ are regular if and only if $(\mathcal{X}, d, \preceq)$ is such.

(ii) If $F : \mathcal{X}^3 \to \mathcal{X}$ is non-decreasing (w.r.t. $\preceq$), then the mapping $T_F^1 : \mathcal{X}^3 \to \mathcal{X}^3$ given by

$$T_F^1 Y = (F(x,y,z), F(y,x,z), F(z,y,x)), \quad Y = (x,y,z) \in \mathcal{X}^3$$

is non-decreasing w.r.t. $\sqsubseteq_1$.

(iii) If $F : \mathcal{X}^3 \to \mathcal{X}$ is mixed-monotone, then the mapping $T_F^2 : \mathcal{X}^3 \to \mathcal{X}^3$ given by

$$T_F^2 Y = (F(x,y,z), F(y,x,y), F(z,y,x)), \quad Y = (x,y,z) \in \mathcal{X}^3$$

is non-decreasing w.r.t. $\sqsubseteq_2$.

(iv) The mappings T_F^i, $i = 1, 2$ are continuous if and only if F is continuous.

(v) The mapping F has a tripled fixed point of the first (resp. of the second) kind if and only if the mapping T_F^1 (resp. T_F^2) has a fixed point in $\mathcal{X}^3$.

If what follows, $Y_0 = (x_0, y_0, z_0)$ will be a fixed element in $\mathcal{X}^3$ and for $Y = (x, y, z) \in \mathcal{X}^3$, we will denote

$$\|Y\| = \|x, y, z\| = D(Y, Y_0) = d(x, x_0) + d(y, y_0) + d(z, z_0).$$

It will be clear that the obtained results do not depend on the particular choice of the point Y_0.

We will prove first some results for monotone mappings and tripled fixed points of the first (Borcut) kind.

Theorem 4.1. *Let $F : \mathcal{X}^3 \to \mathcal{X}$ be a non-decreasing mapping, and suppose that there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \preceq F(y_0, x_0, z_0)$, $z_0 \preceq F(z_0, y_0, x_0)$. Let, for some fixed constants $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$, the inequality*

$$\begin{aligned}(4.3)\quad & d(F(x,y,z), F(u,v,w)) + d(F(y,x,z), F(v,u,w)) + d(F(z,y,x), F(w,v,u)) \\ & \leq (1-\varepsilon)(d(x,u) + d(y,v) + d(z,w)) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x,y,z\| + \|u,v,w\|]^{\beta}\end{aligned}$$

holds for all $\varepsilon \in [0, 1]$ and all $x, y, z, u, v, w \in \mathcal{X}$ with ($x \preceq u$, $y \preceq v$ and $z \preceq w$) or ($x \succeq u$, $y \succeq v$ and $z \succeq w$). Finally, suppose that F is continuous or that the space is regular. Then F has a tripled fixed point $Y^* = (x^*, y^*, z^*) \in \mathcal{X}^3$ of the first kind.

Proof. Consider the space $(\mathcal{X}^3, D, \sqsubseteq_1)$ and the mapping $T_F^1 : \mathcal{X}^3 \to \mathcal{X}^3$, as defined in Lemma 4.1.(i) and (ii). The mapping T_F^1 is non-decreasing w.r.t. $\sqsubseteq_1$. Let $Y = (x, y, z)$ and $V = (u, v, w)$ be comparable w.r.t. $\sqsubseteq_1$, i.e., let ($x \preceq u$, $y \preceq v$ and $z \preceq w$) or ($x \succeq u$, $y \succeq v$ and $z \succeq w$) hold. Then, the condition (4.3) holds, which can be written as

$$D(T_F^1 Y, T_F^1 V) \leq (1-\varepsilon)D(Y, V) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + D(Y, Y_0) + D(V, Y_0)]^{\beta}.$$

In other words, T_F^1 satisfies condition of the type (3.1) in the space $(\mathcal{X}^3, D, \sqsubseteq_1)$. Applying Theorem 3.1, we obtain that T_F^1 has a fixed point $Y^* = (x^*, y^*, z^*) \in \mathcal{X}^3$, which is, by Lemma 4.1.(v), a tripled fixed point of the first kind of mapping F. □

Corollary 4.1. *Let $F : \mathcal{X}^3 \to \mathcal{X}$ be a non-decreasing mapping, and suppose that there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \preceq F(y_0, x_0, z_0)$, $z_0 \preceq F(z_0, y_0, x_0)$. Let, for some fixed constants $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$, the inequality*

$$\begin{aligned}(4.4)\quad & d(F(x,y,z), F(u,v,w)) \\ & \leq \frac{1-\varepsilon}{3}(d(x,u) + d(y,v) + d(z,w)) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x,y,z\| + \|u,v,w\|]^{\beta}\end{aligned}$$

holds for all $\varepsilon \in [0, 1]$ and all $x, y, z, u, v, w \in \mathcal{X}$ with ($x \preceq u$, $y \preceq v$ and $z \preceq w$) or ($x \succeq u$, $y \succeq v$ and $z \succeq w$). Finally, suppose that F is continuous or that the space is regular. Then F has a tripled fixed point $Y^* = (x^*, y^*, z^*) \in \mathcal{X}^3$ of the first kind.

Proof. Suppose that $Y = (x, y, z), V = (u, v, w) \in \mathcal{X}^3$ are comparable w.r.t. $\sqsubseteq_1$. Applying (4.4) to the triples (x, y, z) and (u, v, w), we get that

$$\begin{aligned}(4.5)\quad & d(F(x,y,z), F(u,v,w)) \\ & \leq \frac{1-\varepsilon}{3}D(Y, V) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + D(Y, Y_0) + D(V, Y_0)]^{\beta}.\end{aligned}$$

Applying the same inequality to the triples (y,x,z) and (v,u,w), we obtain

$$\begin{aligned}(4.6)\qquad & d(F(y,x,z),F(v,u,w))\\ &\le \frac{1-\varepsilon}{3}D(Y,V)+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+d(y,x_0)+d(x,y_0)+d(z,z_0)\\ &\qquad +d(v,x_0)+d(u,y_0)+d(w,z_0)]^{\beta}\\ &\le \frac{1-\varepsilon}{3}D(Y,V)+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+D(Y,Y_0)+D(V,Y_0)+4d(x_0,y_0)]^{\beta}.\end{aligned}$$

Finally, applying (4.4) to the triples (z,y,x) and (w,v,u), we get

$$\begin{aligned}(4.7)\qquad & d(F(z,y,x),F(w,v,u))\\ &\le \frac{1-\varepsilon}{3}D(Y,V)+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+d(z,x_0)+d(y,y_0)+d(x,z_0)\\ &\qquad +d(w,x_0)+d(v,y_0)+d(u,z_0)]^{\beta}\\ &\le \frac{1-\varepsilon}{3}D(Y,V)+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+D(Y,Y_0)+D(V,Y_0)+4d(x_0,z_0)]^{\beta}.\end{aligned}$$

Adding up the inequalities (4.5), (4.6) and (4.7), and writing temporarily $A = D(Y,Y_0)+D(V,Y_0)$, we get the following estimate:

$$\begin{aligned}(4.8)\qquad & D(T_F^1Y,T_F^1V)\le(1-\varepsilon)D(Y,V)\\ &\qquad +\Lambda\varepsilon^{\alpha}\psi(\varepsilon)\{[1+A]^{\beta}+[1+A+4d(x_0,y_0)]^{\beta}+[1+A+4d(x_0,z_0)]^{\beta}\}\end{aligned}$$

Now,

$$\begin{aligned}&[1+A]^{\beta}+[1+A+4d(x_0,y_0)]^{\beta}+[1+A+4d(x_0,z_0)]^{\beta}\\ &\quad=[1+A]^{\beta}[1+(1+\frac{4d(x_0,y_0)}{1+A})^{\beta}+(1+\frac{4d(x_0,z_0)}{1+A})^{\beta}]\\ &\quad\le[1+A]^{\beta}[1+(1+4d(x_0,y_0))^{\beta}+(1+4d(x_0,z_0))^{\beta}]\\ &\quad=C[1+A]^{\beta},\end{aligned}$$

where C is a constant (not depending on Y, V and ε). Hence, putting $\Lambda_1=\Lambda C$, (4.8) can be written as

$$D(T_F^1Y,T_F^1V)\le(1-\varepsilon)D(Y,V)+\Lambda_1\varepsilon^{\alpha}\psi(\varepsilon)[1+D(Y,Y_0)+D(V,Y_0)]^{\beta},$$

which means that all the conditions of Theorem 4.1 are fulfilled. □

The following example shows that Theorem 4.1 is strictly stronger than Corollary 4.1.

Example 4.1. Let $\mathcal{X} = \mathbb{R}$ be equipped with the usual metric and order. The mapping $F : \mathcal{X}^3 \to \mathcal{X}$ defined by $F(x, y, z) = \frac{1}{8}(5x + y + z)$ is obviously nondecreasing. It is easy to obtain that

$$\begin{aligned}
&d(F(x,y,z),F(u,v,w)) + d(F(y,x,z),F(v,u,w)) + d(F(z,x,y),F(v,u,w))\\
&= |\frac{5x+y+z}{8} - \frac{5u+v+w}{8}| + |\frac{5y+x+z}{8} - \frac{5v+u+w}{8}|\\
&\quad + |\frac{5z+y+x}{8} - \frac{5w+v+u}{8}|\\
&\le \frac{5}{8}|x-u| + \frac{1}{8}|y-v| + \frac{1}{8}|z-w| + \frac{5}{8}|y-v| + \frac{1}{8}|x-u| + \frac{1}{8}|z-w|\\
&\quad + \frac{5}{8}|z-w| + \frac{1}{8}|y-v| + \frac{1}{8}|x-u|\\
&= \frac{7}{8}[d(x,u) + d(y,v) + d(z,w)],
\end{aligned}$$

i.e., $D(T_F^1 Y, T_F^1 V) \le \lambda D(Y, V)$, where $\lambda = \frac{7}{8}$. Similarly as in [1, §3], it can be proved that also (4.3) holds for appropriate Λ, α and β, all $\varepsilon \in [0, 1]$ and all comparable $Y, V \in \mathcal{X}^3$.

On the other hand, suppose that the condition (4.4) of Corollary 4.1 holds, i.e.,

$$\begin{aligned}
&|\frac{5x+y+z}{8} - \frac{5u+v+w}{8}|\\
&\quad \le \frac{1-\varepsilon}{2}[|x-u| + |y-v| + |z-w|] + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x,y,z\| + \|u,v,w\|]^{\beta}
\end{aligned}$$

is satisfied for each $\varepsilon \in [0, 1]$ and all comparable $Y, V \in \mathcal{X}^3$. Taking $\varepsilon = 0$, $y = v$ and $z = w$, we obtain that

$$\frac{5}{8}|x-u| \le \frac{1}{2}|x-u|$$

which obviously cannot hold (except when $x = u$).

Consider now mixed-monotone mappings and tripled fixed points of the second (Berinde-Borcut)) kind.

Theorem 4.2. *Let $F : \mathcal{X}^3 \to \mathcal{X}$ be a mixed-monotone mapping, and suppose that there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, z_0)$, $z_0 \preceq F(z_0, y_0, x_0)$. Let, for some fixed constants $\Lambda \ge 0$, $\alpha \ge 1$ and $\beta \in [0, \alpha]$, the inequality*

$$\begin{aligned}
&d(F(x,y,z),F(u,v,w)) + d(F(y,x,y),F(v,u,v)) + d(F(z,y,x),F(w,v,u))\\
&\le (1-\varepsilon)(d(x,u) + d(y,v) + d(z,w)) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x,y,z\| + \|u,v,w\|]^{\beta}
\end{aligned}$$

holds for all $\varepsilon \in [0, 1]$ and all $x, y, z, u, v, w \in \mathcal{X}$ with ($x \preceq u$, $y \succeq v$ and $z \preceq w$) or ($x \succeq u$, $y \preceq v$ and $z \succeq w$). Finally, suppose that F is continuous or that the space is regular. Then F has a tripled fixed point $Y^* = (x^*, y^*, z^*) \in \mathcal{X}^3$ of the second kind.

Proof. The proof is similar to the proof of Theorem 4.1, using the mapping T_F^2 of Lemma 4.1.(iii) in the space $(\mathcal{X}^3, D, \sqsubseteq_2)$. □

Corollary 4.2. *Let $F : \mathcal{X}^3 \to \mathcal{X}$ be a mixed-monotone mapping, and suppose that there exist $x_0, y_0, z_0 \in \mathcal{X}$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, z_0)$,*

$z_0 \preceq F(z_0, y_0, x_0)$. *Let, for some fixed constants* $\Lambda \geq 0$, $\alpha \geq 1$ *and* $\beta \in [0, \alpha]$, *the inequality*

$$\begin{aligned} &d(F(x,y,z), F(u,v,w)) \\ &\leq \frac{1-\varepsilon}{3}(d(x,u)+d(y,v)+d(z,w)) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|x,y,z\|+\|u,v,w\|]^{\beta} \end{aligned}$$

holds for all $\varepsilon \in [0,1]$ *and all* $x, y, z, u, v, w \in \mathcal{X}$ *with* ($x \preceq u$, $y \succeq v$ *and* $z \preceq w$) *or* ($x \succeq u$, $y \preceq v$ *and* $z \succeq w$). *Finally, suppose that* F *is continuous or that the space is regular. Then* F *has a tripled fixed point* $Y^* = (x^*, y^*, z^*) \in \mathcal{X}^3$ *of the first kind.*

A similar example as Example 4.1 can be constructed to show that Theorem 4.2 is strictly stronger than Corollary 4.2.

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Conflict of interests

The authors declares that there is no conflict of interests regarding the publication of this article.

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